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A RANDOM WALK AND ITS LIL IN A BANACH SPACE*

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Abstract

Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. Banach space valued random variables with $E[X_n] = 0$ and $E\|X_n\|^2 < \infty$, and let $S_0 = 0$, $S_n = X_1 + X_2 + \dots + X_n$, $n \geq 1$. We prove that if $\{S_n : n \geq 1\}$ satisfies the LIL in B then the sequence $\{\eta_n : n \geq 1\}$ satisfies the LIL in $C([0, 1], B)$, where $\eta_n(t) = S_{[nt]} + (nt - [nt]) X_{[nt]+1}$, $0 \leq t \leq 1$ and $C([0, 1], B) = \{f : [0, 1] \rightarrow B \mid f \text{ is continuous}\}$. We also use this result to give an alternative to the proof of the LIL of Brownian motion in Banach spaces.

1. Introduction

Let B be a real separable Banach space with the norm $\|\cdot\|$ and B^* be its topological dual. Throughout, $\{X_n : n \geq 1\}$ always denotes a sequence of i.i.d. B -valued random variables on a probability space $(\Omega, \mathcal{A}, P)$ with $E(X_n) = 0$ and $E\|X_n\|^2 < \infty$. Note that $E\|X_n\|^2 < \infty$ assures the existence of a covariance operator

$$T(f, g) = E[f(X_n)g(X_n)], \quad f, g \in B^*.$$

Let μ denote the mean zero Gaussian measure on B with the given covariance operator whenever this measure exists. Let $H_\mu \subseteq B$ denote the reproducing kernel Hilbert space of μ . This pair of spaces (B, H_μ) is often referred to as an abstract Wiener space [4]. Perhaps one of the most important properties of abstract Wiener space is the existence of a constant $M > 0$ such that $\|x\| \leq M\|x\|_\mu$ for every x in H_μ , where $\|\cdot\|_\mu$ is the norm of H_μ . Consequently, through the continuous injection $i : H_\mu \rightarrow B$ and the restriction map $i^* : B^* \rightarrow H_\mu^*$ we have the relation $B^* \subseteq H_\mu^* \approx H_\mu \subseteq B$. Let $\{W(t) : t \geq 0\}$ denote μ -Brownian motion with the transition probability $P_t(a, A) = \mu((A - a)/t^{1/2})$. It is known that $\{W(t) : 0 \leq t \leq 1\}$ induces a mean zero, Gaussian measure P_w on the

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measure space $(C_B, \mathcal{F})$, where C_B is the space of continuous functions w from $[0, 1]$ into B with $w(0)=0$, and $\mathcal{F}$ is the σ -field generated by the functions $w \rightarrow w(t)$. P_w is called abstract Wiener measure. See [4], [5] and [13] for expositions of concepts of μ -Brownian motion.

In this paper, we are interested in the random walk $\{\eta_n : n \geq 1\}$ defined by

$$(1) \quad \eta_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad 0 \leq t \leq 1,$$

where $S_0=0$,

$$S_k = X_1 + X_2 + \cdots + X_k$$

for $k \geq 1$ and $[r]$ mean the greatest integer which is less than or equal to r . We say that the sequence $\{X_n : n \geq 1\}$ satisfies the central limit theorem (CLT) in B if the distribution of $S_n/n^{1/2}$ converges weakly to $\mu(\mathcal{L}(S_n/n^{1/2}) \Rightarrow \mu)$. We say that the sequence $\{S_n : n \geq 1\}$ satisfies the law of iterated logarithm (LIL) in B if there exists a compact, symmetric convex $K \subset B$ such that

