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A RANDOM WALK AND ITS LIL IN A BANACH SPACE*

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Abstract

Let $\{X_n : n \geqq 1\}$ be a sequence of i.i.d. Banach space valued random variables with $E[X_n]=0$ and $E\|X_n\|^2<\infty$, and let $S_0=0$, $S_n=X_1+X_2+\ldots+X_n$, $n\geqq 1$. We prove that if $\{S_n : n\geqq 1\}$ satisfies the LIL in B then the sequence $\{\eta_n : n\geqq 1\}$ satisfies the LIL in $C([0,1], B)$, where $\eta_n(t)=S_{[nt]}+(nt-[nt])\ X_{[nt]+1}$, $0\leqq t\leqq 1$ and $C([0,1], B)=\{f : [0,1]\rightarrow B \mid f$ is continuous$\}$. We also use this result to give an alternative to the proof of the LIL of Brownian motion in Banach spaces.

1. Introduction

Let B be a real separable Banach space with the norm $\|\cdot\|$ and B^* be its topological dual. Throughout, $\{X_n : n\geqq 1\}$ always denotes a sequence of i.i.d. B-valued random variables on a probability space $(\Omega, \mathcal{A}, P)$ with $E(X_n)=0$ and $E\|X_n\|^2<\infty$. Note that $E\|X_n\|^2<\infty$ assures the existence of a covariance operator

$$T(f, g)=E[f(X_n)g(X_n)],\quad f, g\in B^*.$$

Let μ denote the mean zero Gaussian measure on B with the given covariance operator whenever this measure exists. Let $H\mu\subseteqq B$ denote the reproducing kernel Hilbert space of μ. This pair of spaces $(B, H\mu)$ is often referred to as an abstract Wiener space [4]. Perhaps one of the most important properties of abstract Wiener space is the existence of a constant $M>0$ such that $\|x\|\leqq M\|x\|_\mu$ for every x in H_μ, where $\|\cdot\|_\mu$ is the norm of H_μ. Consequently, through the continuous injection $i : H_\mu\rightarrow B$ and the restriction map $i^* : B^*\rightarrow H_\mu^*$ we have the relation $B^*\subseteqq H_\mu^*\approx H_\mu\subseteqq B$. Let $\{W(t) : t\geqq 0\}$ denote μ-Brownian motion with the transition probability $P_t(a, A)=\mu((A-a)/t^{1/2})$. It is known that $\{W(t) : 0\leqq t\leqq 1\}$ induces a mean zero, Gaussian measure P_w on the

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measure space $(C_B, \mathscr{F})$, where C_B is the space of continuous functions w from $[0, 1]$ into B with $w(0)=0$, and $\mathscr{F}$ is the σ-field generated by the functions $w \to w(t)$. P_w is called abstract Wiener measure. See [4], [5] and [13] for expositions of concepts of μ-Brownian motion.

In this paper, we are interested in the random walk $\{\eta_n : n \geqq 1\}$ defined by

$$(1) \qquad \eta_n(t) = S_{[nt]} + (nt-[nt])X_{[nt]+1}, \quad 0 \leqq t \leqq 1,$$

where $S_0=0$,

$$S_k = X_1 + X_2 + \cdots + X_k$$

for $k \geqq 1$ and $[r]$ mean the greatest integer which is less than or equal to r. We say that the sequence $\{X_n : n \geqq 1\}$ satisfies the central limit theorem (CLT) in B if the distribution of $S_n/n^{1/2}$ converges weakly to $\mu (\mathscr{L}(S_n/n^{1/2}) \Rightarrow \mu)$. We say that the sequence $\{S_n : n \geqq 1\}$ satisfies the law of iterated logarithm (LIL) in B if there exists a compact, symmetric convex $K \subset B$ such that

