A TRUNCATED PLAY-THE-WINNER PROCEDURE FOR SELECTING THE BEST OF $ k \geq 3 $ BINOMIAL POPULATION

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A TRUNCATED PLAY-THE-WINNER PROCEDURE FOR SELECTING THE BEST OF $k \geq 3$ BINOMIAL POPULATION

By

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Summary

A truncated play-the-winner procedure for selecting the best of $k \geq 3$ binomial populations is proposed. Sampling always terminates and the maximal number of trials up to a final decision is equal to $k \cdot (r+1-c)$. The large expected sample sizes of the Sobel-Weiss-procedure [4] when the success parameters are small are avoided and the problem of a never ending sampling (using the Sobel-Weiss-procedure) when all success parameters are equal to 0 doesn't occur.

1. Introduction

The following procedure is a generalization of the Sobel-Berry-procedure [2] for $k \geq 3$ binomial populations. At the outset it puts the $k$ populations in a random order. Let $A_1$ denote the best population, that means the population with the largest success parameter, $A_2$ the one following $A_1$ in the initial randomization, e. t. c. (continuing in cyclic order) and let $p_i$ denote the success parameter of population $A_i$, $i \in \{1, 2, \ldots, k\}$. We use play-the-winner sampling with the ordered populations. Sampling terminates whenever one of the $k$ populations yields $r$ successes or all yield $c$ failures (i.e. whenever play-the-winner procedure has gone through $c$ cycles). In either case the population with the larger number of successes is selected. If the number of successes is the same for at least two populations (which can only occur after $c$ cycles), one of these populations is selected randomly. We must determine $r$ and $c$ so that the probability of a correct selection ($P(CS)$) satisfies the following $(P*; \Delta*)$-condition:

$$P(CS) \geq P*$$

whenever

$$p_1 - \max_{i \geq 1} p_i = : \Delta \geq \Delta*$$

with $1/k < P* < 1$ and $\Delta* > 0$ \hspace{1cm} \(1.1\)

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The maximal number of trials up to a final decision is equal to \( k(r+c-1) \). This may for example occur if \( k-1 \) of the \( k \) populations yield \( c \) failures and \( r-1 \) successes and the \( k \)-th population yields \( c-1 \) failures and \( r \) successes.

2. Exact results for \( P(CS) \)

Let \( F_i(r) \) denote the number of \( A_i \)-failures preceding the \( r \)-th success of \( A_i \), and let \( S_i(c) \) denote the number of \( A_i \)-successes preceding the \( c \)-th failure of \( A_i \); \( i \in \{1, 2, \ldots, k\} \).

The probability of a correct selection is a sum of two terms \( N, M \) which are determined by \( (F_i(r))_{i \in \{1, 2, \ldots, k\}} \) and \( (S_i(c))_{i \in \{1, 2, \ldots, k\}} \) respectively. \( N \) is the probability of selecting \( A_1 \) before \( c \) cycles and \( M \) is the probability of selecting \( A_1 \) in exactly \( c \) cycles. We obtain:

\[
N = P(F_1(r) < F_2(r), \ldots, F_1(r) < F_k(r), 0 \leq F_1(r) < c) + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \sum_{v=0}^{k-n-1} \left[ \frac{(k-1-n)!(k-v-1)!}{k!(k-1-n-v)!} P(F_1(r) = F_{1_1}(r) = \cdots) \right] (2.1)
\]

\[
= F_{1_1}(r), F_1(r) < F_{j_1}(r), \ldots, F_1(r) < F_{j_{k-n-1}}(r); 0 \leq F_1(r) < c \]

where \( S_n \) is defined as follows:

\[
S_n := \{ \omega = (\omega_1, \omega_2) \mid \omega_1 = (i_1, \ldots, i_n), \omega_2 = (j_1, \ldots, j_{k-n-1}), i_1 < \cdots < i_n, j_1 < \cdots < j_{k-n-1}, i_l, j_l \in \{2, \ldots, k\} \text{ and } \{i_1, \ldots, i_n \} \cap \{j_1, \ldots, j_{k-n-1} \} = \emptyset \}.
\]

In the first term of (2.1) the conditional probability of a correct selection is equal to 1; in the second term of (2.1) the conditional probability of a correct selection equals the probability that in the initial randomization \( A_1 \) precedes all \( A_i \) with \( l \in \{i_1, \ldots, i_n\} \).

