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A TRUNCATED PLAY-THE-WINNER PROCEDURE FOR SELECTING THE BEST OF $k \geq 3$ BINOMIAL POPULATION

By

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Summary

A truncated play-the-winner procedure for selecting the best of $k \geq 3$ binomial populations is proposed. Sampling always terminates and the maximal number of trials up to a final decision is equal to $k \cdot (r+c-1)$. The large expected sample sizes of the Sobel-Weiss-procedure [4] when the success parameters are small are avoided and the problem of a never ending sampling (using the Sobel-Weiss-procedure) when all success parameters are equal to 0 doesn't occur.

1. Introduction

The following procedure is a generalization of the Sobel-Berry-procedure [2] for $k \geq 3$ binomial populations. At the outset it puts the k populations in a random order. Let A_1 denote the best population, that means the population with the largest success parameter, A_2 the one following A_1 in the initial randomization, e.t.c. (continuing in cyclic order) and let p_i denote the success parameter of population A_i , $i \in \{1, 2, \dots, k\}$. We use play-the-winner sampling with the ordered populations. Sampling terminates whenever one of the k populations yields r successes or all yield c failures (i.e. whenever play-the-winner procedure has gone through c cycles). In either case the population with the larger number of successes is selected. If the number of successes is the same for at least two populations (which can only occur after c cycles), one of these populations is selected randomly. We must determine r and c so that the probability of a correct selection ($P(CS)$) satisfies the following $(P^*; A^*)$ -condition:

$$P(CS) \geq P^* \quad \text{whenever} \quad p_1 - \max_{i>1} p_i = :A \geq A^* \\ \text{with } 1/k < P^* < 1 \quad \text{and } A^* > 0 \quad (1.1)$$

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The maximal number of trials up to a final decision is equal to $k(r+c-1)$. This may for example occur if $k-1$ of the k populations yield c failures and $r-1$ successes and the k -th population yields $c-1$ failures and r successes.

2. Exact results for $P(CS)$

Let $F_i(r)$ denote the number of A_i -failures preceding the r -th success of A_i , and let $S_i(c)$ denote the number of A_i -successes preceding the c -th failure of A_i ; $i \in \{1, 2, \dots, k\}$.

The probability of a correct selection is a sum of two terms N, M which are determined by $(F_i(r))_{i \in \{1, \dots, k\}}$ and $(S_i(c))_{i \in \{1, \dots, k\}}$ respectively. N is the probability of selecting A_1 before c cycles and M is the probability of selecting A_1 in exactly c cycles. We obtain :

$$\begin{aligned} N &= P(F_1(r) < F_2(r), \dots, F_1(r) < F_k(r), 0 \leq F_1(r) < c) \\ &\quad + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \sum_{\nu=0}^{k-n-1} \left[\frac{(k-1-n)! (k-\nu-1)!}{k! (k-1-n-\nu)!} P(F_1(r) = F_{i_1}(r) = \dots \right. \\ &\quad \left. = F_{i_n}(r), F_1(r) < F_{j_1}(r), \dots, F_1(r) < F_{j_{k-n-1}}(r); 0 \leq F_1(r) < c) \right] \end{aligned} \quad (2.1)$$

where S_n is defined als follows :

$$\begin{aligned} S_n := \{\omega = (\omega_1, \omega_2) \mid &\omega_1 = (i_1, \dots, i_n), \omega_2 = (j_1, \dots, j_{k-n-1}), i_1 < \dots < i_n, \\ &j_1 < \dots < j_{k-n-1}, i_1, \dots, i_n, j_1, \dots, j_{k-n-1} \in \{2, \dots, k\} \text{ and} \\ &\{i_1, \dots, i_n\} \cap \{j_1, \dots, j_{k-n-1}\} = \emptyset\}. \end{aligned}$$

In the first term of (2.1) the conditional probability of a correct selection is equal to 1; in the second term of (2.1) the conditional probability of a correct selection equals the probability that in the initial randomization A_1 precedes all A_l with $l \in \{i_1, \dots, i_n\}$.

The contribution of $(S_i(c))_{i \in \{1, \dots, k\}}$ is much simpler to evaluate. We obtain :

$$\begin{aligned} M &= P(S_2(c) < S_1(c), \dots, S_k(c) < S_1(c), 0 < S_1(c) < r) \\ &\quad + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \left[\frac{1}{n+1} P(S_1(c) = S_{i_1}(c) = \dots = S_{i_n}(c), \right. \\ &\quad \left. S_{j_1}(c) < S_1(c), \dots, S_{j_{k-n-1}}(c) < S_1(c), 0 < S_1(c) < r) \right] \end{aligned} \quad (2.2)$$

The conditional probability of a correct selection equals 1, $\frac{1}{n+1}$ in the first and second term of (2.2) respectively.