$$(2) \qquad P\{\lim_n d(S_n/(2nLLn)^{1/2}, K)=0\}=1$$

and

$$(3) \qquad P\{C(\{S_n/(2nLLn)^{1/2}\})=K\}=1,$$

where

$$d(x, K) = \inf_{y \in K} \|x-y\|,$$

$C(\{X_n\})$ means the set of strong limit points of the sequence $\{X_n : n \geqq 1\}$ in B and $LLn=1$ if $n=1, 2, =\log\log n$ if $n \geqq 3$. The equivalence among the boundedness of $E\|X_n\|^2$, CLT and LIL are well known in [6], [15] and [16] when $B=\boldsymbol{R}^k$. However, when B is a general Banach space, there is no implication among those three concepts as can be seen in [2], [7] and [10]. The main purpose of this paper is to show that the LIL of $\{S_n : n \geqq 1\}$ in B implies the LIL of $\{\eta_n : n \geqq 1\}$ in C_B. We also use this result to give an alternative to the proof of the LIL of Brownian motion in Banach spaces. A work of the same spirit but different content is [8] in which $\mathscr{L}(S_n/n^{1/2}) \Rightarrow \mu$ implies $\mathscr{L}(\eta_n/n^{1/2}) \Rightarrow P_w$ has been established.

The following necessary and sufficient condition for LIL in B will be used in proving our main result.

THEOREM 1. (Kuelbs [11, p. 745]) *Let* $X_1, X_2, \cdots$ *be i.i.d. B-valued such that* $E(X_n)=0$ *and* $E\|X_n\|^2<\infty$. *Then the sequence* $\{S_n : n \geqq 1\}$ *satisfies the LIL in B if and only if*

$$(4) \qquad P\{\{S_n/(2nLLn)^{1/2} : n \geqq 1\} \text{ is relatively compact in } B\}=1.$$

For other results on LIL in abstract spaces, see [9], [12], [13] and [14].

2. Main result

Let $C([0,1],B)$ be the space of continuous functions f from $[0,1]$ into B. $C([0,1],B)$ is a real separable Banach space under the norm

$$\|f\|_\infty = \sup_{0\le t\le 1}\|f(t)\|.$$

A family

$$\{f_\alpha : \alpha\in A\}\subset C([0,1],B)$$

is said to be uniformly equicontinuous if

$$\lim_{\delta\to 0}\sup_{\alpha\in A}\sup_{|s-t|<\delta}\|f_\alpha(s)-f_\alpha(t)\|=0 \tag{5}$$

The following lemma (without condition (ii)) is known as the Arzelá-Ascoli theorem when $\dim(B)<\infty$. However in a general Banach space B, its proof does not seem to be available in literature. Therefore, we include its proof in the Appendix for the sake of completeness.

LEMMA 1. (Arzelá-Ascoli Theorem) *A subset $\{f_\alpha : \alpha\in A\}$ is relatively compact in $C([0,1],B)$ if and only if*

(i) $\sup_{\alpha\in A}\|f_\alpha(0)\|<\infty$,

(ii) *for each t in $[0,1]$, the set $\{f_\alpha(t) : \alpha\in A\}$ is relatively compact in B*

and

(iii) *$\{f_\alpha : \alpha\in A\}$ is uniformly equicontinuous in $C([0,1],B)$.*

THEOREM 2. (Main Result) *Let $\{X_n : n\geqq 1\}$ be a sequence of i.i.d. B-valued random variables such that $E(X_n)=0$ and $E\|X_n\|^2<\infty$, and let $\{\eta_n : n\geqq 1\}$ be as in* (1). *Then $\{S_n : n\geqq 1\}$ satisfies the LIL in B if and only if $\{\eta_n : n\geqq 1\}$ satisfies the LIL in C_B.*

PROOF. The sufficiency of the theorem is trivial. Therefore we shall only prove the necessity. By Lemma 1 and Theorem 1, this is equivalent to show that the following three conditions hold.

(i) $P\{\{\eta_n(0)/2nLLn)^{1/2} : n\geqq 1\}$ is bounded in $B\}=1$;

(ii) $P\{\{\eta_n(t)/2nLLn)^{1/2} : n\geqq 1\}$ is relatively compact in $B\}=1$ for each t in $[0,1]$;

and

(iii) $P\{\{\eta_n/(2nLLn)^{1/2} : n\geqq 1\}$ is uniformly equicontinuous in $C_B\}=1$.