The contribution of \( (S_i(c))_{i \in \{1, 2, \ldots, k\}} \) is much simpler to evaluate. We obtain:

\[
M = P(S_2(c) < S_1(c), \ldots, S_k(c) < S_1(c), 0 < S_1(c) < r) + \sum_{n=1}^{k-1} \sum_{\omega \in S_n} \left[ \frac{1}{n+1} P(S_1(c) = S_{1_1}(c) = \cdots = S_{1_n}(c), S_{j_1}(c) < S_1(c), \ldots, S_{j_{k-n-1}}(c) < S_1(c), 0 < S_1(c) < r) \right] (2.2)
\]

The conditional probability of a correct selection equals \( 1, \frac{1}{n+1} \) in the first and second term of (2.2) respectively.

Using the fact that \( F_i(r) \) and \( S_i(c) \) are all negative binomial chance variables, we obtain for \( N \),
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\[ N = \sum_{j=0}^{c-1} \binom{r+j-1}{j} p_i q_i J_{q_1}(j+1, r) \cdots J_{q_h}(j+1, r) \]

\[ + \sum_{n=1}^{k-1} \sum_{a=\mathbb{Z}_n}^{k-n-1} \sum_{v=0}^{c-1} \binom{k-n-1}{k-v-1} \frac{(r+j-1)!}{k!(k-1-n-v)!} \frac{(r+j-1)!}{j!} p_i q_i, \]

\[ \cdot (J_{q_1}(j, r) - J_{q_h}(j+1, r)) \cdots (J_{q_{k-n}}(j, r) - J_{q_h}(j+1, r)) \]

\[ \cdot J_{q_{k-n-1}}(j+1, r) \]

where \( J_q(\cdot, \cdot) \) denotes an incomplete beta-function.

Let \( X_{r,p_1} \) denote a negative binomial chance variable with index \( r>0 \) and success parameter \( p_1 \) and let \( E^{c-1}(\gamma) \) denote the expectation of the random variable \( \gamma \) truncated at \( c-1 \), then \( N \) may be written as follows:

\[ N = E^{c-1} \left( \prod_{l=2}^{k} J_{q_1}(X_{r,p_1}+1, r) \right) + \sum_{n=1}^{k-1} \sum_{a=\mathbb{Z}_n}^{k-n-1} \sum_{v=0}^{c-1} \frac{(k-1-n)!}{k!(k-1-n-v)!} \binom{(r+j-1)!}{j!} p_i q_i \]

\[ \cdot E^{c-1} \left( \prod_{l=1}^{n} (J_{q_1}(X_{r,p_1}+1, r) - J_{q_1}(X_{r,p_1}+1, r)) \right) \cdot \prod_{l=1}^{k-n-1} J_{q_{k-n-1}}(X_{r,p_1}+1, r) \].

In the same way we obtain for \( M \) (using (2.2)):

\[ M = \sum_{j=0}^{c-1} \binom{c+j+1}{j} q_i p_i J_{q_2}(c, j) \cdots J_{q_h}(c, j) \]

\[ + \sum_{n=1}^{k-1} \sum_{a=\mathbb{Z}_n}^{k-n-1} \sum_{j=0}^{c-1} \frac{1}{n+1} \binom{c+j+1}{j} q_i p_i (J_{p_1}(j, c) - J_{p_1}(j+1, c)) \cdots \]

\[ \cdot (J_{p_{k-n}}(j, c) - J_{p_{k-n}}(j+1, c)) J_{q_1}(c, j) \cdots J_{q_{k-n-1}}(c, j) \]

Using the same notation as above we obtain for \( M \):

\[ M = E^{c-1} \left( \prod_{l=2}^{k} J_{q_1}(c, X_{c,q_1}) \right) \]

\[ + \sum_{n=1}^{k-1} \sum_{a=\mathbb{Z}_n}^{k-n-1} \frac{1}{n+1} E^{c-1} \left( \prod_{l=1}^{n} (J_{p_1}(X_{c,q_1}, c) - J_{p_1}(X_{c,q_1}+1, c)) \right) \]

\[ \cdot \prod_{l=1}^{k-n-1} J_{q_{k-n}}(c, X_{c,q_1}) \]

