A TRUNCATED PLAY-THE-WINNER PROCEDURE FOR SELECTING THE BEST OF $ k geqq 3 $ BINOMIAL POPULATION

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A TRUNCATED PLAY-THE-WINNER PROCEDURE FOR SELECTING THE BEST OF $k \geqq 3$ BINOMIAL POPULATION

By

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Summary

A truncated play-the-winner procedure for selecting the best of $k \geqq 3$ binomial populations is proposed. Sampling always terminates and the maximal number of trials up to a final decision is equal to $k \cdot (r+c-1)$. The large expected sample sizes of the Sobel-Weiss-procedure [4] when the success parameters are small are avoided and the problem of a never ending sampling (using the Sobel-Weiss-procedure) when all success parameters are equal to 0 doesn't occur.

1. Introduction

The following procedure is a generalization of the Sobel-Berry-procedure [2] for $k \geqq 3$ binomial populations. At the outset it puts the k populations in a random order. Let A_1 denote the best population, that means the population with the largest success parameter, A_2 the one following A_1 in the initial randomization, e.t.c. (continuing in cyclic order) and let p_i denote the success parameter of population A_i, $i \in \{1, 2, \cdots, k\}$. We use play-the-winner sampling with the orderd populations. Sampling terminates whenever one of the k populations yields r successes or all yield c failures (i.e. whenever play-the-winner procedure has gone through c cycles). In either case the population with the larger number of successes is selected. If the number of successes is the same for at least two populations (which can only occur after c cycles), one of these populations is selected randomly. We must determine r and c so that the probability of a correct selection ($P(CS)$) satisfies the following $(P^*; \Delta^*)$-condition:

$$P(CS) \geqq P^* \quad \text{whenever} \quad p_1 - \max_{i>1} p_i = : \Delta \geqq \Delta^*$$
$$\text{with} \quad 1/k < P^* < 1 \quad \text{and} \quad \Delta^* > 0 \tag{1.1}$$

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The maximal number of trials up to a final decision is equal to $k(r+c-1)$. This may for example occur if $k-1$ of the k populations yield c failures and $r-1$ successes and the k-th population yields $c-1$ failures and r successes.

2. Exact results for $P(CS)$

Let $F_i(r)$ denote the number of A_i-failures preceding the r-th success of A_i, and let $S_i(c)$ denote the number of A_i-successes preceding the c-th failure of A_i; $i\epsilon\{1, 2, \cdots, k\}$.

The probability of a correct selection is a sum of two terms N, M which are determined by $(F_i(r))_{i\in\{1,\cdots,k\}}$ and $(S_i(c))_{i\in\{1,\cdots,k\}}$ respectively. N is the probability of selecting A_1 before c cycles and M is the probability of selecting A_1 in exactly c cycles. We obtain:

$$\begin{aligned}N=&P(F_1(r)<F_2(r), \cdots, F_1(r)<F_k(r), 0\leqq F_1(r)<c)\\&+\sum_{n=1}^{k-1}\sum_{\omega\in S_n}\sum_{\nu=0}^{k-n-1}\left[\frac{(k-1-n)!(k-\nu-1)!}{k!(k-1-n-\nu)!}P(F_1(r)=F_{i_1}(r)=\cdots\right.\\&\left.=F_{i_n}(r), F_1(r)<F_{j_1}(r), \cdots, F_1(r)<F_{j_{k-n-1}}(r)\,;\ 0\leqq F_1(r)<c)\right]\end{aligned} \tag{2.1}$$

where S_n is defined als follows:

$$\begin{aligned}S_n:=\{&\omega=(\omega_1, \omega_2)\,|\,\omega_1=(i_1, \cdots, i_n), \omega_2=(j_1, \cdots, j_{k-n-1}), i_1<\cdots<i_n,\\&j_1<\cdots<j_{k-n-1}, i_1, \cdots, i_n, j_1, \cdots, j_{k-n-1}\in\{2, \cdots, k\} \quad\text{and}\\&\{i_1, \cdots, i_n\}\cap\{j_1, \cdots, j_{k-n-1}\}=\phi\}.\end{aligned}$$

In the first term of (2.1) the conditional probability of a correct selection is equal to 1; in the second term of (2.1) the conditional probability of a correct selection equals the probability that in the initial randomization A_1 precedes all A_l with $l\in\{i_1, \cdots, i_n\}$.