Condition (i) follows immediately from the fact that for $n=1,2,\cdots$ $P\{\eta_n(0)=0\}=1$. The validity of (ii) and (iii) are treated in Lemma 2 and Lemma 3 respectively.

LEMMA 2. *Let $\{X_n : n\geqq 1\}$ and $\{\eta_n : n\geqq 1\}$ be as in Theorem 2. If $\{S_n : n\geqq 1\}$ satisfies the LIL in B then for each t in $[0,1]$*

$$P\{\{\eta_n(t)/(2nLLn)^{1/2} : n\geqq 1\} \text{ is relatively compact in } B\}=1. \tag{6}$$

PROOF. By Theorem 1, there exists a compact, symmetric, convex $K \subset B$ such that $P(\Omega_1) = P(\Omega_2) = 1$, where

$$\Omega_1 = \{\lim_n d(S_n/(2nLLn)^{1/2}, K) = 0\}$$

and

$$\Omega_2 = \{C(\{S_n/(2nLLn)^{1/2}\}) = K\}.$$

Now let $\omega \in \Omega_1 \cap \Omega_2$ and $t \in [0, 1]$ be given (when $t=0$ the conclusion is trivial), and let $\tilde{K} = a_t K = \{a_t y : y \in K\}$, where

$$a_t = \lim_n ([nt] LL[nt]/nLLn)^{1/2}.$$

We claim that

$$\lim_n d(\eta_n(t, \omega)/(2nLLn)^{1/2}, \tilde{K}) = 0 \tag{7}$$

and

$$C(\{\eta_n(t, \omega)/(2nLLn)^{1/2}\}) = \tilde{K}. \tag{8}$$

These two conclusions will then imply (6), by Theorem 1.

For the proof of (7), we simply observe that

$$\lim_n \| X_{[nt]+1}(\omega)/(2nLLn)^{1/2} \| = 0.$$

Then

$$\lim_n \inf_{y \in K} \| \eta_n(t, \omega)/(2nLLn)^{1/2} - z \|$$

$$\leqq \lim_n \inf_{y \in K} a_t \| S_{[nt]}(\omega)/(2[nt] LL[nt])^{1/2} - y \|$$

$$+ \lim_n \| X_{[nt]+1}(\omega)/(2nLLn)^{1/2} \| = 0.$$

For the proof of (8), let $z \in \tilde{K}$. Then $z = a_t y$ for some $y \in K$. By the assumption, we have a subsequence $\{n(k) : k \geqq 1\}$ of $\{n : n \geqq 1\}$ such that

$$\lim_k \| S_{n(k)}(\omega)/(2n(k) LLn(k))^{1/2} - y \| = 0.$$

This implies that

$$\lim_k \| \eta_{n(k)}(t)/(2n(k) LLn(k))^{1/2} - z \| = 0$$

by the same argument as in the proof of (7).

REMARK 1. The proof of Lemma 2 actually yields that if $\{S_n : n \geqq 1\}$ satisfies LIL in B then

(9) $P\{\{(\eta_n(t) - \eta_n(s))/\lambda(n, t, s) : n \geqq 1\}$ is relatively compact in $B\} = 1$

for any s, t in $[0, 1]$ with $s < t$, where

$$\lambda(n, t, s) = \{2([nt] - [ns]) LL([nt] - [ns])\}^{1/2}.$$

This remark will be used in the proof of Lemma 3.

LEMMA 3. *Let* $\{X_n : n \geqq 1\}$ *and* $\{\eta_n : n \geqq 1\}$ *be as in Theorem* 2. *If* $\{S_n : n \geqq 1\}$ *satisfies the LIL in B then*

(10) $P\{\{\eta_n/(2nLLn)^{1/2} : n \geqq 1\}$ *is uniformly equicontinuous in* $C_B\} = 1$.

PROOF. Let $\beta > 1$ be fixed, and let $n_r = [\beta^r]$. We shall prove that

(11) $P\{\{\eta_{n_r}/(2n_r LLn_r)^{1/2} : r \geqq 1\}$ is uniformly equicontinuous in $C_B\} = 1$.

