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Abstract

Let \( \{X_n : n \geq 1\} \) be a sequence of i.i.d. Banach space valued random variables with \( E[X_n] = 0 \) and \( E\|X_n\|^2 < \infty \), and let \( S_0 = 0, \quad S_n = X_1 + X_2 + \ldots + X_n, \quad n \geq 1 \). We prove that if \( \{S_n : n \geq 1\} \) satisfies the LIL in \( B \) then the sequence \( \{\tau_n : n \geq 1\} \) satisfies the LIL in \( C([0, 1], B) \), where \( \tau_n(t) = S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor} \), \( 0 \leq t \leq 1 \) and \( C([0, 1], B) = \{f : [0, 1] \to B \mid f \text{ is continuous}\} \). We also use this result to give an alternative to the proof of the LIL of Brownian motion in Banach spaces.

1. Introduction

Let \( B \) be a real separable Banach space with the norm \( \| \cdot \| \) and \( B^* \) be its topological dual. Throughout, \( \{X_n : n \geq 1\} \) always denotes a sequence of i.i.d. \( B \)-valued random variables on a probability space \( (\Omega, \mathcal{F}, P) \) with \( E(X_n) = 0 \) and \( E\|X_n\|^2 < \infty \). Note that \( E\|X_n\|^2 < \infty \) assures the existence of a covariance operator

\[
T(f, g) = E[f(X_n)g(X_n)], \quad f, g \in B^*.
\]

Let \( \mu \) denote the mean zero Gaussian measure on \( B \) with the given covariance operator whenever this measure exists. Let \( H_\mu \subseteq B \) denote the reproducing kernel Hilbert space of \( \mu \). This pair of spaces \( (B, H_\mu) \) is often referred to as an abstract Wiener space [4]. Perhaps one of the most important properties of abstract Wiener space is the existence of a constant \( M > 0 \) such that \( \|x\| \leq M\|x\|_\mu \) for every \( x \in H_\mu \), where \( \| \cdot \|_\mu \) is the norm of \( H_\mu \). Consequently, through the continuous injection \( i : H_\mu \to B \) and the restriction map \( i^* : B^* \to H_\mu^* \) we have the relation \( B^* \subseteq H_\mu^* \approx H_\mu \subseteq B \). Let \( \{W(t) : t \geq 0\} \) denote \( \mu \)-Brownian motion with the transition probability \( P_t(a, A) = \mu((A - a)/t^{1/2}) \). It is known that \( \{W(t) : 0 \leq t \leq 1\} \) induces a mean zero, Gaussian measure \( P_w \) on the

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measure space \((C_B, \mathcal{F})\), where \(C_B\) is the space of continuous functions \(w\) from \([0, 1]\) into \(B\) with \(w(0) = 0\), and \(\mathcal{F}\) is the \(\sigma\)-field generated by the functions \(w \rightarrow w(t)\). \(P_w\) is called abstract Wiener measure. See [4], [5] and [13] for expositions of concepts of \(\mu\)-Brownian motion.

In this paper, we are interested in the random walk \(\{\eta_n : n \geq 1\}\) defined by

\[
\eta_n(t) = S_{nt} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}, \quad 0 \leq t \leq 1,
\]

where \(S_0 = 0, S_k = X_1 + X_2 + \cdots + X_k\) for \(k \geq 1\) and \(\lfloor r \rfloor\) mean the greatest integer which is less than or equal to \(r\). We say that the sequence \(\{X_n : n \geq 1\}\) satisfies the central limit theorem (CLT) in \(B\) if the distribution of \(S_n/n^{1/2}\) converges weakly to \(\mu(\mathcal{L}(S_n/n^{1/2}) \Rightarrow \mu)\). We say that the sequence \(\{S_n : n \geq 1\}\) satisfies the law of iterated logarithm (LIL) in \(B\) if there exists a compact, symmetric convex \(K \subset B\) such that

\[
P\{\lim_{n} d(S_n/(2n \ln n)^{1/2}, K) = 0\} = 1
\]

and

\[
P\{C(S_n/(2n \ln n)^{1/2}) = K\} = 1,
\]

where

\[
d(x, K) = \inf_{y \in K} \|x - y\|,
\]