\( P(CS) \) is the sum of (2.4) and (2.6).
3. Exact results for $E(N_i)$ and $E(N)$

Our next step consists in the derivation of the expectations $E(N_i)$; where $N_i$ denotes the number of "patients on treatment $i$"; $i \in \{1, 2, \ldots, k\}$. For this purpose we define a random vector $T=(T_1, \ldots, T_k, T_{k+1})$, where $T_i$ denotes the number of additional successes needed to declare population $A_i$ as best for $i \in \{1, \ldots, k\}$, and $T_{k+1}$ denotes the number of additional failures needed to finish the $c$-th cycle. In a second step we define:

$$U'_j(n_1, \ldots, n_k, f) := E(N_i|T=(n_1, \ldots, n_k, f), NT=A_j)$$

(3.1)

where "NT=A_j" means: "the next trial is on $A_j$".

We obtain the following recursion formulas:

$$U'_j(n_1, \ldots, n_k, f) = p_j U'_j(n_1, \ldots, n_{j-1}, n_j-1, n_{j+1}, \ldots, n_k, f) + q_j U'_{j+1}(n_1, \ldots, n_k, f-1) + \delta_{i,j}$$

(3.2)

with $\delta_{i,j}=1$ for $i=j$ and $0$ for $i \neq j$; $U'_{k+1} := U'_1$. The boundary conditions for (3.2) are given by:

$$U'_j(n_i, \ldots, n_{j-1}, 0, n_{j+1}, \ldots, n_k, f) = 0 \text{ for } n_i > 0 \text{ and } f > 0;$$

$$U'_j(n_i, \ldots, n_k, 0) = 0 \text{ for } n_i > 0 \text{ and } i \neq j \text{ for } i, j \in \{1, \ldots, k\}$$

(3.3)

To find a solution of (3.2) satisfying (3.3) we use generating functions $T'_j$ defined by

$$T'_j := T'_j(x_1, \ldots, x_k, y) := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \sum_{f=1}^{\infty} U'_j(n_1, \ldots, n_k, f) x_1^{n_1} \cdots x_k^{n_k} y^f$$

(3.4)

Having solved (3.2) we obtain for the expectations:

$$E(N_i) = \frac{1}{k} \sum_{j=1}^{k} E(N_i) = \frac{1}{k} \sum_{j=1}^{k} \sum_{f=1}^{\infty} U'_j(r, \ldots, r, k \cdot c)$$

(3.5)

$$E(N) = E(N_1 + \ldots + N_k) = \frac{1}{k} \sum_{j=1}^{k} \sum_{f=1}^{\infty} U'_j(r, \ldots, r, k \cdot c).$$

Solving (3.2) we have to distinguish two cases; If $j=i$ we obtain:

$$T'_j(1-p_i x_i) = q_i y T'_{i+1} + \frac{x_i}{1-x_i} \cdots \frac{x_k}{1-x_k} \frac{y}{1-y}.$$  

(3.6)

If $j \neq i$ we obtain:

$$T'_j = \frac{q_j y}{1-p_j x_j} T'_{j+1}.$$  

(3.7)
Using (3.7) we obtain from (3.6):

\[ T_i^j = \frac{1}{D^*} \prod_{\nu=1}^{k} (1-p_\nu x_\nu) \frac{y}{1-y} \prod_{\nu=1}^{k} \frac{x_\nu}{1-x_\nu} \]  
(3.8)

with

\[ D^* = (1-p_1 x_1) \cdot \ldots \cdot (1-p_k x_k) - q_1 \cdot \ldots \cdot q_k y^k \]

and from (3.7) follows:

\[ T_i^j = T_i \prod_{\nu=1}^{i-1} \frac{q_\nu y}{1-p_\nu x_\nu} \quad \text{if } j < i, \]  
(3.9)

\[ T_i^j = T_i \prod_{\nu=1}^{i-1} \frac{q_\nu y}{1-p_\nu x_\nu} \prod_{\nu=1}^{k} \frac{q_\nu y}{1-p_\nu x_\nu} \quad \text{if } j > i. \]  
(3.10)