Let $\Gamma > 0$ be such that for $n \geqq 1$ and $s, t \in [0, 1]$

(12) $$P\{\|\eta_n(t) - \eta_n(s)\| > \Gamma\lambda(n, t, s)\} = 0.$$

This Γ exists by Remark 1. For $r = 1, 2, \cdots$ and m_0 an arbitrary positive integer, define A_r by

$$A_r = \bigcup_{m=m_0}^{\infty} \bigcup_{k=1}^{2m} A_{rkm},$$

where

$$A_{rkm} = \{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\| > \varepsilon_m(2n_r LLn_r)^{1/2}\}$$

and $\varepsilon_m = 2^{-m/2}\Gamma$. It is clear that (11) holds if $P\{A_r \text{ i.o. in } r\} = 0$ since $\varepsilon_m \downarrow 0$ as $m \to \infty$. Now

$$P(A_{rkm}) = P\{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\|/\lambda(n_r, k2^{-m}, (k-1)2^{-m}) \geqq \varepsilon_m \theta(r, m, k)\}$$

where

$$\theta(r, m, k) = (2n_r LLn_r)^{1/2}/\lambda(n_r, k2^{-m}, (k-1)2^{-m})$$

and $\lambda(n, t, s)$ is as in (9). We choose m_0 sufficiently large such for $m \geqq m_0$

$$\lambda(n_r, k2^{-m}, (k-1)2^{-m}) \leqq 2^{-m/2}\{2n_r LLn_r\}^{1/2}.$$

Then

$$\begin{aligned} P(A_r) &\leqq \sum_{m=m_0}^{\infty} \sum_{k=1}^{2m} P(A_{rkm}) \\ &\leqq \sum_{m=m_0}^{\infty} \sum_{k=1}^{2m} P\{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\|/\lambda(n_r, k2^{-m}, (k-1)2^{-m})\varepsilon^m 2^{-m/2}\} \\ &= \sum_{m=m_0}^{\infty} \sum_{k=1}^{2m} P\{\|\eta_{n_r}(k2^{-m}) - \eta_{n_r}((k-1)2^{-m})\| > \Gamma\lambda(n_r, k2^{-m}, (k-1)2^{-m})\} \\ &= 0 \quad \text{by Remark 1 for } r = 1, 2, \cdots. \end{aligned}$$

Consequently

$$\sum_{r=1}^{\infty} P(A_r) = 0.$$

Thus by Borel-Cantelli's lemma, we have proved $P\{A_r \text{ i.o. in } r\} = 0$.

We next want to prove that for each $\varepsilon > 0$ there exists a fixed $\beta_0 > 1$ such that for all β satisfying $1 < \beta \leqq \beta_0$

(13) $$P\{C_r \text{ i.o. in } r\} = 0,$$

where

$$C_r=\{\max_{n_{r-1}\leqq k\leqq n_r}\|\eta_{n_{r-1}}/(2n_{r-1}LLn_{r-1})^{1/2}-\eta_k/(2kLLk)^{1/2}\|_\infty>\varepsilon\}.$$

This together with (11) will then conclude the lemma. Define

$$D_r=\{\max_{n_{r-1}\leqq k\leqq n_r}\|\eta_{n_{r-1}}-\eta_k\|_\infty>\varepsilon(2n_{r-1}LLn_{r-1})^{1/2}\}$$

and

$$E_r=\{\max_{n_{r-1}\leqq k\leqq n_r}\|\eta_k((2kLLk)^{-1/2}-(2n_{r-1}LLn_{r-1})^{-1/2}\|_\infty>\varepsilon/2\}.$$

Let $\alpha=\varepsilon/[4(\beta^{1/2}-1)]$. We have

$$\begin{aligned}P(E_r)&\leqq P\{\max_{k\leqq n_r}\|\eta_k\|_\infty>\alpha(2n_rLLn_r)^{1/2}\}\\ &\leqq\sum_{k=1}^{n_r}P\{\|\eta_k\|_\infty>\alpha(2kLLk)^{1/2}\}\\ &\leqq\sum_{k=1}^{n_r}P\{\sup_{0<|s-t|\leqq 1}\|\eta_k(t)-\eta_k(s)\|>\alpha(2kLLk)^{1/2}\}\\ &\leqq\sum_{k=1}^{n_r}P\{\sup_{0<|s-t|\leqq 2^{-m}}\|\eta_k(t)-\eta_k(s)\|>\alpha(2kLLk)^{1/2}/2^m\}\end{aligned}$$

for any $m\geqq 1$.