$$(2) \quad P\{\lim_n d(S_n/(2nLLn)^{1/2}, K) = 0\} = 1$$

and

$$(3) \quad P\{C(\{S_n/(2nLLn)^{1/2}\}) = K\} = 1,$$

where

$$d(x, K) = \inf_{y \in K} \|x - y\|,$$

$C(\{X_n\})$ means the set of strong limit points of the sequence $\{X_n : n \geq 1\}$ in B and $LLn=1$ if $n=1, 2$, $=\log \log n$ if $n \geq 3$. The equivalence among the boundedness of $E\|X_n\|^2$, CLT and LIL are well known in [6], [15] and [16] when $B=\mathbf{R}^k$. However, when B is a general Banach space, there is no implication among those three concepts as can be seen in [2], [7] and [10]. The main purpose of this paper is to show that the LIL of $\{S_n : n \geq 1\}$ in B implies the LIL of $\{\eta_n : n \geq 1\}$ in C_B . We also use this result to give an alternative to the proof of the LIL of Brownian motion in Banach spaces. A work of the same spirit but different content is [8] in which $\mathcal{L}(S_n/n^{1/2}) \Rightarrow \mu$ implies $\mathcal{L}(\eta_n/n^{1/2}) \Rightarrow P_w$ has been established.

The following necessary and sufficient condition for LIL in B will be used in proving our main result.

THEOREM 1. (Kuelbs [11, p.745]) *Let $X_1, X_2, \dots$ be i.i.d. B -valued such that $E(X_n)=0$ and $E\|X_n\|^2 < \infty$. Then the sequence $\{S_n : n \geq 1\}$ satisfies the LIL in B if and only if*

$$(4) \quad P\{\{S_n/(2nLLn)^{1/2} : n \geq 1\} \text{ is relatively compact in } B\} = 1.$$

For other results on LIL in abstract spaces, see [9], [12], [13] and [14].

2. Main result

Let $C([0, 1], B)$ be the space of continuous functions f from $[0, 1]$ into B . $C([0, 1], B)$ is a real separable Banach space under the norm

$$\|f\|_\infty = \sup_{0 \leq t \leq 1} \|f(t)\|.$$

A family

$$\{f_\alpha : \alpha \in A\} \subset C([0, 1], B)$$

is said to be uniformly equicontinuous if

$$(5) \quad \lim_{\delta \rightarrow 0} \sup_{\alpha \in A} \sup_{|s-t| < \delta} \|f_\alpha(s) - f_\alpha(t)\| = 0$$

The following lemma (without condition (ii)) is known as the Arzelá-Ascoli theorem when $\dim(B) < \infty$. However in a general Banach space B , its proof does not seem to be available in literature. Therefore, we include its proof in the Appendix for the sake of completeness.

LEMMA 1. (Arzelá-Ascoli Theorem) *A subset $\{f_\alpha : \alpha \in A\}$ is relatively compact in $C([0, 1], B)$ if and only if*

$$(i) \quad \sup_{\alpha \in A} \|f_\alpha(0)\| < \infty,$$

$$(ii) \quad \text{for each } t \text{ in } [0, 1], \text{ the set } \{f_\alpha(t) : \alpha \in A\} \text{ is relatively compact in } B$$

and

$$(iii) \quad \{f_\alpha : \alpha \in A\} \text{ is uniformly equicontinuous in } C([0, 1], B).$$

THEOREM 2. (Main Result) *Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. B -valued random variables such that $E(X_n) = 0$ and $E\|X_n\|^2 < \infty$, and let $\{\eta_n : n \geq 1\}$ be as in (1). Then $\{S_n : n \geq 1\}$ satisfies the LIL in B if and only if $\{\eta_n : n \geq 1\}$ satisfies the LIL in C_B .*

PROOF. The sufficiency of the theorem is trivial. Therefore we shall only prove the necessity. By Lemma 1 and Theorem 1, this is equivalent to show that the following three conditions hold.

$$(i) \quad P\{\{\eta_n(0)/(2nLLn)^{1/2} : n \geq 1\} \text{ is bounded in } B\} = 1;$$

$$(ii) \quad P\{\{\eta_n(t)/(2nLLn)^{1/2} : n \geq 1\} \text{ is relatively compact in } B\} = 1 \text{ for each } t \text{ in } [0, 1];$$

and

$$(iii) \quad P\{\{\eta_n/(2nLLn)^{1/2} : n \geq 1\} \text{ is uniformly equicontinuous in } C_B\} = 1.$$

Condition (i) follows immediately from the fact that for $n = 1, 2, \dots$ $P\{\eta_n(0) = 0\} = 1$. The validity of (ii) and (iii) are treated in Lemma 2 and Lemma 3 respectively.