Using the fact that $F_i(r)$ and $S_i(c)$ are all negative binomial chance variables, we obtain for N ,

$$\begin{aligned}
N = & \sum_{j=0}^{c-1} \binom{r+j-1}{j} p_1^r q_1^j J_{q_2}(j+1, r) \cdot \dots \cdot J_{q_k}(j+1, r) \\
& + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \sum_{\nu=0}^{k-n-1} \sum_{j=0}^{c-1} \left[\frac{(k-1-n)! (k-\nu-1)!}{k! (k-1-n-\nu)!} \binom{r+j-1}{j} p_1^r q_1^j \cdot \right. \\
& \cdot (J_{q_{i_1}}(j, r) - J_{q_{i_1}}(j+1, r)) \cdot \dots \cdot (J_{q_{i_n}}(j, r) - J_{q_{i_n}}(j+1, r)) \cdot \\
& \left. \cdot J_{q_{j_1}}(j+1, r) \cdot \dots \cdot J_{q_{j_{k-n-1}}}(j+1, r) \right],
\end{aligned} \tag{2.3}$$

where $J_q(\cdot, \cdot)$ denotes an incomplete beta-function.

Let X_{r, p_1} denote a negative binomial chance variable with index $r > 0$ and success parameter p_1 and let $E^{c-1}(\gamma)$ denote the expectation of the random variable γ truncated at $c-1$, then N may be written as follows:

$$\begin{aligned}
N = & E^{c-1} \left(\prod_{l=2}^k J_{q_l}(X_{r, p_1} + 1, r) \right) + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \sum_{\nu=0}^{k-n-1} \left[\frac{(k-1-n)! (k-\nu-1)!}{k! (k-1-n-\nu)!} \cdot \right. \\
& \cdot E^{c-1} \left(\prod_{l=1}^n (J_{q_{i_l}}(X_{r, p_1}, r) - J_{q_{i_l}}(X_{r, p_1} + 1, r)) \cdot \prod_{l=1}^{k-n-1} J_{q_{j_l}}(X_{r, p_1} + 1, r) \right) \left. \right].
\end{aligned} \tag{2.4}$$

In the same way we obtain for M (using (2.2)):

$$\begin{aligned}
M = & \sum_{j=0}^{r-1} \binom{c+j+1}{j} q_1^r p_1^j J_{q_2}(c, j) \cdot \dots \cdot J_{q_k}(c, j) \\
& + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \sum_{j=0}^{r-1} \left[\frac{1}{n+1} \binom{c+j+1}{j} q_1^r p_1^j (J_{p_{i_1}}(j, c) - J_{p_{i_1}}(j+1, c)) \cdot \dots \cdot \right. \\
& \cdot (J_{p_{i_n}}(j, c) - J_{p_{i_n}}(j+1, c)) J_{q_{j_1}}(c, j) \cdot \dots \cdot J_{q_{j_{k-n-1}}}(c, j) \left. \right]
\end{aligned} \tag{2.5}$$

Using the same notation as above we obtain for M :

$$\begin{aligned}
M = & E^{r-1} \left(\prod_{l=2}^k J_{q_l}(c, X_{c, q_1}) \right) \\
& + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \frac{1}{n+1} E^{r-1} \left(\prod_{l=1}^n (J_{p_{i_l}}(X_{c, q_1}, c) - J_{p_{i_l}}(X_{c, q_1} + 1, c)) \right. \\
& \left. \cdot \prod_{l=1}^{k-n-1} J_{q_{j_l}}(c, X_{c, q_1}) \right)
\end{aligned} \tag{2.6}$$

$P(CS)$ is the sum of (2.4) and (2.6).

3. Exact results for $E(N_i)$ and $E(N)$

Our next step consists in the derivation of the expectations $E(N_i)$; where N_i denotes the number of "patients on treatment i "; $i \in \{1, 2, \dots, k\}$. For this purpose we define a random vector $T = (T_1, \dots, T_k, T_{k+1})$, where T_i denotes the number of additional successes needed to declare population A_i as best for $i \in \{1, \dots, k\}$, and T_{k+1} denotes the number of additional failures needed to finish the c -th cycle. In a second step we define:

$$U_j^i(n_1, \dots, n_k, f) := E(N_i | T = (n_1, \dots, n_k, f), NT = A_j) \quad (3.1)$$

where " $NT = A_j$ " means: "the next trial is on A_j ".