\(C(\{X_n\})\) means the set of strong limit points of the sequence \(\{X_n : n \geq 1\}\) in \(B\) and \(LLn=1\) if \(n=1, 2, =\log \log n\) if \(n \geq 3\). The equivalence among the boundedness of \(E\|X_n\|^2\), CLT and LIL are well known in [6], [15] and [16] when \(B = \mathbb{R}^k\). However, when \(B\) is a general Banach space, there is no implication among those three concepts as can be seen in [2], [7] and [10]. The main purpose of this paper is to show that the LIL of \(\{S_n : n \geq 1\}\) in \(B\) implies the LIL of \(\{\eta_n : n \geq 1\}\) in \(C_B\). We also use this result to give an alternative to the proof of the LIL of Brownian motion in Banach spaces. A work of the same spirit but different content is [8] in which \(\mathcal{L}(S_n/n^{1/2}) \Rightarrow \mu\) implies \(\mathcal{L}(\eta_n/n^{1/2}) \Rightarrow P_w\) has been established.

The following necessary and sufficient condition for LIL in \(B\) will be used in proving our main result.

**Theorem 1.** (Kuelbs [11, p. 745]) Let \(X_1, X_2, \ldots\) be i.i.d. \(B\)-valued such that \(E(X_1) = 0\) and \(E\|X_1\|^2 < \infty\). Then the sequence \(\{S_n : n \geq 1\}\) satisfies the LIL in \(B\) if and only if

\[
P\{\{S_n/(2n \ln n)^{1/2} : n \geq 1\} \text{ is relatively compact in } B\} = 1.
\]

For other results on LIL in abstract spaces, see [9], [12], [13] and [14].
2. Main result

Let $C([0, 1], B)$ be the space of continuous functions $f$ from $[0, 1]$ into $B$. $C([0, 1], B)$ is a real separable Banach space under the norm

$$
\|f\|_\infty = \sup_{t \in [0, 1]} \|f(t)\|.
$$

A family

$$
\{f_\alpha : \alpha \in \mathcal{A}\} \subset C([0, 1], B)
$$

is said to be uniformly equicontinuous if

$$
\lim_{\delta \to 0} \sup_{\alpha \in \mathcal{A}} \sup_{\|s-t\| < \delta} \|f_\alpha(s) - f_\alpha(t)\| = 0.
$$

The following lemma (without condition (ii)) is known as the Arzelá-Ascoli theorem when $\dim(B) < \infty$. However in a general Banach space $B$, its proof does not seem to be available in literature. Therefore, we include its proof in the Appendix for the sake of completeness.

**LEMMA 1. (Arzelá-Ascoli Theorem)** A subset $\{f_\alpha : \alpha \in \mathcal{A}\}$ is relatively compact in $C([0, 1], B)$ if and only if

(i) $\sup_{\alpha \in \mathcal{A}} \|f_\alpha(0)\| < \infty$,

(ii) for each $t$ in $[0, 1]$, the set $\{f_\alpha(t) : \alpha \in \mathcal{A}\}$ is relatively compact in $B$ and

(iii) $\{f_\alpha : \alpha \in \mathcal{A}\}$ is uniformly equicontinuous in $C([0, 1], B)$.

**THEOREM 2. (Main Result)** Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. $B$-valued random variables such that $E(X_1) = 0$ and $E\|X_1\|^2 < \infty$, and let $\{\eta_n : n \geq 1\}$ be as in (1). Then $\{S_n : n \geq 1\}$ satisfies the LIL in $B$ if and only if $\{\eta_n : n \geq 1\}$ satisfies the LIL in $C_B$.

**PROOF.** The sufficiency of the theorem is trivial. Therefore we shall only prove the necessity. By Lemma 1 and Theorem 1, this is equivalent to show that the following three conditions hold.

(i) $P\{(\eta_n(0)/2nLLn)^{1/2} : n \geq 1\}$ is bounded in $B$ = 1;

(ii) $P\{(\eta_n(t)/2nLLn)^{1/2} : n \geq 1\}$ is relatively compact in $B$ = 1 for each $t$ in $[0, 1]$;

and

(iii) $P\{(\eta_n/(2nLLn))^{1/2} : n \geq 1\}$ is uniformly equicontinuous in $C_B$ = 1.

Condition (i) follows immediately from the fact that for $n=1, 2, \ldots$ $P\{\eta_n(0) = 0\} = 1$. The validity of (ii) and (iii) are treated in Lemma 2 and Lemma 3 respectively.