LEMMA 2. *Let $\{X_n : n \geq 1\}$ and $\{\eta_n : n \geq 1\}$ be as in Theorem 2. If $\{S_n : n \geq 1\}$ satisfies the LIL in B then for each t in $[0, 1]$*

$$(6) \quad P\{\{\eta_n(t)/(2nLLn)^{1/2} : n \geq 1\} \text{ is relatively compact in } B\} = 1.$$

PROOF. By Theorem 1, there exists a compact, symmetric, convex $K \subset B$ such that $P(\mathcal{Q}_1) = P(\mathcal{Q}_2) = 1$, where

$$\mathcal{Q}_1 = \{\lim_n d(S_n / (2nLLn)^{1/2}, K) = 0\}$$

and

$$\mathcal{Q}_2 = \{C(\{S_n / (2nLLn)^{1/2}\}) = K\}.$$

Now let $\omega \in \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $t \in [0, 1]$ be given (when $t=0$ the conclusion is trivial), and let $\tilde{K} = a_t K = \{a_t y : y \in K\}$, where

$$a_t = \lim_n ([nt] LL[nt] / nLLn)^{1/2}.$$

We claim that

$$(7) \quad \lim_n d(\eta_n(t, \omega) / (2nLLn)^{1/2}, \tilde{K}) = 0$$

and

$$(8) \quad C(\{\eta_n(t, \omega) / (2nLLn)^{1/2}\}) = \tilde{K}.$$

These two conclusions will then imply (6), by Theorem 1.

For the proof of (7), we simply observe that

$$\lim_n \|X_{[nt]+1}(\omega) / (2nLLn)^{1/2}\| = 0.$$

Then

$$\begin{aligned} & \lim_n \inf_{y \in K} \|\eta_n(t, \omega) / (2nLLn)^{1/2} - z\| \\ & \leq \lim_n \inf_{y \in K} a_t \|S_{[nt]}(\omega) / (2[nt]LL[nt])^{1/2} - y\| \\ & + \lim_n \|X_{[nt]+1}(\omega) / (2nLLn)^{1/2}\| = 0. \end{aligned}$$

For the proof of (8), let $z \in \tilde{K}$. Then $z = a_t y$ for some $y \in K$. By the assumption, we have a subsequence $\{n(k) : k \geq 1\}$ of $\{n : n \geq 1\}$ such that

$$\lim_k \|S_{n(k)}(\omega) / (2n(k)LLn(k))^{1/2} - y\| = 0.$$

This implies that

$$\lim_k \|\eta_{n(k)}(t) / (2n(k)LLn(k))^{1/2} - z\| = 0$$

by the same argument as in the proof of (7).

REMARK 1. The proof of Lemma 2 actually yields that if $\{S_n : n \geq 1\}$ satisfies LIL in B then

$$(9) \quad P\{\{(\eta_n(t) - \eta_n(s)) / \lambda(n, t, s) : n \geq 1\} \text{ is relatively compact in } B\} = 1$$

for any s, t in $[0, 1]$ with $s < t$, where

$$\lambda(n, t, s) = \{2([nt] - [ns])LL([nt] - [ns])\}^{1/2}.$$

This remark will be used in the proof of Lemma 3.

LEMMA 3. Let $\{X_n : n \geq 1\}$ and $\{\eta_n : n \geq 1\}$ be as in Theorem 2. If $\{S_n : n \geq 1\}$ satisfies the LIL in B then

$$(10) \quad P\{\{\eta_n / (2nLLn)^{1/2} : n \geq 1\} \text{ is uniformly equicontinuous in } C_B\} = 1.$$

PROOF. Let $\beta > 1$ be fixed, and let $n_r = [\beta^r]$. We shall prove that

$$(11) \quad P\{\{\eta_{n_r} / (2n_r LLn_r)^{1/2} : r \geq 1\} \text{ is uniformly equicontinuous in } C_B\} = 1.$$