Let $\beta>1$ be sufficiently close to one such that

$$\alpha/2^m=\varepsilon/2^{m+2}(\beta^{1/2}-1)>\Gamma,$$

where Γ is as in (12). Then

$$\begin{aligned}(14)\qquad &P(E_r)\\ &\leqq\sum_{k=1}^{n_r}P\{\sup_{0<|s-t|<2^{-m}}\|\eta_k(t)-\eta_k(s)\|>\Gamma(2kLLk)^{1/2}\}\\ &\leqq\sum_{k=1}^{n_r}\sum_{j=1}^{2^m}P\{\|\eta_k(j2^{-m})-\eta_k((j-1)2^{-m})\|>\Gamma(2kLLk)^{1/2}\}\\ &=0\quad\text{by Remark 1.}\end{aligned}$$

Now

$$\begin{aligned}&P(D_r)\\ &\leqq\sum_{k=n_{r-1}}^{n_r}\sum_{j=1}^{k-n_{r-1}+1}P\{\|Y_{rj}\|_\infty>\gamma(r,j)\}\\ &\leqq\sum_{k=n_{r-1}}^{n_r}\sum_{j=1}^{k-n_{r-k}+1}P\{\sup_{0<|s-t|\leqq 2^{-m}}\|Y_{rj}(t)-Y_{rj}(s)\|>\gamma(r,j)/2^m\},\end{aligned}$$

where

$$Y_{rj}=\eta_{n_{r-1}+j}-\eta_{n_{r-1}+j-1}$$

and

$$\gamma(r, j)=(2(n_r-n_{r-1})LL(n_r-n_{r-1}))^{1/2}/4j(\beta-1)^{1/2}.$$

Choosing $\beta>1$ sufficiently close to 1 so that

$$\varepsilon/j2^{m+2}(\beta-1)^{1/2}>\Gamma,$$

where Γ is as in (12), and applying Remark 1 and argument as in (14), we have

$$\begin{aligned}
&P\{\sup_{0<|s-t|<2^{-m}}\|Y_{rj}(t)-Y_{rj}(s)\|>\gamma(r, j)/2^m\}\\
&\leqq P\{\sup_{0<|s-t|\leqq 2^{-m}}\|Y_{rj}(t)-Y_{rj}(s)\|>\gamma(r, j)/2^m\}\\
&\leqq P\{\sup_{0<|s-t|\leqq 2^{-m}}\|Y_{rj}(t)-Y_{rj}(s)\|>\phi(r)\}\\
&\leqq \sum_{l=1}^{2^m} P\{\|\eta_{n_{r-1}+j}(l2^{-m})-\eta_{n_{r-1}+j}((l-1)2^{-m})\|>\phi(r)/2\}\\
&\leqq \sum_{l=1}^{2^m} P\{\|\eta_{n_{r-1}+j-1}(l2^{-m})-\eta_{n_{r-1}+j-1}((l-1)2^{-m})\|>\phi(r)/2\}\\
&=0,
\end{aligned}$$

where

$$\phi(r)=\Gamma(2(n_r-n_{r-1})LL(n_r-n_{r-1}))^{1/2}.$$

Thus $P(D_r)=0$. Now

$$\sum_{r=1}^{\infty}P(C_r)\leqq\sum_{r=1}^{\infty}P(D_r)+\sum_{r=1}^{\infty}P(E_r)=0.$$

By the Borel-Cantelli lemma, we have $P(C_r \text{ i.o. in } r)=0$. This completes the proof of Lemma 3.