**LEMMA 2.** Let $\{X_n : n \geq 1\}$ and $\{\eta_n : n \geq 1\}$ be as in Theorem 2. If $\{S_n : n \geq 1\}$ satisfies the LIL in $B$ then for each $t$ in $[0, 1]$

$$
P\{(\eta_n(t)/(2nLLn))^{1/2} : n \geq 1\} \text{ is relatively compact in } B = 1.
$$
PROOF. By Theorem 1, there exists a compact, symmetric, convex \(K \subseteq B\) such that \(P(Q_1) = P(Q_2) = 1\), where
\[
\Omega_1 = \{ \lim_n d(S_n/(2nLLn)^{1/2}, K) = 0 \}
\]
and
\[
\Omega_2 = \{ C(\{S_n/(2nLLn)^{1/2}\}) = K \}.
\]
Now let \(\omega \in \Omega_1 \cap \Omega_2\) and \(t \in [0, 1]\) be given (when \(t = 0\) the conclusion is trivial), and let \(\tilde{K} = a_t K = \{ a_t y : y \in K \}\), where
\[
a_t = \lim_n \left( \frac{\lfloor nt \rfloor}{\lfloor nt \rfloor + 1} \right)^{1/2}.
\]
We claim that
\[
(7) \lim_n d(\eta_n(t, \omega)/(2nLLn)^{1/2}, \tilde{K}) = 0
\]
and
\[
(8) C(\{\eta_n(t, \omega)/(2nLLn)^{1/2}\}) = \tilde{K}.
\]
These two conclusions will then imply (6), by Theorem 1.

For the proof of (7), we simply observe that
\[
\lim_n \inf \left\| X_{[nt]+1}(\omega)/(2nLLn)^{1/2} - z \right\| = 0.
\]
Then
\[
\lim_n \inf \left\| \eta_n(t, \omega)/(2nLLn)^{1/2} - z \right\| \leq \lim_n \inf \left\| a_t \frac{S_{\lfloor nt \rfloor}((\omega)}{(2\lfloor nt \rfloor LL\lfloor nt \rfloor)^{1/2} - y} \right\| + \lim_n \left\| X_{[nt]+1}(\omega)/(2nLLn)^{1/2} \right\| = 0.
\]

For the proof of (8), let \(z \in \tilde{K}\). Then \(z = a_t y\) for some \(y \in K\). By the assumption, we have a subsequence \(\{ n(k) : k \geq 1 \}\) of \(\{ n : n \geq 1 \}\) such that
\[
\lim_k \frac{\left\| S_{n(k)}(t, \omega)/(2n(k)LLn(k))^{1/2} - y \right\| = 0.
\]
This implies that
\[
\lim_k \frac{\left\| \eta_{n(k)}(t)/(2n(k)LLn(k))^{1/2} - z \right\| = 0
\]
by the same argument as in the proof of (7).

REMARK 1. The proof of Lemma 2 actually yields that if \(\{ S_n : n \geq 1 \}\) satisfies LIL in \(B\) then
\[
(9) P\{ \{(\eta_n(t) - \eta_n(s))/\lambda(n, t, s) : n \geq 1\} \text{ is relatively compact in } B \} = 1
\]
for any \(s, t\) in \([0, 1]\) with \(s < t\), where
\[
\lambda(n, t, s) = \{ 2(\lfloor nt \rfloor - \lfloor ns \rfloor) LL(\lfloor nt \rfloor - \lfloor ns \rfloor) \}^{1/2}.
\]
This remark will be used in the proof of Lemma 3.
LEMMA 3. Let \( \{X_n : n \geq 1\} \) and \( \{\eta_n : n \geq 1\} \) be as in Theorem 2. If \( \{S_n : n \geq 1\} \) satisfies the LIL in \( B \) then

\[
(10) \quad P\left\{ \frac{\eta_n}{(2nLLn)^{1/2}} : n \geq 1 \right\} \text{ is uniformly equicontinuous in } C_B = 1.
\]

**Proof.** Let \( \beta > 1 \) be fixed, and let \( n_r = \lfloor \beta^r \rfloor \). We shall prove that

\[
(11) \quad P\left\{ \frac{\eta_n}{(2n_r LLn_r)^{1/2}} : r \geq 1 \right\} \text{ is uniformly equicontinuous in } C_B = 1.
\]