The contribution of $(S_i(c))_{i\in\{1,\cdots,k\}}$ is much simpler to evaluate. We obtain:

$$\begin{aligned}M=&P(S_2(c)<S_1(c), \cdots, S_k(c)<S_1(c), 0<S_1(c)<r)\\&+\sum_{n=1}^{k-1}\sum_{\omega\in S_n}\left[\frac{1}{n+1}P(S_1(c)=S_{i_1}(c)=\cdots=S_{i_n}(c),\right.\\&\left.S_{j_1}(c)<S_1(c), \cdots, S_{j_{k-n-1}}(c)<S_1(c), 0<S_1(c)<r)\right]\end{aligned} \tag{2.2}$$

The conditional probability of a correct selection equals 1, $\dfrac{1}{n+1}$ in the first and second term of (2.2) respectively.

Using the fact that $F_i(r)$ and $S_i(c)$ are all negative binomial chance variables, we obtain for N,

$$N=\sum_{j=0}^{c-1}\binom{r+j-1}{j}p_1^r q_1^j J_{q_2}(j+1,r)\cdot\cdots\cdot J_{q_k}(j+1,r)$$

$$+\sum_{n=1}^{k-1}\sum_{\omega\in S_n}\sum_{\nu=0}^{k-n-1}\sum_{j=0}^{c-1}\Big[\frac{(k-1-n)!\,(k-\nu-1)!}{k!(k-1-n-\nu)!}\binom{r+j-1}{j}p_1^r q_1^j\cdot \tag{2.3}$$

$$\cdot(J_{q_{i_1}}(j,r)-J_{q_{i_1}}(j+1,r))\cdot\cdots\cdot(J_{q_{i_n}}(j,r)-J_{q_{i_n}}(j+1,r))\cdot$$

$$\cdot J_{q_{j_1}}(j+1,r)\cdot\cdots\cdot J_{q_{j_{k-n-1}}}(j+1,r)\Big],$$

where $J_q(\cdot,\cdot)$ denotes an incomplete beta-function.

Let X_{r,p_1} denote a negative binomial chance variable with index $r>0$ and success parameter p_1 and let $E^{c-1}(\gamma)$ denote the expectation of the random variable γ truncated at $c-1$, then N may be written as follows:

$$N=E^{c-1}\Big(\prod_{l=2}^{k}J_{q_l}(X_{r,p_1}+1,r)\Big)+\sum_{n=1}^{k-1}\sum_{\omega\in S_n}\sum_{\nu=0}^{k-n-1}\Big[\frac{(k-1-n)!\,(k-\nu-1)!}{k!\,(k-1-n-\nu)!}\cdot$$

$$\cdot E^{c-1}\Big(\prod_{l=1}^{n}(J_{q_{i_l}}(X_{r,p_1},r)-J_{q_{i_l}}(X_{r,p_1}+1,r))\cdot\prod_{l=1}^{k-n-1}J_{q_{j_l}}(X_{r,p_1}+1,r)\Big)\Big]. \tag{2.4}$$

In the same way we obtain for M (using (2.2)):

$$M=\sum_{j=0}^{r-1}\binom{c+j+1}{j}q_1^c p_1^j J_{q_2}(c,j)\cdot\cdots\cdot J_{q_k}(c,j)$$

$$+\sum_{n=1}^{k-1}\sum_{\omega\in S_n}\sum_{j=0}^{r-1}\Big[\frac{1}{n+1}\binom{c+j+1}{j}q_1^c p_1^j\,(J_{p_{i_1}}(j,c)-J_{p_{i_1}}(j+1,c))\cdot\cdots\cdot \tag{2.5}$$

$$\cdot(J_{p_{i_n}}(j,c)-J_{p_{i_n}}(j+1,c))J_{q_{j_1}}(c,j)\cdot\cdots\cdot J_{q_{j_{k-n-1}}}(c,j)\Big]$$

Using the same notation as above we obtain for M:

$$M=E^{r-1}\Big(\prod_{l=2}^{k}J_{q_l}(c,X_{c,q_1})\Big)$$

$$+\sum_{n=1}^{k-1}\sum_{\omega\in S_n}\frac{1}{n+1}E^{r-1}\Big(\prod_{l=1}^{n}(J_{p_{i_l}}(X_{c,q_1},c)-J_{p_{i_l}}(X_{c,q_1}+1,c)) \tag{2.6}$$

$$\cdot\prod_{l=1}^{k-n-1}J_{q_{j_l}}(c,X_{c,q_1})\Big)$$

$P(CS)$ is the sum of (2.4) and (2.6).