We next determine the power series expansions of (3.8), (3.9) and (3.10). The expansion of \( \frac{1}{D^*} \) is given by:

\[ \frac{1}{D^*} = \sum_{n_1=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} \frac{1}{n_1! \ldots n_k!} (n_1 \cdot n_1) \cdot \ldots \cdot (n_k \cdot n_k) p_1^{n_1} \cdots p_k^{n_k} q_i^{k} \]  
(3.11)

\[ \cdot q_1 x_1 \cdots x_k^{k} \cdot y^{k-i}. \]

We have to expand the other factors of the product given in (3.8) and to multiply them with (3.11). The coefficients of the resulting power series are as follows:

\[ U_i(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}) = \sum_{i=0}^{\sigma(k, \lambda_{k+1})} \left[ q_1 q_2 \ldots q_i \right] \left[ q_1 q_2 \ldots q_i \right] \left[ q_1 q_2 \ldots q_i \right] \]  
(3.12)

\( q_i \neq 0 \) and \( \sigma(k, \lambda_{k+1}) \) is defined by:

\[ \sigma(k, \lambda_{k+1}) = \begin{cases} \left\lfloor \frac{\lambda_{k+1}}{k} \right\rfloor - 1 & \text{if } \gcd(k, \lambda_{k+1}) = k \\ \left\lfloor \frac{\lambda_{k+1}}{k} \right\rfloor & \text{if } \gcd(k, \lambda_{k+1}) < k \end{cases} \]  
(3.13)

where \( \gcd(a, b) \) denotes the greatest common divisor of \( a \) and \( b \) and \( \lfloor a \rfloor \) denotes the greatest integral number not greater than \( a \). We finally obtain:

\[ U_i^j(r, \ldots, r, k \cdot c) = \sum_{l=0}^{c-1} \left[ \frac{1}{q_1 q_2 \ldots q_l} \right] \left[ q_1 q_2 \ldots q_l \right] \left[ q_1 q_2 \ldots q_l \right] \]  
(3.14)

In the same way we obtain for \( j < i, (q_i \neq 0) \):

\[ U_j^i(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}) = \sum_{l=0}^{c-1} \left[ q_1 q_2 \ldots q_l \right] \left[ q_1 q_2 \ldots q_l \right] \left[ q_1 q_2 \ldots q_l \right] \]  
(3.15)
\[ U_j(r, \ldots, r, k; c) = \prod_{i=0}^{k-1} \left[ \prod_{v=1}^{j-1} \frac{1}{q_i} \frac{I_{q_v} (l+1, r)}{I_{q_v} (l, r)} \right], \quad (3.16) \]

and the last expressions of interest are as follows:

\[ j > i, \quad q_i \neq 0 \]

\[ U_j(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}) = \sum_{l=0}^{k-1} \left[ \prod_{v=1}^{j-1} \frac{1}{q_i} \frac{I_{q_v} (l+1, \lambda_v)}{I_{q_v} (l, \lambda_v)} \right], \quad (3.17) \]

\[ U_j(r, \ldots, r, k; c) = \prod_{i=0}^{k-1} \left[ \prod_{v=1}^{j-1} \frac{1}{q_i} \frac{I_{q_v} (l+1, r)}{I_{q_v} (l, r)} \right]. \quad (3.18) \]

\[ E(N_i) \text{ and } E(N) \text{ are now available from } (3.5), (3.14), (3.16) \text{ and } (3.18). \]

4. Asymptotic analysis

\( F_i(r) \) is a negative binomial chance variable with index \( r \) and success parameter \( p_i; i \in \{1, \ldots, k\} \). \( S_i(c) \) is a negative binomial chance variable with index \( c \) and success parameter \( q_i, i \in \{1, 2, \ldots, k\} \). It follows immediately from their definition that \( F_i(r) \) and \( F_j(r) \) are independent for \( i \neq j \); the same is true for the random variables \( S_i(c) \).