Let \( \Gamma > 0 \) be such that for \( n \geq 1 \) and \( s, t \in [0, 1] \)

\[
(12) \quad P\{\|\eta_n(t) - \eta_n(s)\| > \Gamma \lambda(n, t, s)\} = 0.
\]

This \( \Gamma \) exists by Remark 1. For \( r = 1, 2, \cdots \) and \( m_0 \) an arbitrary positive integer, define \( A_r \) by

\[
A_r = \bigcup_{m = m_0}^{\infty} \bigcup_{k = 1}^{m} A_{r,km},
\]

where

\[
A_{r,km} = \left\{ \frac{\eta_n(k2^{-m}) - \eta_n((k-1)2^{-m})}{\sqrt{\lambda(n, k2^{-m}, (k-1)2^{-m})}} \right\}
\]

and \( \varepsilon_m = 2^{-m/2} \Gamma \). It is clear that (11) holds if \( P\{A_r \text{ i.o. in } r\} = 0 \) since \( \varepsilon_m \downarrow 0 \) as \( m \to \infty \).

Now

\[
P(A_{r,km}) = P\{\|\eta_n(k2^{-m}) - \eta_n((k-1)2^{-m})\|/\lambda(n, k2^{-m}, (k-1)2^{-m}) \geq \varepsilon_m \theta(r, m, k)\}
\]

where

\[
\theta(r, m, k) = (2n_r LLn_r)^{1/2}/\lambda(n_r, k2^{-m}, (k-1)2^{-m})
\]

and \( \lambda(n, t, s) \) is as in (9). We choose \( m_0 \) sufficiently large such for \( m \geq m_0 \)

\[
\lambda(n_r, k2^{-m}, (k-1)2^{-m}) \leq 2^{-m/2} (2n_r LLn_r)^{1/2}.
\]

Then

\[
P(A_r) \leq \sum_{m = m_0}^{\infty} \sum_{k = 1}^{m} P(A_{r,km})
\]

\[
\leq \sum_{m = m_0}^{\infty} \sum_{k = 1}^{m} P\{\|\eta_n(k2^{-m}) - \eta_n((k-1)2^{-m})\|/\lambda(n, k2^{-m}, (k-1)2^{-m}) \geq \varepsilon_m \theta(r, m, k)\}
\]

\[
= \sum_{m = m_0}^{\infty} \sum_{k = 1}^{m} P\{\|\eta_n(k2^{-m}) - \eta_n((k-1)2^{-m})\| > \Gamma \lambda(n_r, k2^{-m}, (k-1)2^{-m})\}
\]

\[
= 0 \quad \text{by Remark 1 for } r = 1, 2, \cdots.
\]

Consequently

\[
\sum_{r = 1}^{\infty} P(A_r) = 0.
\]

Thus by Borel-Cantelli's lemma, we have proved \( P\{A_r \text{ i.o. in } r\} = 0 \).

We next want to prove that for each \( \varepsilon > 0 \) there exists a fixed \( \beta_0 > 1 \) such that for all \( \beta \) satisfying \( 1 < \beta \leq \beta_0 \)

\[
(13) \quad P\{C_r \text{ i.o. in } r\} = 0,
\]
where
\[ C_r = \left\{ \max_{n_{r-1} \leq k \leq n_r} \| \eta_{n_{r-1}} / (2n_{r-1} L L n_{r-1})^{1/2} - \gamma_k / (2k L L k)^{1/2} \|_\infty > \varepsilon \right\}. \]

This together with (11) will then conclude the lemma. Define
\[ D_r = \left\{ \max_{n_{r-1} \leq k \leq n_r} \| \eta_{n_{r-1}} - \gamma_k \|_\infty > \varepsilon (2n_{r-1} L L n_{r-1})^{1/2} \right\} \]
and
\[ E_r = \left\{ \max_{n_{r-1} \leq k \leq n_r} \| \gamma_k ((2k L L k)^{-1/2} - (2n_{r-1} L L n_{r-1})^{-1/2}) \|_\infty > \varepsilon / 2 \right\}. \]