Let $\Gamma > 0$ be such that for $n \geq 1$ and $s, t \in [0, 1]$

$$(12) \quad P\{\|\eta_n(t) - \eta_n(s)\| > \Gamma \lambda(n, t, s)\} = 0.$$

This Γ exists by Remark 1. For $r=1, 2, \dots$ and m_0 an arbitrary positive integer, define A_r by

$$A_r = \bigcup_{m=m_0}^{\infty} \bigcup_{k=1}^{2^m} A_{rkm},$$

where

$$A_{rkm} = \{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\| > \varepsilon_m (2n_r LLn_r)^{1/2}\}$$

and $\varepsilon_m = 2^{-m/2} \Gamma$. It is clear that (11) holds if $P\{A_r \text{ i. o. in } r\} = 0$ since $\varepsilon_m \downarrow 0$ as $m \rightarrow \infty$. Now

$$P(A_{rkm}) = P\{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\| / \lambda(n_r, k2^{-m}, (k-1)2^{-m}) \geq \varepsilon_m \theta(r, m, k)\}$$

where

$$\theta(r, m, k) = (2n_r LLn_r)^{1/2} / \lambda(n_r, k2^{-m}, (k-1)2^{-m})$$

and $\lambda(n, t, s)$ is as in (9). We choose m_0 sufficiently large such for $m \geq m_0$

$$\lambda(n_r, k2^{-m}, (k-1)2^{-m}) \leq 2^{-m/2} \{2n_r LLn_r\}^{1/2}.$$

Then

$$\begin{aligned} P(A_r) &\leq \sum_{m=m_0}^{\infty} \sum_{k=1}^{2^m} P(A_{rkm}) \\ &\leq \sum_{m=m_0}^{\infty} \sum_{k=1}^{2^m} P\{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\| / \lambda(n_r, k2^{-m}, (k-1)2^{-m}) \varepsilon_m 2^{-m/2}\} \\ &= \sum_{m=m_0}^{\infty} \sum_{k=1}^{2^m} P\{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\| > \Gamma \lambda(n_r, k2^{-m}, (k-1)2^{-m})\} \\ &= 0 \quad \text{by Remark 1 for } r=1, 2, \dots \end{aligned}$$

Consequently

$$\sum_{r=1}^{\infty} P(A_r) = 0.$$

Thus by Borel-Cantelli's lemma, we have proved $P\{A_r \text{ i. o. in } r\} = 0$.

We next want to prove that for each $\varepsilon > 0$ there exists a fixed $\beta_0 > 1$ such that for all β satisfying $1 < \beta \leq \beta_0$

$$(13) \quad P\{C_r \text{ i. o. in } r\} = 0,$$

where

$$C_r = \{ \max_{n_{r-1} \leq k \leq n_r} \|\eta_{n_{r-1}} / (2n_{r-1} LLn_{r-1})^{1/2} - \eta_k / (2kLLk)^{1/2}\|_\infty > \varepsilon \}.$$

This together with (11) will then conclude the lemma. Define

$$D_r = \{ \max_{n_{r-1} \leq k \leq n_r} \|\eta_{n_{r-1}} - \eta_k\|_\infty > \varepsilon (2n_{r-1} LLn_{r-1})^{1/2} \}$$

and

$$E_r = \{ \max_{n_{r-1} \leq k \leq n_r} \|\eta_k ((2kLLk)^{-1/2} - (2n_{r-1} LLn_{r-1})^{-1/2})\|_\infty > \varepsilon/2 \}.$$

Let $\alpha = \varepsilon / [4(\beta^{1/2} - 1)]$. We have

$$\begin{aligned} P(E_r) &\leq P\{\max_{k \leq n_r} \|\eta_k\|_\infty > \alpha (2n_r LLn_r)^{1/2}\} \\ &\leq \sum_{k=1}^{n_r} P\{\|\eta_k\|_\infty > \alpha (2kLLk)^{1/2}\} \\ &\leq \sum_{k=1}^{n_r} P\{\sup_{0 < |s-t| \leq 1} \|\eta_k(t) - \eta_k(s)\| > \alpha (2kLLk)^{1/2}\} \\ &\leq \sum_{k=1}^{n_r} P\{\sup_{0 < |s-t| \leq 2^{-m}} \|\eta_k(t) - \eta_k(s)\| > \alpha (2kLLk)^{1/2}/2^m\} \end{aligned}$$

for any $m \geq 1$.