3. Exact results for $E(N_i)$ and $E(N)$

Our next step consists in the derivation of the expectations $E(N_i)$; where N_i denotes the number of "patients on treatment i"; $i\in\{1, 2, \cdots, k\}$. For this purpose we define a random vector $T=(T_1, \cdots, T_k, T_{k+1})$, where T_i denotes the number of additional successes needed to declare population A_i as best for $i\in\{1, \cdots, k\}$, and T_{k+1} denotes the number of additional failures needed to finish the c-th cycle. In a second step we define:

$$U_j^i(n_1, \cdots, n_k, f) := E(N_i \mid T=(n_i, \cdots, n_k, f), NT=A_j) \tag{3.1}$$

where "$NT=A_j$" means: "the next trial is on A_j".

We obtain the following recursion formulas:

$$\begin{aligned} U_j^i(n_1, \cdots, n_k, f) = {} & p_j U_j^i(n_1, \cdots, n_{j-1}, n_j-1, n_{j+1}, \cdots, n_k, f) \\ & + q_j U_{j+1}^i(n_1, \cdots, n_k, f-1) + \delta_{ij} \end{aligned} \tag{3.2}$$

with $\delta_{ij}=1$ for $i=j$ and 0 for $i\neq j$; $U_{k+1}^i := U_1^i$. The boundary conditions for (3.2) are given by:

$$\begin{aligned} & U_j^i(n_1, \cdots, n_{j-1}, 0, n_{j+1}, \cdots, n_k, f)=0 \quad \text{for} \quad n_\nu>0 \quad \forall\nu\neq j \quad \text{and} \quad f>0; \\ & U_j^i(n_1, \cdots, n_k, 0)=0 \quad \text{for} \quad n_\nu>0 \quad \forall\nu\in\{1, \cdots, k\}, \quad \forall i, j\in\{1, \cdots, k\} \end{aligned} \tag{3.3}$$

To find a solution of (3.2) satisfying (3.3) we use generating functions T_j^i defined by

$$T_j^i := T_j^i(x_1, \cdots, x_k, y) := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \sum_{f=1}^{\infty} U_j^i(n_1, \cdots, n_k, f) x_1^{n_1}, \cdots, x_k^{n_k} y^f \tag{3.4}$$

Having solved (3.2) we obtain for the expectations:

$$\begin{aligned} & E(N_i) = \frac{1}{k} \sum_{j=1}^{k} U_j^i(r, \cdots, r, k\cdot c), \\ & E(N) = E(N_1 + \cdots + N_k) = \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k} U_j^i(r, \cdots, r, k\cdot c). \end{aligned} \tag{3.5}$$

Solving (3.2) we have to distinguish two cases; If $j=i$ we obtain:

$$T_i^i(1-p_i x_i) = q_i y T_{i+1}^i + \frac{x_1}{1-x_1} \cdot \cdots \cdot \frac{x_k}{1-x_k} \cdot \frac{y}{1-y}. \tag{3.6}$$

If $j\neq i$ we obtain:

$$T_j^i = \frac{q_j y}{1-p_j x_j} T_{j+1}^i. \tag{3.7}$$

Using (3.7) we obtain from (3.6) :

$$T_i^i=\frac{1}{D^*}\prod_{\substack{\nu=1\\ \nu\neq i}}^{k}(1-p_\nu x_\nu)\frac{y}{1-y}\prod_{\nu=1}^{k}\frac{x_\nu}{1-x_\nu} \tag{3.8}$$

with

$$D^*:=(1-p_1x_1)\cdot\cdots\cdot(1-p_kx_k)-q_1\cdot\cdots\cdot q_ky^k$$

and from (3.7) follows:

$$T_j^i=T_i^i\prod_{\nu=j}^{i-1}\frac{q_\nu y}{1-p_\nu x_\nu} \qquad \text{if } j<i, \tag{3.9}$$

$$T_j^i=T_i^i\prod_{\nu=1}^{i-1}\frac{q_\nu y}{1-p_\nu x_\nu}\prod_{\nu=j}^{k}\frac{q_\nu y}{1-p_\nu x_\nu} \qquad \text{if } j>i. \tag{3.10}$$