Let \( \alpha = \varepsilon / [4(\beta^{1/2} - 1)] \). We have
\[
P(E_r) \leq P\{ \max_{k \leq n_r} \| \eta_k \|_\infty > \alpha (2n_{r-1} L L n_{r-1})^{1/2} \}
\]
\[
\leq \sum_{k=1}^{n_r} P\{ \| \eta_k \|_\infty > \alpha (2k L L k)^{1/2} \}
\]
\[
\leq \sum_{k=1}^{n_r} P\{ \sup_{0 < t < s \leq 1} \| \eta_k (t) - \eta_k (s) \| > \alpha (2k L L k)^{1/2} \}
\]
\[
\leq \sum_{k=1}^{n_r} P\{ \sup_{0 < t < s \leq 1} \| \eta_k (t) - \eta_k (s) \| > \alpha (2k L L k)^{1/2} / 2^{m} \}
\]
for any \( m \geq 1 \).

Let \( \beta > 1 \) be sufficiently close to one such that
\[
\alpha / 2^m = \varepsilon / 2^m (\beta^{1/2} - 1) > \Gamma,
\]
where \( \Gamma \) is as in (12). Then
\[
(14) \quad P(E_r)
\]
\[
\leq \sum_{k=1}^{n_r} P\{ \sup_{0 < t < s < 2^{-m}} \| \eta_k (t) - \eta_k (s) \| > \Gamma (2k L L k)^{1/2} \}
\]
\[
\leq \sum_{k=1}^{n_r} \sum_{j=1}^{2^m} P\{ \| \eta_k (j 2^{-m} - 1) - \eta_k ((j-1) 2^{-m}) \| > \Gamma (2k L L k)^{1/2} \}
\]
\[
= 0 \text{ by Remark 1.}
\]
Now
\[
P(D_r)
\]
\[
\leq \sum_{k=1}^{n_r} \sum_{j=1}^{k-n_{r-1} + 1} P\{ \| Y_{rj} \|_\infty > \gamma (r, j) \}
\]
\[
\leq \sum_{k=1}^{n_r} \sum_{j=1}^{k-n_{r-1} + 1} P\{ \sup_{0 < t < s \leq 2^{-m}} \| Y_{rj} (t) - Y_{rj} (s) \| > \gamma (r, j) / 2^{m} \},
\]
where

\[ Y_{r,j} = \eta_{n_{r-1}+j} - \eta_{n_{r-1}+j-1} \]

and

\[ \tau(r, j) = (2(n_r - n_{r-1}) LL (n_r - n_{r-1}))^{1/2}/4j(\beta - 1)^{1/2}. \]

Choosing \( \beta > 1 \) sufficiently close to 1 so that

\[
e / j^{2m+2}(\beta - 1)^{1/2} > \Gamma,
\]

where \( \Gamma \) is as in (12), and applying Remark 1 and argument as in (14), we have

\[
P\{ \sup_{0 \leq t \leq 2^{-m}} \| Y_{r,j}(t) - Y_{r,j}(s) \| > \tau(r, j)/2^m \}
\]

\[
\leq P\{ \sup_{0 \leq t \leq 2^{-m}} \| Y_{r,j}(t) - Y_{r,j}(s) \| > \tau(r, j)/2^m \}
\]

\[
\leq P\{ \sup_{0 \leq t \leq 2^{-m}} \| Y_{r,j}(t) - Y_{r,j}(s) \| > \phi(r) \}
\]

\[
\leq \sum_{i=1}^{2m} P\{ \| \eta_{n_{r-1}+j} (l2^{-m}) - \eta_{n_{r-1}+j} (l-1) 2^{-m} \| > \phi(r)/2 \}
\]

\[
\leq \sum_{i=1}^{2m} P\{ \| \eta_{n_{r-1}+j-1} (l2^{-m}) - \eta_{n_{r-1}+j-1} (l-1) 2^{-m} \| > \phi(r)/2 \}
\]

\[
= 0,
\]

where

\[ \phi(r) = \Gamma (2(n_r - n_{r-1}) LL (n_r - n_{r-1}))^{1/2}. \]

Thus \( P(D_r) = 0 \). Now

\[
\sum_{r=1}^{\infty} P(C_r) \leq \sum_{r=1}^{\infty} P(D_r) + \sum_{r=1}^{\infty} P(E_r) = 0.
\]

By the Borel-Cantelli lemma, we have \( P(C, \text{i.o. in } r) = 0 \). This completes the proof of Lemma 3.