Let $\beta > 1$ be sufficiently close to one such that

$$\alpha/2^m = \varepsilon/2^{m+2}(\beta^{1/2} - 1) > \Gamma,$$

where Γ is as in (12). Then

$$\begin{aligned} (14) \quad P(E_r) &\leq \sum_{k=1}^{n_r} P\{\sup_{0 < |s-t| \leq 2^{-m}} \|\eta_k(t) - \eta_k(s)\| > \Gamma (2kLLk)^{1/2}\} \\ &\leq \sum_{k=1}^{n_r} \sum_{j=1}^{2^m} P\{\|\eta_k(j2^{-m}) - \eta_k((j-1)2^{-m})\| > \Gamma (2kLLk)^{1/2}\} \\ &= 0 \quad \text{by Remark 1.} \end{aligned}$$

Now

$$\begin{aligned} P(D_r) &\leq \sum_{k=n_{r-1}}^{n_r} \sum_{j=1}^{k-n_{r-1}+1} P\{\|Y_{rj}\|_\infty > \gamma(r, j)\} \\ &\leq \sum_{k=n_{r-1}}^{n_r} \sum_{j=1}^{k-n_{r-1}+1} P\{\sup_{0 < |s-t| \leq 2^{-m}} \|Y_{rj}(t) - Y_{rj}(s)\| > \gamma(r, j)/2^m\}, \end{aligned}$$

where

$$Y_{rj} = \eta_{n_{r-1}+j} - \eta_{n_{r-1}+j-1}$$

and

$$\gamma(r, j) = (2(n_r - n_{r-1}) LL(n_r - n_{r-1}))^{1/2} / 4j(\beta - 1)^{1/2}.$$

Choosing $\beta > 1$ sufficiently close to 1 so that

$$\varepsilon / j 2^{m+2} (\beta - 1)^{1/2} > \Gamma,$$

where Γ is as in (12), and applying Remark 1 and argument as in (14), we have

$$\begin{aligned} & P\left\{ \sup_{0 < |s-t| < 2^{-m}} \|Y_{rj}(t) - Y_{rj}(s)\| > \gamma(r, j) / 2^m \right\} \\ & \leq P\left\{ \sup_{0 < |s-t| \leq 2^{-m}} \|Y_{rj}(t) - Y_{rj}(s)\| > \gamma(r, j) / 2^m \right\} \\ & \leq P\left\{ \sup_{0 < |s-t| \leq 2^{-m}} \|Y_{rj}(t) - Y_{rj}(s)\| > \phi(r) \right\} \\ & \leq \sum_{l=1}^{2^m} P\{\|\eta_{n_{r-1}+j}(l2^{-m}) - \eta_{n_{r-1}+j}((l-1)2^{-m})\| > \phi(r) / 2\} \\ & \leq \sum_{l=1}^{2^m} P\{\|\eta_{n_{r-1}+j-1}(l2^{-m}) - \eta_{n_{r-1}+j-1}((l-1)2^{-m})\| > \phi(r) / 2\} \\ & = 0, \end{aligned}$$

where

$$\phi(r) = \Gamma (2(n_r - n_{r-1}) LL(n_r - n_{r-1}))^{1/2}.$$

Thus $P(D_r) = 0$. Now

$$\sum_{r=1}^{\infty} P(C_r) \leq \sum_{r=1}^{\infty} P(D_r) + \sum_{r=1}^{\infty} P(E_r) = 0.$$

By the Borel-Cantelli lemma, we have $P(C_r \text{ i. o. in } r) = 0$. This completes the proof of Lemma 3.