We obtain the following recursion formulas:

$$\begin{aligned} U_j^i(n_1, \dots, n_k, f) &= p_j U_j^i(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_k, f) \\ &\quad + q_j U_{j+1}^i(n_1, \dots, n_k, f-1) + \delta_{ij} \end{aligned} \quad (3.2)$$

with $\delta_{ij}=1$ for $i=j$ and 0 for $i \neq j$; $U_{k+1}^i := U_1^i$. The boundary conditions for (3.2) are given by:

$$\begin{aligned} U_j^i(n_1, \dots, n_{j-1}, 0, n_{j+1}, \dots, n_k, f) &= 0 \quad \text{for } n_\nu > 0 \quad \forall \nu \neq j \quad \text{and } f > 0; \\ U_j^i(n_1, \dots, n_k, 0) &= 0 \quad \text{for } n_\nu > 0 \quad \forall \nu \in \{1, \dots, k\}, \quad \forall i, j \in \{1, \dots, k\} \end{aligned} \quad (3.3)$$

To find a solution of (3.2) satisfying (3.3) we use generating functions T_j^i defined by

$$T_j^i := T_j^i(x_1, \dots, x_k, y) := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \sum_{f=1}^{\infty} U_j^i(n_1, \dots, n_k, f) x_1^{n_1} \cdots x_k^{n_k} y^f \quad (3.4)$$

Having solved (3.2) we obtain for the expectations:

$$\begin{aligned} E(N_i) &= \frac{1}{k} \sum_{j=1}^k U_j^i(r, \dots, r, k \cdot c), \\ E(N) &= E(N_1 + \dots + N_k) = \frac{1}{k} \sum_{i=1}^k \sum_{j=1}^k U_j^i(r, \dots, r, k \cdot c). \end{aligned} \quad (3.5)$$

Solving (3.2) we have to distinguish two cases; If $j=i$ we obtain:

$$T_i^i(1 - p_i x_i) = q_i y T_{i+1}^i + \frac{x_1}{1 - x_1} \cdot \dots \cdot \frac{x_k}{1 - x_k} \cdot \frac{y}{1 - y}. \quad (3.6)$$

If $j \neq i$ we obtain:

$$T_j^i = \frac{q_j y}{1 - p_j x_j} T_{j+1}^i. \quad (3.7)$$

Using (3.7) we obtain from (3.6) :

$$T_i^i = \frac{1}{D^*} \prod_{\substack{\nu=1 \\ \nu \neq i}}^k (1-p_\nu x_\nu) \frac{y}{1-y} \prod_{\nu=1}^k \frac{x_\nu}{1-x_\nu} \quad (3.8)$$

with

$$D^* := (1-p_1 x_1) \cdot \dots \cdot (1-p_k x_k) - q_1 \cdot \dots \cdot q_k y^k$$

and from (3.7) follows :

$$T_j^i = T_i^i \prod_{\nu=j}^{i-1} \frac{q_\nu y}{1-p_\nu x_\nu} \quad \text{if } j < i, \quad (3.9)$$

$$T_j^i = T_i^i \prod_{\nu=1}^{i-1} \frac{q_\nu y}{1-p_\nu x_\nu} \prod_{\nu=j}^k \frac{q_\nu y}{1-p_\nu x_\nu} \quad \text{if } j > i. \quad (3.10)$$

We next determine the power series expansions of (3.8), (3.9) and (3.10). The expansion of $\frac{1}{D^*}$ is given by :

$$\begin{aligned} \frac{1}{D^*} = & \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \sum_{l=0}^{\infty} \binom{n_1+l}{n_1} \cdots \binom{n_k+l}{n_k} p_1^{n_1} \cdots p_k^{n_k} q_1^l \cdots \\ & \cdot q_k^l x_1^{n_1} \cdots x_k^{n_k} \cdot y^{k+l}. \end{aligned} \quad (3.11)$$