### 3. An application

Let \( \{Z_n : n \geq 1\} \) be a sequence of independent copies of \( Z \), where \( Z \) is the mean zero, \( B \)-valued Gaussian random variable whose distribution is \( \mu \). We define

\[
W_n(t) = \sum_{0 \leq t \leq n} Z_t + (nt - [nt]) Z_{[nt]+1}, \quad 0 \leq t \leq 1.
\]

Note that for each \( n = 1, 2, \cdots \), the stochastic process \( \{W_n(t) : 0 \leq t \leq 1\} \) is essentially the polygonized Brownian motion. That is

\[ W_n \left( \frac{t}{n} \right) = W(t) \]
for $l=1,2,\ldots,n$, and is linear on intervals $[(l-1)/n, l/n]$. It is known from [14] that

$$\left\{ \sum_{i=1}^{n} Z_i : n \geq 1 \right\}$$

satisfies the LIL in $B$. Therefore the sequence $\{W_n : n \geq 1\}$ satisfies the LIL in $C_B$ by Theorem 2. In this section, we shall illustrate an alternative to the proof of the LIL for Brownian motion in Banach space given by Kuelbs and LePage [13].

**Theorem 3.** (Kuelbs and LePage [13]) Let $\{W(t) : t \geq 0\}$ be $\mu$-Brownian motion in $B$. Then the sequence $\{\xi_n : n \geq 1\}$ satisfies the LIL in $C_B$, where $\xi_n(t) = W(nt)$, $0 \leq t \leq 1$. Furthermore the compact, symmetric convex $K \subset C_B$ described in (2) and (3) can be characterized as follows:

$$K = \{ f \in C_B : f(t) \in H_\mu \text{ for each } t \in [0,1] \text{, and} \}
\sum_{j=1}^{j_1} \int_{0}^{1} \| (d/dt)x_j^*(f)(t) \|^2 dt \leq 1 \},$$

where $\{x_j^* : j \geq 1\} \subset B^*$ is such that the set

$$\left\{ \int_B x_j^*(x) x d\mu(x) : j \geq 1 \right\}$$

forms a complete orthonormal system for $H_\mu$.

**Proof.** The characterization of the set $K$ in (16) is from Lemma 4 of [13]. It remains to show that

$$P\{ \lim_{n} \| \xi_n - W_n \|_\infty / (2n LLn)^{1/2} = 0 \} = 1.$$  

This together with the fact that $\{W_n : n \geq 1\}$ satisfies the LIL in $C_B$ will then imply that $\{\xi_n : n \geq 1\}$ satisfies the LIL in $C_B$.

Let $\{\varepsilon_n : n \geq 1\}$ be a sequence of positive real numbers whose precise values will be determined later. We have

$$P\{ \| \xi_n - W_n \|_\infty / (2n LLn)^{1/2} \geq \varepsilon_n \} \leq \sum_{k=0}^{\infty} P\{ \sup_{k/n \leq t < (k+1)/n} \| W(nt) - W_n(t) \|_\infty \geq \varepsilon_n (2n LLn)^{1/2} \}$$

$$= \sum_{k=0}^{\infty} P\{ \sup_{0 \leq t \leq (1/n)^{-1}} \| W(nt) - nt W(1) \|_\infty \geq \varepsilon_n (2n LLn)^{1/2} \},$$

since

$$W_{\frac{k}{n}}(t) = W(t)$$

and is linear on $[(l/n, (l+1)/n)]$ for $l=0,1,2,\ldots,n$; $n=1,2,\ldots$.  

Now

(17) \[ P\left\{ \sup_{b \leq s \leq n} \| W(nt) - nt W(1) \| \geq \varepsilon_n (2nLLn)^{1/2} \right\} \]

\[ \leq \frac{1}{2} P\left\{ \sup_{b \leq s \leq n} \| W(nt) - nt W(nt) \| \geq \varepsilon_n (2nLLn)^{1/2} / 2 \right\} \]

\[ + P\left\{ \sup_{b \leq s \leq n} \| nt W(nt) - nt W(1) \| \geq \varepsilon_n (2nLLn)^{1/2} / 2 \right\} \]

\[ \leq 5 P\left\{ \| W(1) \| \geq \varepsilon_n (2nLLn)^{1/2} / 4 \right\} . \]