We next determine the power series expansions of (3.8), (3.9) and (3.10). The expansion of $\frac{1}{D^*}$ is given by:

$$\frac{1}{D^*}=\sum_{n_1=0}^{\infty}\cdots\sum_{n_k=0}^{\infty}\sum_{l=0}^{\infty}\binom{n_1+l}{n_1}\cdot\cdots\cdot\binom{n_k+l}{n_k}p_1^{n_1}\cdots p_k^{n_k}q_1^l\cdots \tag{3.11}$$
$$\cdot\, q_k^l x_1^{n_1}\cdots x_k^{n_k}\cdot y^{k\cdot l}.$$

We have to expand the other factors of the product given in (3.8) and to multiply them with (3.11). The coefficients of the resulting power series are as follows:

$$U_i^i(\lambda_1,\cdots,\lambda_k,\lambda_{k+1})=\sum_{l=0}^{\sigma(k,\lambda_{k+1})}\left[\frac{1}{q_i}J_{q_i}(l+1,\lambda_i)\prod_{\substack{\nu=1\\ \nu\neq i}}^{k}J_{q_\nu}(l,\lambda_\nu)\right], \tag{3.12}$$

$q_i\neq 0$ and $\sigma(k,\lambda_{k+1})$ is defined by:

$$\sigma(k,\lambda_{k+1}):=\begin{cases}\left[\dfrac{\lambda_{k+1}}{k}\right]-1 & \text{if } \gcd(k,\lambda_{k+1})=k\\[2ex] \left[\dfrac{\lambda_{k+1}}{k}\right] & \text{if } \gcd(k,\lambda_{k+1})<k\end{cases} \tag{3.13}$$

where $\gcd(a,b)$ denotes the greatest common divisor of a and b and $[a]$ denotes the greatest integral number not greater than a. We finally obtain:

$$U_i^i(r,\cdots,r,k\cdot c)=\sum_{l=0}^{c-1}\left[\frac{1}{q_i}J_{q_i}(l+1,r)\prod_{\substack{\nu=1\\ \nu\neq i}}^{k}J_{q_\nu}(l,r)\right] \tag{3.14}$$

In the same way we obtain for $j<i$, $(q_i\neq 0)$:

$$U_j^i(\lambda_1,\cdots,\lambda_k,\lambda_{k+1})=\sum_{l=0}^{\sigma(k,\lambda_{k+1}-(i-j))}\left[\prod_{\nu=1}^{j-1}J_{q_\nu}(l,\lambda_\nu)\frac{1}{q_i}\prod_{\nu=j}^{i}J_{q_\nu}(l+1,\lambda_\nu)\prod_{\nu=i+1}^{k}J_{q_\nu}(l,\lambda_\nu)\right] \tag{3.15}$$

$$U_j^i(r, \cdots, r, k\cdot c) = \prod_{l=0}^{c-1}\left[\prod_{\nu=1}^{j-1} J_{q_\nu}(l, r)\frac{1}{q_i}\prod_{\nu=j}^{i} J_{q_\nu}(l+1, r)\prod_{\nu=i+1}^{k} J_{q_\nu}(l, r)\right], \tag{3.16}$$

and the last expressions of interest are as follows:

$$j>i, \quad q_i \neq 0$$

$$U_j^i(\lambda_1, \cdots, \lambda_k, \lambda_{k+1}) = \sum_{l=0}^{\sigma(k, \lambda_{k+1}-k+(j-i))}\left[\prod_{\nu=1}^{i}\frac{1}{q_i}J_{q_\nu}(l+1, \lambda_\nu)\prod_{\nu=i+1}^{j-1} J_{q_\nu}(l, \lambda_\nu)\prod_{\nu=j}^{k} J_{q_\nu}(l+1, \lambda_\nu)\right], \tag{3.17}$$

$$U_j^i(r, \cdots, r, k\cdot c) = \prod_{l=0}^{c-1}\left[\prod_{\nu=1}^{i}\frac{1}{q_i}J_{q_\nu}(l+1, r)\prod_{\nu=i+1}^{j-1} J_{q_\nu}(l+1, r)\right]. \tag{3.18}$$

$E(N_i)$ and $E(N)$ are now available from (3.5), (3.14), (3.16) and (3.18).

4. Asymptotic analysis

$F_i(r)$ is a negative binomial chance variable with index r and success parameter p_i; $i\in\{1, \cdots, k\}$. $S_i(c)$ is a negative binomial chance variable with index c and success parameter $q_i, i\in\{1, 2, \cdots, k\}$. It follows immediately from their definition that $F_i(r)$ and $F_j(r)$ are independent for $i\neq j$; the same is true for the random variables $S_i(c)$.