We have to expand the other factors of the product given in (3.8) and to multiply them with (3.11). The coefficients of the resulting power series are as follows :

$$U_i^i(\lambda_1, \dots, \lambda_k, \lambda_{k+1}) = \sum_{l=0}^{\sigma(k, \lambda_{k+1})} \left[\frac{1}{q_i} J_{q_i}(l+1, \lambda_i) \prod_{\substack{\nu=1 \\ \nu \neq i}}^k J_{q_\nu}(l, \lambda_\nu) \right], \quad (3.12)$$

$q_i \neq 0$ and $\sigma(k, \lambda_{k+1})$ is defined by :

$$\sigma(k, \lambda_{k+1}) := \begin{cases} \left[\frac{\lambda_{k+1}}{k} \right] - 1 & \text{if } \gcd(k, \lambda_{k+1}) = k \\ \left[\frac{\lambda_{k+1}}{k} \right] & \text{if } \gcd(k, \lambda_{k+1}) < k \end{cases} \quad (3.13)$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b and $[a]$ denotes the greatest integral number not greater than a . We finally obtain :

$$U_i^i(r, \dots, r, k \cdot c) = \sum_{l=0}^{c-1} \left[\frac{1}{q_i} J_{q_i}(l+1, r) \prod_{\substack{\nu=1 \\ \nu \neq i}}^k J_{q_\nu}(l, r) \right] \quad (3.14)$$

In the same way we obtain for $j < i$, ($q_i \neq 0$) :

$$U_j^i(\lambda_1, \dots, \lambda_k, \lambda_{k+1}) = \sum_{l=0}^{\sigma(k, \lambda_{k+1}) - (i-j)} \left[\prod_{\nu=1}^{j-1} J_{q_\nu}(l, \lambda_\nu) \frac{1}{q_i} \prod_{\nu=j}^i J_{q_\nu}(l+1, \lambda_\nu) \prod_{\nu=i+1}^k J_{q_\nu}(l, \lambda_\nu) \right] \quad (3.15)$$

$$U_j^i(r, \dots, r, k \cdot c) = \prod_{l=0}^{c-1} \left[\prod_{\nu=1}^{j-1} J_{q_\nu}(l, r) \frac{1}{q_i} \prod_{\nu=j}^i J_{q_\nu}(l+1, r) \prod_{\nu=i+1}^k J_{q_\nu}(l, r) \right], \quad (3.16)$$

and the last expressions of interest are as follows :

$$j > i, \quad q_i \neq 0$$

$$U_j^i(\lambda_1, \dots, \lambda_k, \lambda_{k+1}) = \sum_{l=0}^{\sigma(k, \lambda_{k+1}-k+(j-i))} \left[\prod_{\nu=1}^i \frac{1}{q_i} J_{q_\nu}(l+1, \lambda_\nu) \prod_{\nu=i+1}^{j-1} J_{q_\nu}(l, \lambda_\nu) \prod_{\nu=j}^k J_{q_\nu}(l+1, \lambda_\nu) \right], \quad (3.17)$$

$$U_j^i(r, \dots, r, k \cdot c) = \prod_{l=0}^{c-1} \left[\prod_{\nu=1}^i \frac{1}{q_i} J_{q_\nu}(l+1, r) \prod_{\nu=i+1}^{j-1} J_{q_\nu}(l+1, r) \right]. \quad (3.18)$$

$E(N_i)$ and $E(N)$ are now available from (3.5), (3.14), (3.16) and (3.18).

4. Asymptotic analysis

$F_i(r)$ is a negative binomial chance variable with index r and success parameter p_i ; $i \in \{1, \dots, k\}$. $S_i(c)$ is a negative binomial chance variable with index c and success parameter q_i , $i \in \{1, 2, \dots, k\}$. It follows immediately from their definition that $F_i(r)$ and $F_j(r)$ are independent for $i \neq j$; the same is true for the random variables $S_i(c)$.

We use the following identities :

$$F_\nu(r) > F_1(r) \longleftrightarrow \frac{F_\nu(r) - rq_\nu/p_\nu}{\sqrt{rq_\nu/p_\nu}} > \frac{p_\nu}{p_1} \sqrt{\frac{q_1}{q_\nu}} \frac{S_1(r) - rq_1/p_1}{\sqrt{rq_1/p_1}} - \frac{A_{1\nu}}{p_1} \sqrt{\frac{r}{q_\nu}}, \quad (4.1)$$

$$S_\nu(c) < S_1(c) \longleftrightarrow \frac{S_\nu(c) - cp_\nu/q_\nu}{\sqrt{cp_\nu/q_\nu}} < \frac{q_\nu}{q_1} \sqrt{\frac{p_1}{p_\nu}} \frac{S_1(c) - cp_1/q_1}{\sqrt{cp_1/q_1}} + \frac{A_{1\nu}}{q_1} \sqrt{\frac{c}{p_\nu}}, \quad (4.2)$$