From Fernique’s estimate [2], there exist constants \( \gamma > 0 \) and \( C > 0 \) such that

\[ \exp \{ \gamma \| W(1) \|^q \} \leq C. \]

Applying Chebyshev’s inequality and Fernique’s estimate to the last expression in (17), we have

\[ P\left\{ \sup_{b \leq s \leq n} \| W(nt) - nt W(1) \| \geq \varepsilon_n (2nLLn)^{1/2} \right\} \]

\[ \leq 5C \exp \{ -\gamma \varepsilon_n^2 nLLn / 8 \} . \]

Choose \( \varepsilon_n = (LLn)^{-1/2} \).

Then \( \varepsilon_n \downarrow 0 \) as \( n \to \infty \) and we have

\[ \sum_{n=1}^{\infty} P\left\{ \| \hat{W}_n - W_n \|_{\infty} / (2nLLn)^{1/2} \leq \varepsilon_n \right\} \]

\[ \leq \sum_{n=1}^{\infty} 5Cn \exp \{ -\gamma n / 8 \} < \infty. \]

By Borel-Cantelli’s lemma, this implies that

\[ P\left\{ \lim_n \| \hat{W}_n - W_n \|_{\infty} / (2nLLn)^{1/2} = 0 \right\} = 1. \]

Appendix: Proof of Lemma 1

The necessity of Lemma 1 follows directly as that of the Arzelá-Ascoli theorem in \( C[0,1] \) (see e.g. [1, p. 221]). We only have to prove the sufficiency.

Now assume that conditions (i)—(iii) hold. Let \( \varepsilon > 0 \) be given. Choose \( k \) large enough that

\[ \sup_{\alpha \in A, 1 \leq i < j \leq k} \sup_{s \leq t < k, k} \| f_\alpha(s) - f_\alpha(t) \| < \varepsilon. \]

Since

\[ \| f_\alpha(t) \| \leq \| f_\alpha(0) \| + \sum_{i=1}^{k} \| f_\alpha(it/k) - f_\alpha((i-1)t/k) \| , \]

it follows that

(18) \[ \sup_{0 \leq t \leq 1} \sup_{\alpha \in A} \| f_\alpha(t) \| = C < \infty. \]

Let \( K = \bigcup_{\alpha \in A} \{ f_\alpha(i/k) : 0 < i < k \}. \)
Note that $K$ is relatively compact (and hence is totally bounded). Therefore there exists a finite set $Q \subset B$ such that for any $x \in K$, $\|x - \tilde{x}\| < \varepsilon$ for some $\tilde{x} \in Q$.

Now let $\phi$ be the set of functions $f$ from $[0, 1]$ into $B$ such that $f(i/k) \in Q$ for $i = 0, 1, 2, \cdots, k$ and $f$ is linear on $[(i-1)/k, i/k]$ for $i = 1, 2, \cdots, k$. Note that $\phi$ is a finite set. We claim that $\phi$ is a $5\varepsilon$-net with respect to $A$. Then $A$ is totally bounded and therefore is relatively compact since $C([0, 1], B)$ is complete. To show this, let $f_\alpha \in A$. Then $\|f_\alpha(i/k)\| \leq C$ for $i = 0, 1, 2, \cdots, k$, and there exists $g \in \phi$ such that

$$\|f_\alpha(i/k) - g(i/k)\| < \varepsilon = 0, 1, 2, \cdots, k.$$ 

Now let $t_\alpha \in [0, 1]$ be such that

$$\|f_\alpha - g\|_{\infty} = \|f_\alpha(t_\alpha) - g(t_\alpha)\|,$$

and let $i_0$ be such that

$$\frac{i_0}{k} \leq t < \frac{i_0 + 1}{k}.$$

Then $\|g(i_0/k) - g(t_\alpha)\| < 3\varepsilon$ and

$$\|f_\alpha - g\|_{\infty} \leq \|f_\alpha(t_\alpha) - f_\alpha(i_0/k)\| + \|f_\alpha(i_0/k) - g(i_0/k)\|
+ \|g(i_0/k) - g(t_\alpha)\| < \varepsilon + \varepsilon + 3\varepsilon = 5\varepsilon.$$

This completes the proof of Lemma 1.

References