We use the following identities:

$$F_\nu(r) > F_1(r) \longleftrightarrow \frac{F_\nu(r) - rq_\nu/p_\nu}{\sqrt{rq_\nu}/p_\nu} > \frac{p_\nu}{p_1}\sqrt{\frac{q_1}{q_\nu}}\frac{F_1(r) - rq_1/p_1}{\sqrt{rq_1}/p_1} - \frac{\Delta_{1\nu}}{p_1}\sqrt{\frac{r}{q_\nu}}, \tag{4.1}$$

$$S_\nu(c) < S_1(c) \longleftrightarrow \frac{S_\nu(c) - cp_\nu/q_\nu}{\sqrt{cp_\nu}/q_\nu} < \frac{q_\nu}{q_1}\sqrt{\frac{p_1}{p_\nu}}\frac{S_1(c) - cp_1/q_1}{\sqrt{cp_1}/q_1} + \frac{\Delta_{1\nu}}{q_1}\sqrt{\frac{c}{p_\nu}}, \tag{4.2}$$

with

$$\Delta_{1\nu} := p_1 - p_\nu; \quad \nu\in\{2, \cdots, k\}.$$

From the central limit theorem follows that for large r and c the random variables

$$F_{\nu r} := \frac{F_\nu(r) - rq_\nu/p_\nu}{\sqrt{rq_\nu}/p_\nu} \quad \text{and} \quad S_{\nu r} := \frac{S_\nu(c) - cp_\nu/q_\nu}{\sqrt{cp_\nu}/q_\nu}$$

may be expressed by standard normal chance variables X_ν, Y_ν respectively. ($X_1 =: X$, $Y_1 =: Y$) Denoting with $V_{\nu r}(\cdot)$, $W_{\nu r}(\cdot)$ the distribution functions of $F_{\nu r}, S_{\nu r}$ respectively, we obtain from (2.1):

$$\begin{aligned} &P(F_1(r) < F_2(r), \cdots, F_1(r) < F_k(r), 0 \leq F_1(r) < c) \\ &= \prod_{\nu=2}^{k} P\left(\frac{p_\nu}{p_1}\sqrt{\frac{q_1}{q_\nu}}F_{1r} - \frac{\Delta_{1\nu}}{p_1}\sqrt{\frac{r}{q_\nu}} < F_{\nu r}, -\sqrt{rq_1} \leq F_{1r} \leq \frac{cp_1 - rq_1}{\sqrt{rq_1}}\right) \\ &= \int_{-\sqrt{rq_1}}^{(cp_1 - rq_1)/\sqrt{rq_1}} \prod_{\nu=2}^{k}\left(1 - V_{\nu r}\left(\frac{p_\nu}{p_1}\sqrt{\frac{q_1}{q_\nu}}x - \frac{\Delta_{1\nu}}{p_1}\sqrt{\frac{r}{q_\nu}}\right)\right)dV_{1r}(x). \end{aligned} \tag{4.3}$$

Using lemma 1 of [4] we obtain:

$$(4.3) \sim \int_{-\infty}^{(cp_1-rq_1)/\sqrt{rq_1}} \prod_{\nu=2}^{k}\left(1-\Phi\left(\frac{p_\nu}{p_1}\sqrt{\frac{q_1}{q_\nu}}x-\frac{\Delta_{1\nu}}{p_1}\sqrt{\frac{r}{q_\nu}}\right)\right)d\Phi(x), \tag{4.4}$$

where "$\sim$" means asymptotic equivalence as $r\to\infty$. Replacing x by $-x$, we get the simpler version of (4.4), that means:

$$(4.3) \sim \int_{(rq_1-cp_1)/\sqrt{rq_1}}^{\infty} \prod_{\nu=2}^{k}\Phi\left(\frac{p_\nu}{p_1}\sqrt{\frac{q_1}{q_\nu}}x+\frac{\Delta_{1\nu}}{p_1}\sqrt{\frac{r}{q_\nu}}\right)d\Phi(x), \tag{4.5}$$