with

$$A_{1\nu} := p_1 - p_\nu; \quad \nu \in \{2, \dots, k\}.$$

From the central limit theorem follows that for large r and c the random variables

$$F_{\nu r} := \frac{F_\nu(r) - rq_\nu/p_\nu}{\sqrt{rq_\nu/p_\nu}} \quad \text{and} \quad S_{\nu r} := \frac{S_\nu(c) - cp_\nu/q_\nu}{\sqrt{cp_\nu/q_\nu}}$$

may be expressed by standard normal chance variables X_ν, Y_ν respectively. ($X_1 = : X$, $Y_1 = : Y$) Denoting with $V_{\nu r}(\cdot), W_{\nu r}(\cdot)$ the distribution functions of $F_{\nu r}, S_{\nu r}$ respectively, we obtain from (2.1) :

$$\begin{aligned} P(F_1(r) < F_2(r), \dots, F_1(r) < F_k(r), 0 \leq F_1(r) < c) \\ &= \prod_{\nu=2}^k P\left(\frac{p_\nu}{p_1} \sqrt{\frac{q_1}{q_\nu}} F_{1r} - \frac{A_{1\nu}}{p_1} \sqrt{\frac{r}{q_\nu}} < F_{\nu r}, -\sqrt{rq_1} \leq F_{1r} \leq \frac{cp_1 - rq_1}{\sqrt{rq_1}}\right) \\ &= \int_{-\sqrt{rq_1}}^{\frac{(cp_1 - rq_1)/\sqrt{rq_1}}{\sqrt{rq_1}}} \prod_{\nu=2}^k \left(1 - V_{\nu r}\left(\frac{p_\nu}{p_1} \sqrt{\frac{q_1}{q_\nu}} x - \frac{A_{1\nu}}{p_1} \sqrt{\frac{r}{q_\nu}}\right)\right) dV_{1r}(x). \end{aligned} \quad (4.3)$$

Using lemma 1 of [4] we obtain :

$$(4.3) \sim \int_{-\infty}^{(cp_1 - rq_1)/\sqrt{rq_1}} \prod_{\nu=2}^k \left(1 - \Phi \left(\frac{p_\nu}{p_1} \sqrt{\frac{q_1}{q_\nu}} x - \frac{A_{1\nu}}{p_1} \sqrt{\frac{r}{q_\nu}} \right) \right) d\Phi(x), \quad (4.4)$$

where “ \sim ” means asymptotic equivalence as $r \rightarrow \infty$. Replacing x by $-x$, we get the simpler version of (4.4), that means :

$$(4.3) \sim \int_{(rq_1 - cp_1)/\sqrt{rq_1}}^{\infty} \prod_{\nu=2}^k \Phi \left(\frac{p_\nu}{p_1} \sqrt{\frac{q_1}{q_\nu}} x + \frac{A_{1\nu}}{p_1} \sqrt{\frac{r}{q_\nu}} \right) d\Phi(x), \quad (4.5)$$

In the same way we obtain :

$$\begin{aligned} P(S_2(c) < S_1(c), \dots, S_k(c) < S_1(c), 0 < S_1(c) < r) &\sim \\ &\sim \int_{-\infty}^{(rq_1 - cp_1)/\sqrt{cp_1}} \prod_{\nu=2}^k \Phi \left(\frac{q_\nu}{q_1} \sqrt{\frac{p_1}{p_\nu}} y + \frac{A_{1\nu}}{q_1} \sqrt{\frac{c}{p_\nu}} \right) d\Phi(y). \end{aligned} \quad (4.6)$$