3. An application

Let $\{Z_n : n \geq 1\}$ be a sequence of independent copies of Z , where Z is the mean zero, B -valued Gaussian random variable whose distribution is μ . We define

$$(15) \quad W_n(t) = \sum_{i=0}^{\lfloor nt \rfloor} Z_i + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}, \quad 0 \leq t \leq 1.$$

Note that for each $n=1, 2, \dots$ the stochastic process $\{W_n(t) : 0 \leq t \leq 1\}$ is essentially the polygonalized Brownian motion. That is

$$W_n\left(\frac{l}{n}\right) = W(l)$$

for $l=1, 2, \dots, n$, and is linear on intervals $[(l-1)/n, l/n]$. It is known from [14] that

$$\left\{ \sum_{i=1}^n Z_i : n \geq 1 \right\}$$

satisfies the LIL in B . Therefore the sequence $\{W_n : n \geq 1\}$ satisfies the LIL in C_B by Theorem 2. In this section, we shall illustrate an alternative to the proof of the LIL for Brownian motion in Banach space given by Kuelbs and LePage [13].

THEOREM 3. (Kuelbs and LePage [13]) *Let $\{W(t) : t \geq 0\}$ be μ -Brownian motion in B . Then the sequence $\{\xi_n : n \geq 1\}$ satisfies the LIL in C_B , where $\xi_n(t) = W(nt)$, $0 \leq t \leq 1$. Furthermore the compact, symmetric convex $K \subset C_B$ described in (2) and (3) can be characterized as follows:*

$$(16) \quad K = \{f \in C_B : f(t) \in H_\mu \text{ for each } t \in [0, 1], \text{ and}$$

$$\sum_j \int_0^1 [(d/dt) x_j^*(f)(t)]^2 dt \leq 1\},$$

where $\{x_j^* : j \geq 1\} \subset B^*$ is such that the set

$$\left\{ \int_B x_j^*(x) x d\mu(x) : j \geq 1 \right\}$$

forms a complete orthonormal system for H_μ .

PROOF. The characterization of the set K in (16) is from Lemma 4 of [13]. It remains to show that

$$P\{\lim_n \|\xi_n - W_n\|_\infty / (2nLLn)^{1/2} = 0\} = 1.$$

This together with the fact that $\{W_n : n \geq 1\}$ satisfies the LIL in C_B will then imply that $\{\xi_n : n \geq 1\}$ satisfies the LIL in C_B .

Let $\{\varepsilon_n : n \geq 1\}$ be a sequence of positive real numbers whose precise values will be determined later. We have

$$\begin{aligned} & P\{\|\xi_n - W_n\|_\infty / (2nLLn)^{1/2} \geq \varepsilon_n\} \\ & \leq \sum_{k=0}^{n-1} P\left\{ \sup_{k/n \leq t \leq (k+1)/n} \|W(nt) - W_n(t)\| \geq \varepsilon_n (2nLLn)^{1/2} \right\} \\ & = \sum_{k=0}^{n-1} P\left\{ \sup_{0 \leq t \leq n^{-1}} \|W(nt) - nt W(1)\| \geq \varepsilon_n (2nLLn)^{1/2} \right\}, \end{aligned}$$

since

$$W_n\left(\frac{l}{n}\right) = W(l)$$

and is linear on $[l/n, (l+1)/n]$ for $l=0, 1, 2, \dots, n$; $n=1, 2, \dots$.

Now

$$\begin{aligned}
 (17) \quad & P\{ \sup_{0 \leq t \leq n^{-1}} \|W(nt) - nt W(1)\| \geq \varepsilon_n (2nLLn)^{1/2} \} \\
 & \leq P\{ \sup_{0 \leq t \leq n^{-1}} \|W(nt) - nt W(n^{-1})\| \geq \varepsilon_n (2nLLn)^{1/2}/2 \} \\
 & \quad + P\{ \sup_{0 \leq t \leq n^{-1}} \|nt W(n^{-1}) - nt W(1)\| \geq \varepsilon_n (2nLLn)^{1/2}/2 \} \\
 & \leq 5P\{\|W(1)\| \geq \varepsilon_n (2nLLn)^{1/2}/4\}.
 \end{aligned}$$