In the same way we obtain:

$$P(S_2(c)<S_1(c),\cdots,S_k(c)<S_1(c),0<S_1(c)<r)\sim$$

$$\sim \int_{-\infty}^{(rq_1-cp_1)/\sqrt{cp_1}} \prod_{\nu=2}^{k}\Phi\left(\frac{q_\nu}{q_1}\sqrt{\frac{p_1}{p_\nu}}y+\frac{\Delta_{1\nu}}{q_1}\sqrt{\frac{c}{p_\nu}}\right)d\Phi(y). \tag{4.6}$$

The second terms of (2.5) and (2.6) tend to 0 using this approximation, and may be neglected. Our first result is therefore:

$$P(CS)\sim(4.5)+(4.6). \tag{4.7}$$

It follows from the integral expression of the incomplete beta-function $J_q(s,t)$, that for fixed s and t $J_q(s,t)$ is increasing with q. That is why the incomplete beta-functions $J_{q_\nu}(\cdot,\cdot)$ in the first term of (2.4) and (2.6) get their smallest value if we make q_ν as small as possible, and that will be the case if we define $p_\nu:=p_2^*$ for all $\nu\in\{2,\cdots,k\}$. p_2^* is the second largest success parameter. (p_2 is the parameter of the population A_2 and will be usually different from p_2^*). We obtain:

$$\begin{aligned}P(CS) \sim & \int_{(rq_1-cp_1)/\sqrt{rq_1}}^{\infty}\left(\Phi\left(\frac{p_2^*}{p_1}\sqrt{\frac{q_1}{q_2^*}}x+\frac{\Delta}{p_1}\sqrt{\frac{r}{q_2^*}}\right)\right)^{k-1}d\Phi(x)\\ &+\int_{-\infty}^{(rq_1-cp_1)/\sqrt{cp_1}}\left(\Phi\left(\frac{q_2^*}{q_1}\sqrt{\frac{p_1}{p_2^*}}y+\frac{\Delta}{q_1}\sqrt{\frac{c}{p_2^*}}\right)\right)^{k-1}d\Phi(y),\end{aligned} \tag{4.8}$$

with $\Delta:=p_1-p_2^*$. Letting $c\to\infty$ and holding r fixed, we obtain:

$$P(CS) \sim \int_{-\infty}^{+\infty}\left(\Phi\left(\frac{p_2^*}{p_1}\sqrt{\frac{q_1}{q_2^*}}x+\frac{\Delta}{p_1}\sqrt{\frac{r}{q_2^*}}\right)\right)^{k-1}d\Phi(x). \tag{4.9}$$

Disregarding the fact, that the random variables

$$Z_\nu:=\frac{p_2^*}{p_1}\sqrt{\frac{q_1}{q_2^*}}X_1+\frac{\Delta}{p_1}\sqrt{\frac{r}{q_2^*}}-X_\nu$$

and

$$Z_\mu := \frac{p_2^*}{p_1}\sqrt{\frac{q_1}{q_2^*}}X_1 + \frac{\Delta}{p_1}\sqrt{\frac{r}{q_2^*}} - X_\mu$$

are not independent in general, we get the much simpler expression:

$$P(CS) \sim \left(\Phi\left(\frac{\Delta\sqrt{r}}{\sqrt{p_1^2 q_2^* + q_1 p_2^{*2}}}\right)\right)^{k-1}. \tag{4.10}$$

From

$$P(CS) = P^* \longleftrightarrow \Phi\left(\frac{\Delta\sqrt{r}}{\sqrt{p_1^2 q_2^* + q_1 p_2^{*2}}}\right) = P^{*(1/(k-1))}$$

follows that the least favorable configuration is obtained in the same way as in [3], [2], that means:

$$\min P(CS) = \left(\Phi\left(\Delta^*\sqrt{\frac{27r}{8}}\right)\right)^{k-1}. \tag{4.11}$$

We obtain from (4.11):

$$\Delta^* \cdot \sqrt{\frac{27r}{8}} = \Phi^{-1}(\sqrt[k-1]{P^*}) =: \lambda^*(P^*), \tag{4.12}$$

where $\lambda^*(P^*)$ is the $100\,P^{*1/(k-1)}$-percentile of the standard normal distribution. From (4.12) we obtain the required r value:

$$r^* = r = \frac{8}{27}\cdot\left(\frac{\lambda^*(P^*)}{\Delta^*}\right)^2. \tag{4.13}$$

In this special case the least favorable configuration is obtained by centering p_1 and p_2^* about 2/3.