We use the following identities:

\[ F_i(r) > F_j(r) \quad \iff \quad F_i(r) - F_j(r) > 0 \]

\[ S_i(c) < S_j(c) \quad \iff \quad S_i(c) - S_j(c) < 0 \]

\[ A_{i\nu} := p_i - p_\nu, \quad \nu \in \{2, \ldots, k\}, \]

From the central limit theorem follows that for large \( r \) and \( c \) the random variables

\[ F_{i\nu} := \frac{F_i(r) - A_{i\nu}}{r/\sqrt{q_\nu}} \quad \text{and} \quad S_{i\nu} := \frac{S_i(c) - A_{i\nu}}{c/\sqrt{p_\nu}} \]

may be expressed by standard normal chance variables \( X_{i\nu}, Y_{i\nu} \) respectively. \( (X_1 = : X, Y_1 = : Y) \) Denoting with \( V_{i\nu}(\cdot), W_{i\nu}(\cdot) \) the distribution functions of \( F_{i\nu}, S_{i\nu} \) respectively, we obtain from (2.1):

\[ P(F_1(r) < F_2(r), \ldots, F_i(r) < F_k(r), 0 \leq F_1(r) < c) \]

\[ = \prod_{\nu=2}^{k-1} P \left( \frac{p_\nu}{p_1} \sqrt{\frac{q_1}{q_\nu}} F_{1\nu} - A_{1\nu} \sqrt{\frac{r}{q_\nu}} < F_{i\nu}, -\sqrt{r/q_1} \leq F_{i\nu} \leq c - \frac{p_1 - r q_1}{\sqrt{r q_1}} \right) \]

\[ = \int_{-\sqrt{r q_1}}^{(c - r q_1)/\sqrt{r q_1}} \prod_{\nu=2}^{k-1} \left( 1 - V_{i\nu} \left( \frac{p_\nu}{p_1} \sqrt{\frac{q_1}{q_\nu}} x - \frac{A_{1\nu}}{p_1} \sqrt{\frac{r}{q_\nu}} \right) \right) dV_{i\nu}(x). \]

(4.3)
Using lemma 1 of [4] we obtain:

\[(4.3) \sim \int_{-\infty}^{\infty} \prod_{r=2}^{k} \left( 1 - \Phi \left( \frac{p_{r}}{p_{1}} \sqrt{\frac{q_{1}}{q_{r}}} x - \frac{A_{r}}{p_{1}} \sqrt{\frac{r}{q_{r}}} \right) \right) d\Phi(x), \quad (4.4)\]

where "\(\sim\)" means asymptotic equivalence as \(r \to \infty\). Replacing \(x\) by \(-x\), we get the simpler version of (4.4), that means:

\[(4.3) \sim \int_{-\infty}^{\infty} \prod_{r=2}^{k} \Phi \left( \frac{p_{r}}{p_{1}} \sqrt{\frac{q_{1}}{q_{r}}} x + \frac{A_{r}}{p_{1}} \sqrt{\frac{r}{q_{r}}} \right) d\Phi(x), \quad (4.5)\]

In the same way we obtain:

\[P(S_{2}(c) < S_{1}(c), \ldots, S_{k}(c) < S_{1}(c), 0 < S_{1}(c) < r) \sim \int_{-\infty}^{\infty} \prod_{r=2}^{k} \Phi \left( \frac{q_{r}}{q_{1}} \sqrt{\frac{p_{1}}{p_{r}}} y + \frac{A_{r}}{p_{1}} \sqrt{\frac{c}{q_{r}}} \right) d\Phi(y). \quad (4.6)\]

The second terms of (2.5) and (2.6) tend to 0 using this approximation, and may be neglected. Our first result is therefore:

\[P(CS) \sim (4.5) + (4.6). \quad (4.7)\]

It follows from the integral expression of the incomplete beta-function \(J_{q}(s, t)\), that for fixed \(s\) and \(t\) \(J_{q}(s, t)\) is increasing with \(q\). That is why the incomplete beta-functions \(h_{(r, s)}(\cdot, \cdot)\) in the first term of (2.4) and (2.6) get their smallest value if we make \(q_{r}\) as small as possible, and that will be the case if we define \(p_{r}^{*} := p_{r}^{*} \) for all \(\nu \in \{2, \ldots, k\}\). \(p_{r}^{*}\) is the second largest success parameter. \((p_{2}\) is the parameter of the population \(A_{2}\) and will be usually different from \(p_{r}^{*}\)). We obtain:

\[P(CS) \sim \int_{-\infty}^{\infty} \Phi \left( \frac{p_{r}^{*}}{p_{1}} \sqrt{\frac{q_{1}}{q_{r}}} x + \frac{A}{p_{1}} \sqrt{\frac{r}{q_{r}}} \right)^{k-1} d\Phi(x) \quad (4.8)\]

\[\quad + \int_{-\infty}^{\infty} \Phi \left( \frac{q_{r}^{*}}{q_{1}} \sqrt{\frac{p_{1}}{p_{r}}} y + \frac{A}{q_{1}} \sqrt{\frac{c}{q_{r}}} \right)^{k-1} d\Phi(y),\]

with \(A := p_{1} - p_{r}^{*}\). Letting \(c \to \infty\) and holding \(r\) fixed, we obtain:

\[P(CS) \sim \int_{-\infty}^{+\infty} \Phi \left( \frac{p_{r}^{*}}{p_{1}} \sqrt{\frac{q_{1}}{q_{r}}} x + \frac{A}{p_{1}} \sqrt{\frac{r}{q_{r}}} \right)^{k-1} d\Phi(x). \quad (4.9)\]

Disregarding the fact, that the random variables

\[Z_{r} := \frac{p_{r}^{*}}{p_{1}} \sqrt{\frac{q_{1}}{q_{r}}} X_{1} + \frac{A}{p_{1}} \sqrt{\frac{r}{q_{r}}} - X_{r}\]
and

\[ Z_\mu := \frac{p_1^*}{p_1} \sqrt{\frac{q_1}{q_2}} X_1 + \frac{A}{p_1} \sqrt{\frac{r}{q_2}} X_\mu \]

are not independent in general, we get the much simpler expression:

\[ P(CS) \sim \left( \Phi \left( \frac{A \sqrt{r}}{\sqrt{p_1 q_2^* + q_1 p_2^*}} \right) \right)^{k-1}. \]  (4.10)

From

\[ P(CS) = \Phi \left( \frac{A \sqrt{r}}{\sqrt{p_1 q_2^* + q_1 p_2^*}} \right) = P^* \left( 1/ (k-1) \right) \]

follows that the least favorable configuration is obtained in the same way as in [3],[2], that means:

\[ \min P(CS) = \left( \Phi \left( \frac{2 \sqrt{r}}{8} \right) \right)^{k-1}. \]  (4.11)

We obtain from (4.11):

\[ A^* \cdot \sqrt{\frac{2 \sqrt{r}}{8}} = \Phi^{-1} \left( k-1, \sqrt{P^*} \right) = \lambda^* \left( P^* \right), \]  (4.12)

where \( \lambda^* \left( P^* \right) \) is the 100 \( P^{1/(k-1)} \)-percentile of the standard normal distribution. From (4.12) we obtain the required \( r \) value:

\[ r^* = \frac{8}{27} \left( \frac{\lambda^* \left( P^* \right)}{A^*} \right)^2. \]  (4.13)

In this special case the least favorable configuration is obtained by centering \( p_1 \) and \( p_2^* \) about 2/3.

Letting \( r \to \infty \), holding \( c \) fixed and disregarding the fact that \( Z_\nu \) and \( Z_\mu \) are not independent in general, we obtain:

\[ P(CS) \sim \left( \Phi \left( \frac{A \sqrt{c}}{\sqrt{q_1 p_2^* + q_2 p_1}} \right) \right)^{k-1} \]  (4.14)

The argument in (4.10) was minimized by setting \( p_0 := \frac{1}{2} \left( p_1 + p_2^* \right) \). Now we set \( q_0 := 1 - p_0 = \frac{1}{2} \left( q_1 + q_2^* \right) \). With fixed \( q_0 \) we obtain in a first step \( A = A^* \) and then a least favorable configuration \( q_1 \) and \( q_2^* \) centered about \( q_0 = 2/3 \), that means \( p_1 \) and \( p_2^* \) are centered about \( p_0 = 1/3 \). It follows immediately that \( \min P(CS) \) is the same as given in (4.11) with \( c \) instead of \( r \). The required \( c \) value is thus:

\[ c^* = c = \frac{8}{27} \left( \frac{\lambda^* \left( P^* \right)}{A^*} \right)^2. \]  (4.15)