From Fernique's estimate [2], there exist constants $\gamma > 0$ and $C > 0$ such that

$$\exp \{ \gamma \|W(1)\|^2 \} \leq C.$$

Applying Chebyshev's inequality and Fernique's estimate to the last expression in (17), we have

$$\begin{aligned}
 & P\{ \sup_{0 \leq t \leq n^{-1}} \|W(nt) - nt W(1)\| \geq \varepsilon_n (2nLLn)^{1/2} \} \\
 & \leq 5C \exp \{ -\gamma \varepsilon_n^2 nLLn/8 \}. \quad \text{Choose } \varepsilon_n = (LLn)^{-1/2}.
 \end{aligned}$$

Then $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$ and we have

$$\begin{aligned}
 & \sum_{n=1}^{\infty} P\{ \|\xi_n - W_n\|_{\infty} / (2nLLn)^{1/2} \geq \varepsilon_n \} \\
 & \leq \sum_{n=1}^{\infty} 5Cn \exp \{ -\gamma n/8 \} < \infty.
 \end{aligned}$$

By Borel-Cantelli's lemma, this implies that

$$P\{\lim_n \|\xi_n - W_n\|_{\infty} / (2nLLn)^{1/2} = 0\} = 1.$$

Appendix: Proof of Lemma 1

The necessity of Lemma 1 follows exactly as that of the Arzelá-Ascoli theorem in $C[0, 1]$ (see e. g. [1, p. 221]). We only have to prove the sufficiency.

Now assume that conditions (i)–(iii) hold. Let $\varepsilon > 0$ be given. Choose k large enough that

$$\sup_{\alpha \in A} \sup_{|s-t| < 1/k} \|f_{\alpha}(s) - f_{\alpha}(t)\| < \varepsilon.$$

Since

$$\|f_{\alpha}(t)\| \leq \|f_{\alpha}(0)\| + \sum_{i=1}^k \|f_{\alpha}(it/k) - f_{\alpha}((i-1)t/k)\|,$$

it follows that

$$(18) \quad \sup_{0 \leq t \leq 1} \sup_{\alpha \in A} \|f_{\alpha}(t)\| \equiv C < \infty. \quad \text{Let } K = \bigcup_{i=0}^k \{f_{\alpha}(i/k) : \alpha \in A\}.$$

Note that K is relatively compact (and hence is totally bounded). Therefore there exists a finite set $Q \subset B$ such that for any $x \in K$, $\|x - \tilde{x}\| < \varepsilon$ for some $\tilde{x} \in Q$.

Now let ϕ be the set of functions f from $[0, 1]$ into B such that $f(i/k) \in Q$ for $i=0, 1, 2, \dots, k$ and f is linear on $[(i-1)/k, i/k]$ for $i=1, 2, \dots, k$. Note that ϕ is a finite set. We claim that ϕ is a 5ε -net with respect to A . Then A is totally bounded and therefore is relatively compact since $C([0, 1], B)$ is complete. To show this, let $f_\alpha \in A$. Then $\|f_\alpha(i/k)\| \leq C$ for $i=0, 1, 2, \dots, k$, and there exists $g \in \phi$ such that

$$\|f_\alpha(i/k) - g(i/k)\| < \varepsilon, \quad i=0, 1, 2, \dots, k.$$

Now let $t_0 \in [0, 1]$ be such that

$$\|f_\alpha - g\|_\infty = \|f_\alpha(t_0) - g(t_0)\|,$$

and let i_0 be such that

$$i_0/k \leq t < (i_0+1)/k.$$

Then $\|g(i_0/k) - g(t_0)\| < 3\varepsilon$ and

$$\begin{aligned} \|f_\alpha - g\|_\infty &\leq \|f_\alpha(t_0) - f_\alpha(i_0/k)\| + \|f_\alpha(i_0/k) - g(i_0/k)\| \\ &\quad + \|g(i_0/k) - g(t_0)\| < \varepsilon + \varepsilon + 3\varepsilon = 5\varepsilon. \end{aligned}$$

This completes the proof of Lemma 1.

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