The second terms of (2.5) and (2.6) tend to 0 using this approximation, and may be neglected. Our first result is therefore :

$$P(CS) \sim (4.5) + (4.6). \quad (4.7)$$

It follows from the integral expression of the incomplete beta-function $J_q(s, t)$, that for fixed s and t $J_q(s, t)$ is increasing with q . That is why the incomplete beta-functions $J_{q_\nu}(\cdot, \cdot)$ in the first term of (2.4) and (2.6) get their smallest value if we make q_ν as small as possible, and that will be the case if we define $p_\nu := p_2^*$ for all $\nu \in \{2, \dots, k\}$. p_2^* is the second largest success parameter. (p_2 is the parameter of the population A_2 and will be usually different from p_2^*). We obtain :

$$\begin{aligned} P(CS) &\sim \int_{(rq_1 - cp_1)/\sqrt{rq_1}}^{\infty} \left(\Phi \left(\frac{p_2^*}{p_1} \sqrt{\frac{q_1}{q_2^*}} x + \frac{A}{p_1} \sqrt{\frac{r}{q_2^*}} \right) \right)^{k-1} d\Phi(x) \\ &+ \int_{-\infty}^{(rq_1 - cp_1)/\sqrt{cp_1}} \left(\Phi \left(\frac{q_2^*}{q_1} \sqrt{\frac{p_1}{p_2^*}} y + \frac{A}{q_1} \sqrt{\frac{c}{p_2^*}} \right) \right)^{k-1} d\Phi(y), \end{aligned} \quad (4.8)$$

with $A := p_1 - p_2^*$. Letting $c \rightarrow \infty$ and holding r fixed, we obtain :

$$P(CS) \sim \int_{-\infty}^{+\infty} \left(\Phi \left(\frac{p_2^*}{p_1} \sqrt{\frac{q_1}{q_2^*}} x + \frac{A}{p_1} \sqrt{\frac{r}{q_2^*}} \right) \right)^{k-1} d\Phi(x). \quad (4.9)$$

Disregarding the fact, that the random variables

$$Z_\nu := \frac{p_2^*}{p_1} \sqrt{\frac{q_1}{q_2^*}} X_1 + \frac{A}{p_1} \sqrt{\frac{r}{q_2^*}} - X_\nu$$

and

$$Z_\mu := \frac{p_2^*}{p_1} \sqrt{\frac{q_1}{q_2^*}} X_1 + \frac{A}{p_1} \sqrt{\frac{r}{q_2^*}} - X_\mu$$

are not independent in general, we get the much simpler expression :

$$P(CS) \sim \left(\Phi \left(\frac{A\sqrt{r}}{\sqrt{p_1^2 q_2^* + q_1 p_2^{*2}}} \right) \right)^{k-1}. \quad (4.10)$$

From

$$P(CS) = P^* \longleftrightarrow \Phi \left(\frac{A\sqrt{r}}{\sqrt{p_1^2 q_2^* + q_1 p_2^{*2}}} \right) = P^{*(1/(k-1))}$$

follows that the least favorable configuration is obtained in the same way as in [3], [2], that means :

$$\min P(CS) = \left(\Phi \left(A^* \sqrt{\frac{27r}{8}} \right) \right)^{k-1}. \quad (4.11)$$

We obtain from (4.11) :

$$A^* \cdot \sqrt{\frac{27r}{8}} = \Phi^{-1}(\sqrt[k]{P^*}) = : \lambda^*(P^*), \quad (4.12)$$

where $\lambda^*(P^*)$ is the $100 P^{*(1/(k-1))}$ -percentile of the standard normal distribution. From (4.12) we obtain the required r value :

$$r^* = r = \frac{8}{27} \cdot \left(\frac{\lambda^*(P^*)}{A^*} \right)^2. \quad (4.13)$$

In this special case the least favorable configuration is obtained by centering p_1 and p_2^* about $2/3$.

Letting $r \rightarrow \infty$, holding c fixed and disregarding the fact that Z_ν and Z_μ are not independent in general, we obtain :

$$P(CS) \sim \left(\Phi \left(\frac{A\sqrt{c}}{\sqrt{q_1^2 p_2^* + q_2^* p_1}} \right) \right)^{k-1} \quad (4.14)$$

The argument in (4.10) was minimized by setting $p_0 := \frac{1}{2}(p_1 + p_2^*)$. Now we set $q_0 := 1 - p_0 = \frac{1}{2}(q_1 + q_2^*)$. With fixed q_0 we obtain in a first step $A = A^*$ and then a least favorable configuration q_1 and q_2^* centered about $q_0 = 2/3$, that means p_1 and p_2^* are centered about $p_0 = 1/3$. It follows immediately that $\min P(CS)$ is the same as given in (4.11) with c instead of r . The required c value is thus :

$$c^* = c = \frac{8}{27} \cdot \left(\frac{\lambda^*(P^*)}{A^*} \right)^2. \quad (4.15)$$

From (3.14), (3.16) and (3.18) follows that $E(N_i)$ and $E(N)$ are monotone increasing with r and c . That is why we conjecture that the best choice among all pairs