3. An application

Let $\{Z_n : n\geqq 1\}$ be a sequence of independent copies of Z, where Z is the mean zero, B-valued Gaussian random variable whose distribution is μ. We define

$$W_n(t)=\sum_{i=0}^{[nt]}Z_i+(nt-[nt])Z_{[nt]+1},\ 0\leqq t\leqq 1. \tag{15}$$

Note that for each $n=1, 2, \cdots$ the stochastic process $\{W_n(t) : 0\leqq t\leqq 1\}$ is essentially the polygonalized Brownian motion. That is

$$W_n\left(\frac{l}{n}\right)=W(l)$$

for $l=1, 2, \cdots, n$, and is linear on intervals $[(l-1)/n, l/n]$. It is known from [14] that

$$\left\{\sum_{i=1}^{n} Z_i : n \geqq 1\right\}$$

satisfies the LIL in B. Therefore the sequence $\{W_n : n \geqq 1\}$ satisfies the LIL in C_B by Theorem 2. In this section, we shall illustrate an alternative to the proof of the LIL for Brownian motion in Banach space given by Kuelbs and LePage [13].

THEOREM 3. (Kuelbs and LePage [13]) *Let* $\{W(t) : t \geqq 0\}$ *be* μ*-Brownian motion in* B. *Then the sequence* $\{\xi_n : n \geqq 1\}$ *satisfies the LIL in* C_B, *where* $\xi_n(t) = W(nt)$, $0 \leqq t \leqq 1$. *Furthermore the compact, symmetric convex* $K \subset C_B$ *described in* (2) *and* (3) *can be characterized as follows:*

$$K = \{f \in C_B : f(t) \in H_\mu \text{ for each } t \in [0, 1], \text{ and} \quad (16)$$

$$\sum_j \int_0^1 [(d/dt)\, x_j^*(f)(t)]^2 dt \leqq 1\},$$

where $\{x_j^* : j \geqq 1\} \subset B^*$ *is such that the set*

$$\left\{\int_B x_j^*(x)\, x\, d\mu(x) : j \geqq 1\right\}$$

forms a complete orthonormal system for H_μ.

PROOF. The characterization of the set K in (16) is from Lemma 4 of [13]. It remains to show that

$$P\{\lim_n \|\xi_n - W_n\|_\infty / (2nLLn)^{1/2} = 0\} = 1.$$

This together with the fact that $\{W_n : n \geqq 1\}$ satisfies the LIL in C_B will then imply that $\{\xi_n : n \geqq 1\}$ satisfies the LIL in C_B.

Let $\{\varepsilon_n : n \geqq 1\}$ be a sequence of positive real numbers whose precise values will be determined later. We have

$$P\{\|\xi_n - W_n\|_\infty / (2nLLn)^{1/2} \geqq \varepsilon_n\}$$

$$\leqq \sum_{k=0}^{n-1} P\{\sup_{k/n \leq t \leq (k+1)/n} \|W(nt) - W_n(t)\| \geqq \varepsilon_n (2nLLn)^{1/2}\}$$

$$= \sum_{k=0}^{n-1} P\{\sup_{0 \leq t \leq n^{-1}} \|W(nt) - nt\ W(1)\| \geqq \varepsilon_n (2nLLn)^{1/2}\},$$

since

$$W_n\left(\frac{l}{n}\right) = W(l)$$

and is linear on $[l/n, (l+1)/n]$ for $l = 0, 1, 2, \cdots, n$; $n = 1, 2, \cdots$.