Letting $r \to \infty$, holding c fixed and disregarding the fact that Z_ν and Z_μ are not independent in general, we obtain:

$$P(CS) \sim \left(\Phi\left(\frac{\Delta\sqrt{c}}{\sqrt{q_1^2 p_2^* + q_2^{*2} p_1}}\right)\right)^{k-1} \tag{4.14}$$

The argument in (4.10) was minimized by setting $p_0 := \frac{1}{2}(p_1 + p_2^*)$. Now we set $q_0 := 1 - p_0 = \frac{1}{2}(q_1 + q_2^*)$. With fixed q_0 we obtain in a first step $\Delta = \Delta^*$ and then a least favorable configuration q_1 and q_2^* centered about $q_0 = 2/3$, that means p_1 and p_2^* are centered about $p_0 = 1/3$. It follows immediately that $\min P(CS)$ is the same as given in (4.11) with c instead of r. The required c value is thus:

$$c^* = c = \frac{8}{27}\cdot\left(\frac{\lambda^*(P^*)}{\Delta^*}\right)^2. \tag{4.15}$$

From (3.14), (3.16) and (3.18) follows that $E(N_i)$ and $E(N)$ are monotone increasing with r and c. That is why we conjecture that the best choice among all pairs

(r, c) satisfying the (P^*, Δ^*)-condition (1.1) consists in setting $r=c$. With this we obtain from (4.8) :

$$P(CS) \sim \int_{\sqrt{r}(q_1-p_1)/\sqrt{q_1}}^{\infty} \left(\Phi\left(\frac{p_2^*}{p_1}\sqrt{\frac{q_1}{q_2^*}}x+\frac{\Delta}{p_1}\sqrt{\frac{r}{q_2^*}}\right)\right)^{k-1} d\Phi(x)$$
$$+\int_{-\infty}^{\sqrt{r}(q_1-p_1)/\sqrt{p_1}} \left(\Phi\left(\frac{q_2^*}{q_1}\sqrt{\frac{p_1}{p_2^*}}y+\frac{\Delta}{q_1}\sqrt{\frac{r}{p_2^*}}\right)\right)^{k-1} d\Phi(y). \tag{4.16}$$

If $p_1 \neq q_1$ $P(CS)$ is given by (4.10) or (4.14), provided r is large enough. Thus we have only to investigate the special case $p_1=q_1=\frac{1}{2}$. We obtain:

$$P(CS) \sim \int_0^{\infty}\left(\Phi\left(p_2^*\sqrt{\frac{2}{q_2^*}}x+2\Delta\sqrt{\frac{r}{q_2^*}}\right)\right)^{k-1} d\Phi(x)$$
$$+\int_{-\infty}^{0}\left(\Phi\left(q_2^*\sqrt{\frac{2}{p_2^*}}y+2\Delta\sqrt{\frac{r}{p_2^*}}\right)\right)^{k-1} d\Phi(y). \tag{4.17}$$

Using the fact that,

$$q_2^* \gtrless p_2^* \longrightarrow q_2^*\sqrt{\frac{2}{p_2^*}}x+2\Delta\sqrt{\frac{r}{p_2^*}} \gtrless p_2^*\sqrt{\frac{2}{q_2^*}}x+2\Delta\sqrt{\frac{2}{q_2^*}},$$

we obtain:

$$P(CS) \geqq \begin{cases} \left(\Phi\left(\dfrac{\Delta\sqrt{r}}{\sqrt{\left(\frac{1}{2}\right)^2 q_2^*+\frac{1}{2}p_2^{*2}}}\right)\right)^{k-1} & \text{if } q_2^* \geqq p_2^*, \\[2ex] \left(\Phi\left(\dfrac{\Delta\sqrt{r}}{\sqrt{\left(\frac{1}{2}\right)^2 p_2^*+\frac{1}{2}q_2^{*2}}}\right)\right)^{k-1} & \text{if } p_2^* > q_2^*. \end{cases} \tag{4.18}$$

From (4.10) and (4.14) follow immediately that in both cases ($q_2^* \geqq p_2^*$ and $p_2^* > q_2^*$)

$$\min P(CS) \geqq \left(\Phi\left(\Delta^*\sqrt{\frac{27r}{8}}\right)\right)^{k-1},$$

and from this we obtain that the pair (r^*, r^*), given by (4.13), satisfies the (P^*, Δ^*)-condition (1.1).