(r, c) satisfying the (P^*, Δ^*) -condition (1.1) consists in setting $r=c$. With this we obtain from (4.8) :

$$\begin{aligned} P(CS) &\sim \int_{\sqrt{r}(q_1-p_1)/\sqrt{q_1}}^{\infty} \left(\Phi\left(\frac{p_2^*}{p_1}\sqrt{\frac{q_1}{q_2^*}}x + \frac{\Delta}{p_1}\sqrt{\frac{r}{q_2^*}}\right) \right)^{k-1} d\Phi(x) \\ &+ \int_{-\infty}^{\sqrt{r}(q_1-p_1)/\sqrt{p_1}} \left(\Phi\left(\frac{q_2^*}{q_1}\sqrt{\frac{p_1}{p_2^*}}y + \frac{\Delta}{q_1}\sqrt{\frac{r}{p_2^*}}\right) \right)^{k-1} d\Phi(y). \end{aligned} \quad (4.16)$$

If $p_1 \neq q_1$ $P(CS)$ is given by (4.10) or (4.14), provided r is large enough. Thus we have only to investigate the special case $p_1 = q_1 = \frac{1}{2}$. We obtain :

$$\begin{aligned} P(CS) &\sim \int_0^{\infty} \left(\Phi\left(p_2^*\sqrt{\frac{2}{q_2^*}}x + 2\Delta\sqrt{\frac{r}{q_2^*}}\right) \right)^{k-1} d\Phi(x) \\ &+ \int_{-\infty}^0 \left(\Phi\left(q_2^*\sqrt{\frac{2}{p_2^*}}y + 2\Delta\sqrt{\frac{r}{p_2^*}}\right) \right)^{k-1} d\Phi(y). \end{aligned} \quad (4.17)$$

Using the fact that,

$$q_2^* \geq p_2^* \implies q_2^*\sqrt{\frac{2}{p_2^*}}x + 2\Delta\sqrt{\frac{r}{p_2^*}} \geq p_2^*\sqrt{\frac{2}{q_2^*}}x + 2\Delta\sqrt{\frac{r}{q_2^*}},$$

we obtain :

$$P(CS) \geq \begin{cases} \left(\Phi\left(\frac{\Delta\sqrt{r}}{\sqrt{(\frac{1}{2})^2 q_2^* + \frac{1}{2} p_2^{*2}}} \right) \right)^{k-1} & \text{if } q_2^* \geq p_2^*, \\ \left(\Phi\left(\frac{\Delta\sqrt{r}}{\sqrt{(\frac{1}{2})^2 p_2^* + \frac{1}{2} q_2^{*2}}} \right) \right)^{k-1} & \text{if } p_2^* > q_2^*. \end{cases} \quad (4.18)$$

From (4.10) and (4.14) follow immediately that in both cases $(q_2^* \geq p_2^*$ and $p_2^* > q_2^*$)

$$\min P(CS) \geq \left(\Phi\left(\Delta^*\sqrt{\frac{27r}{8}}\right) \right)^{k-1},$$

and from this we obtain that the pair (r^*, r^*) , given by (4.13), satisfies the (P^*, Δ^*) -condition (1.1).

5. Numerical results

$k=3; P^*=0.90$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N A^*=0.1)$	275	311	358	396	342	281	231	179	79
$E(N A^*=0.2)$	—	73	82	86	78	64	51	40	20
$E(N A^*=0.3)$	—	—	34	34	32	27	22	17	9
$E(N A^*=0.4)$	—	—	—	17	16	14	12	9	5

$k=3; P^*=0.95$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N A^*=0.1)$	396	448	516	577	495	408	335	258	114
$E(N A^*=0.2)$	—	106	120	127	113	93	75	57	29
$E(N A^*=0.3)$	—	—	49	51	46	39	31	24	13
$E(N A^*=0.4)$	—	—	—	28	26	23	18	15	8

$k=3; P^*=0.99$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N A^*=0.1)$	685	774	892	1010	856	706	578	443	197
$E(N A^*=0.2)$	—	183	208	225	196	159	128	96	50
$E(N A^*=0.3)$	—	—	85	88	79	65	52	40	22
$E(N A^*=0.4)$	—	—	—	47	43	36	29	23	13