Now

$$(17)\qquad P\{\sup_{0\leq t\leq n^{-1}}\|W(nt)-nt\ W(1)\|\geqq\varepsilon_n(2nLLn)^{1/2}\}$$

$$\leqq P\{\sup_{0\leq t\leq n^{-1}}\|W(nt)-nt\ W(nt)\|\geqq\varepsilon_n(2nLLn)^{1/2}/2\}$$

$$+P\{\sup_{0\leq t\leq n^{-1}}\|nt\ W(nt)-nt\ W(1)\|\geqq\varepsilon_n(2nLLn)^{1/2}/2\}$$

$$\leqq 5P\{\|W(1)\|\geqq\varepsilon_n(2nLLn)^{1/2}/4\}.$$

From Fernique's estimate [2], there exist constants $\gamma>0$ and $C>0$ such that

$$\exp\{\gamma\|W(1)\|^2\}\leqq C.$$

Applying Chevyshev's inequality and Fernique's estimate to the last expression in (17), we have

$$P\{\sup_{0\leq t\leq n^{-1}}\|W(nt)-nt\ W(1)\|\geqq\varepsilon_n(2nLLn)^{1/2}\}$$

$$\leqq 5C\exp\{-\gamma\varepsilon_n{}^2 nLLn/8\}.\quad \text{Choose}\quad \varepsilon_n=(LLn)^{-1/2}.$$

Then $\varepsilon_n\downarrow 0$ as $n\to\infty$ and we have

$$\sum_{n=1}^{\infty}P\{\|\xi_n-W_n\|_\infty/(2nLLn)^{1/2}\geqq\varepsilon_n\}$$

$$\leqq\sum_{n=1}^{\infty}5Cn\exp\{-\gamma n/8\}<\infty.$$

By Borel-Cantelli's lemma, this implies that

$$P\{\lim_n\|\xi_n-W_n\|_\infty/(2nLLn)^{1/2}=0\}=1.$$

Appendix: Proof of Lemma 1

The necessity of Lemma 1 follows exactly as that of the Arzelá-Ascoli theorem in $C[0,1]$ (see e.g. [1, p. 221]). We only have to prove the sufficiency.

Now assume that conditions (i)—(iii) hold. Let $\varepsilon>0$ be given. Choose k large enough that

$$\sup_{\alpha\in A}\sup_{|s-t|<1/k}\|f_\alpha(s)-f_\alpha(t)\|<\varepsilon.$$

Since

$$\|f_\alpha(t)\|\leqq\|f_\alpha(0)\|+\sum_{i=1}^{k}\|f_\alpha(it/k)-f_\alpha((i-1)t/k)\|,$$

it follows that

$$(18)\qquad \sup_{0\leq t\leq 1}\sup_{\alpha\in A}\|f_\alpha(t)\|\equiv C<\infty.\qquad \text{Let}\quad K=\bigcup_{i=0}^{k}\{f_\alpha(i/k):\alpha\in A\}.$$

Note that K is relatively compact (and hence is totally bounded). Therefore there exists a finite set $Q \subset B$ such that for any $x \in K$, $\|x-\tilde{x}\|<\varepsilon$ for some $\tilde{x} \in Q$.

Now let ϕ be the set of functions f from $[0, 1]$ into B such that $f(i/k) \in Q$ for $i=0, 1, 2, \cdots, k$ and f is linear on $[(i-1)/k, i/k]$ for $i=1, 2, \cdots, k$. Note that ϕ is a finite set. We claim that ϕ is a 5ε-net with respect to A. Then A is totally bounded and therefore is relatively compact since $C([0, 1], B)$ is complete. To show this, let $f_\alpha \in A$. Then $\|f_\alpha(i/k)\| \leqq C$ for $i=0, 1, 2, \cdots, k$, and there exists $g \in \phi$ such that

$$\|f_\alpha(i/k)-g(i/k)\|<i=0, 1, 2, \cdots, k.$$

Now let $t_0 \in [0, 1]$ be such that

$$\|f_\alpha - g\|_\infty = \|f_\alpha(t_0)-g(t_0)\|,$$

and let i_0 be such that

$$i_0/k \leqq t < (i_0+1)/k.$$

Then $\|g(i_0/k)-g(t_0)\|<3\varepsilon$ and

$$\|f_\alpha - g\|_\infty \leqq \|f_\alpha(t_0)-f_\alpha(i_0/k)\| + \|f_\alpha(i_0/k)-g(i_0/k)\| + \|g(i_0/k)-g(t_0)\| < \varepsilon+\varepsilon+3\varepsilon=5\varepsilon.$$

This completes the proof of Lemma 1.

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