5. Numerical results

$k=3$; $P^*=0.90$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	275	311	358	396	342	281	231	179	79
$E(N\mid\Delta^*=0.2)$	—	73	82	86	78	64	51	40	20
$E(N\mid\Delta^*=0.3)$	—	—	34	34	32	27	22	17	9
$E(N\mid\Delta^*=0.4)$	—	—	—	17	16	14	12	9	5

$k=3$; $P^*=0.95$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	396	448	516	577	495	408	335	258	114
$E(N\mid\Delta^*=0.2)$	—	106	120	127	113	93	75	57	29
$E(N\mid\Delta^*=0.3)$	—	—	49	51	46	39	31	24	13
$E(N\mid\Delta^*=0.4)$	—	—	—	28	26	23	18	15	8

$k=3$; $P^*=0.99$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	685	774	892	1010	856	706	578	443	197
$E(N\mid\Delta^*=0.2)$	—	183	208	225	196	159	128	96	50
$E(N\mid\Delta^*=0.3)$	—	—	85	88	79	65	52	40	22
$E(N\mid\Delta^*=0.4)$	—	—	—	47	43	36	29	23	13

$k=4$; $P^*=0.90$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	450	508	584	649	555	454	369	278	98
$E(N\mid\Delta^*=0.2)$	—	120	134	141	125	101	80	60	25
$E(N\mid\Delta^*=0.3)$	—	—	53	54	49	41	32	24	11
$E(N\mid\Delta^*=0.4)$	—	—	—	31	29	24	20	15	7

$k=4$; $P^*=0.95$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	615	694	798	894	761	624	506	379	134
$E(N\mid\Delta^*=0.2)$	—	162	184	195	171	138	109	80	34
$E(N\mid\Delta^*=0.3)$	—	—	73	75	68	56	44	33	15
$E(N\mid\Delta^*=0.4)$	—	—	—	40	37	31	25	19	9

$k=4$; $P^*=0.99$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	1000	1129	1298	1469	1238	1016	823	613	218
$E(N\mid\Delta^*=0.2)$	—	262	298	321	277	223	176	128	55
$E(N\mid\Delta^*=0.3)$	—	—	124	129	114	92	72	53	25
$E(N\mid\Delta^*=0.4)$	—	—	—	65	58	48	38	28	14

$k=5$; $P^*=0.90$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	638	720	827	921	783	640	515	381	112
$E(N\mid\Delta^*=0.2)$	—	165	186	195	172	138	108	78	28
$E(N\mid\Delta^*=0.3)$	—	—	77	79	71	58	45	33	13
$E(N\mid\Delta^*=0.4)$	—	—	—	37	34	29	23	17	7

$k=5$; $P^*=0.95$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	843	952	1093	1225	1038	848	683	503	148
$E(N\mid\Delta^*=0.2)$	—	218	246	261	228	183	143	102	37
$E(N\mid\Delta^*=0.3)$	—	—	102	105	94	76	59	43	17
$E(N\mid\Delta^*=0.4)$	—	—	—	55	50	41	32	24	10

$k=5$; $P^*=0.99$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	1333	1505	1728	1955	1642	1342	1079	790	234
$E(N\mid\Delta^*=0.2)$	—	347	393	424	364	291	226	160	59
$E(N\mid\Delta^*=0.3)$	—	—	158	164	144	115	89	63	26
$E(N\mid\Delta^*=0.4)$	—	—	—	84	75	61	47	34	15

$k=6$; $P^*=0.90$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	838	945	1084	1209	1025	834	669	488	123
$E(N\mid\Delta^*=0.2)$	—	217	244	257	225	180	139	99	31
$E(N\mid\Delta^*=0.3)$	—	—	99	100	90	73	56	40	14
$E(N\mid\Delta^*=0.4)$	—	—	—	50	46	39	30	22	8

$k=6$; $P^*=0.95$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N\mid\Delta^*=0.1)$	1089	1229	1410	1582	1336	1089	872	634	160
$E(N\mid\Delta^*=0.2)$	—	280	316	335	292	232	180	126	40
$E(N\mid\Delta^*=0.3)$	—	—	128	131	117	94	72	51	18
$E(N\mid\Delta^*=0.4)$	—	—	—	64	58	48	37	27	10

The expected sample size $E(N)$ is increasing with k. $E(N)$ is already greater than 1000 for $k=3$, $P^*=0.99$, $p_1=0.5$ and $\Delta^*=0.1$. $E(N)$ is relatively small, even for large k, if the difference between the success parameters of the best and second best population is significant, that means greater than 0.4.

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