$k=4; P^*=0.90$									
p_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N A^*=0.1)$	450	508	584	649	555	454	369	278	98
$E(N A^*=0.2)$	—	120	134	141	125	101	80	60	25
$E(N A^*=0.3)$	—	—	53	54	49	41	32	24	11
$E(N A^*=0.4)$	—	—	—	31	29	24	20	15	7

$k=4; P^*=0.95$		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N \Delta^*=0.1)$		615	694	798	894	761	624	506	379	134
$E(N \Delta^*=0.2)$		—	162	184	195	171	138	109	80	34
$E(N \Delta^*=0.3)$		—	—	73	75	68	56	44	33	15
$E(N \Delta^*=0.4)$		—	—	—	40	37	31	25	19	9

$k=4; P^*=0.99$		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N \Delta^*=0.1)$		1000	1129	1298	1469	1238	1016	823	613	218
$E(N \Delta^*=0.2)$		—	262	298	321	277	223	176	128	55
$E(N \Delta^*=0.3)$		—	—	124	129	114	92	72	53	25
$E(N \Delta^*=0.4)$		—	—	—	65	58	48	38	28	14

$k=5; P^*=0.90$		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N \Delta^*=0.1)$		638	720	827	921	783	640	515	381	112
$E(N \Delta^*=0.2)$		—	165	186	195	172	138	108	78	28
$E(N \Delta^*=0.3)$		—	—	77	79	71	58	45	33	13
$E(N \Delta^*=0.4)$		—	—	—	37	34	29	23	17	7

$k=5; P^*=0.95$		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N \Delta^*=0.1)$		843	952	1093	1225	1038	848	683	503	148
$E(N \Delta^*=0.2)$		—	218	246	261	228	183	143	102	37
$E(N \Delta^*=0.3)$		—	—	102	105	94	76	59	43	17
$E(N \Delta^*=0.4)$		—	—	—	55	50	41	32	24	10

$k=5; P^*=0.99$										
p_1		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N \Delta^*=0.1)$		1333	1505	1728	1955	1642	1342	1079	790	234
$E(N \Delta^*=0.2)$		—	347	393	424	364	291	226	160	59
$E(N \Delta^*=0.3)$		—	—	158	164	144	115	89	63	26
$E(N \Delta^*=0.4)$		—	—	—	84	75	61	47	34	15

$k=6; P^*=0.90$										
p_1		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N \Delta^*=0.1)$		838	945	1084	1209	1025	834	669	488	123
$E(N \Delta^*=0.2)$		—	217	244	257	225	180	139	99	31
$E(N \Delta^*=0.3)$		—	—	99	100	90	73	56	40	14
$E(N \Delta^*=0.4)$		—	—	—	50	46	39	30	22	8

$k=6; P^*=0.95$										
p_1		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	r
$E(N \Delta^*=0.1)$		1089	1229	1410	1582	1336	1089	872	634	160
$E(N \Delta^*=0.2)$		—	280	316	335	292	232	180	126	40
$E(N \Delta^*=0.3)$		—	—	128	131	117	94	72	51	18
$E(N \Delta^*=0.4)$		—	—	—	64	58	48	37	27	10

The expected sample size $E(N)$ is increasing with k . $E(N)$ is already greater than 1000 for $k=3$, $P^*=0.99$, $p_1=0.5$ and $\Delta^*=0.1$. $E(N)$ is relatively small, even for large k , if the difference between the success parameters of the best and second best population is significant, that means greater than 0.4.

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References

- [1] ASANO, CH., JOJIMA, K. and SUGIMURA, M., *Extended versions of sequential optimum selection plan with Play-the-Winner sampling and the stopping rules in a finite population*, Proceedings in Computational Statistics, 1976 (W. Berlin), Physica-Verlag, 277-284.
- [2] BERRY, D.A. and SOBEL, M., *An improved procedure for selecting the better of two Bernoulli populations*, Jasa, 68 (1973), 979-984.
- [3] SCHRIEVER, K.-H., *A Play-the-Winner Procedure for selecting the best of $k \geq 3$ binomial populations*. Preprint series of the Institute of Statistics and Mathematical Economics University of Karlsruhe, Federal Republic of Germany, No. 23, May 1977.
- [4] SOBEL, M. and WEISS, G.H., *Play-the-Winner rule and inverse sampling for selecting the best of $k \geq 3$ binomial populations*. Ann. Math. Statist., 48 (1972), 1808-1826.
- [5] SOBEL, M. and WEISS, G.H., *Play-the-Winner rule and inverse sampling in selecting the better of two binomial populations*. Jasa, 66 (1971), 545-551.