From (3.14), (3.16) and (3.18) follows that \( E(N_i) \) and \( E(N) \) are monotone increasing with \( r \) and \( c \). That is why we conjecture that the best choice among all pairs
(r, c) satisfying the \((P^*, \Delta^*)\)-condition (1.1) consists in setting \(r = c\). With this we obtain from (4.8):

\[
P(CS) \sim \int_{-\infty}^{\sqrt{(q_1 - p_1)/q_1}} \left( \phi\left( \frac{p_1^*}{q_1} \sqrt{\frac{r_{1^*}}{p_1^*} x + \frac{\Delta^*}{q_1} \sqrt{\frac{r}{q_1^*}}} \right) \right)^{k-1} \, d\Phi(x)
\]

(4.16)

\[
+ \int_{-\infty}^{\infty} \left( \phi\left( \frac{q_1^*}{p_1^*} \sqrt{\frac{p_1^*}{q_1^*} y + \frac{\Delta^*}{q_1^*} \sqrt{\frac{r}{q_1^*}}} \right) \right)^{k-1} \, d\Phi(y).
\]

If \(p_1 \neq q_1\) \(P(CS)\) is given by (4.10) or (4.14), provided \(r\) is large enough. Thus we have only to investigate the special case \(p_1 = q_1 = \frac{1}{2}\). We obtain:

\[
P(CS) \sim \int_{0}^{\infty} \left( \phi\left( \frac{p_1^*}{q_1^*} \sqrt{\frac{2}{p_1^*} x + 2\Delta \sqrt{\frac{r}{q_1^*}}} \right) \right)^{k-1} \, d\Phi(x)
\]

(4.17)

\[
+ \int_{-\infty}^{\infty} \left( \phi\left( \frac{q_1^*}{p_1^*} \sqrt{\frac{2}{p_1^*} y + 2\Delta \sqrt{\frac{r}{q_1^*}}} \right) \right)^{k-1} \, d\Phi(y).
\]

Using the fact that,

\[
q_{1^*} \geq p_{1^*} \rightarrow q_{1^*} \sqrt{\frac{2}{p_{1^*}} x + 2\Delta \sqrt{\frac{r}{p_{1^*}^2}}} \geq p_{1^*} \sqrt{\frac{2}{q_{1^*}} x + 2\Delta \sqrt{\frac{2}{q_{1^*}^2}}},
\]

we obtain:

\[
P(CS) \geq \begin{cases} 
\left( \phi\left( \frac{\Delta \sqrt{r}}{\sqrt{(1/2)^2 q_{1^*}^2} + 1/2 p_{1^*}^2}} \right) \right)^{k-1} & \text{if } q_{1^*} \geq p_{1^*}, \\
\left( \phi\left( \frac{\Delta \sqrt{r}}{\sqrt{(1/2)^2 p_{1^*}^2} + 1/2 q_{1^*}^2}} \right) \right)^{k-1} & \text{if } p_{1^*} > q_{1^*}.
\end{cases}
\]

(4.18)

From (4.10) and (4.14) follow immediately that in both cases \((q_{1^*} \geq p_{1^*} \text{ and } p_{1^*} > q_{1^*})\)

\[
\min P(CS) \geq \left( \phi\left( \frac{\Delta^* \sqrt{27r}}{8} \right) \right)^{k-1},
\]

and from this we obtain that the pair \((r^*, r^*)\), given by (4.13), satisfies the \((P^*, \Delta^*)\)-condition (1.1).
5. Numerical results

$k = 3; \ P^* = 0.90$

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>0.2</th>
<th>0.3</th>
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$k = 3; \ P^* = 0.95$

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$k = 3; \ P^* = 0.99$

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## A Truncated Play-the-winner Procedure for Selecting

### $k = 4; \quad P^* = 0.95$

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### $k = 5; \quad P^* = 0.90$

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### $k = 5; \quad P^* = 0.95$

<table>
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<th>0.9</th>
<th>$r$</th>
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<tbody>
<tr>
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<td>55</td>
<td>50</td>
<td>41</td>
<td>32</td>
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</table>
The expected sample size $E(N)$ is increasing with $k$. $E(N)$ is already greater than 1000 for $k=3$, $P^*=0.99$, $p_1=0.5$ and $\xi^*=0.1$. $E(N)$ is relatively small, even for large $k$, if the difference between the success parameters of the best and second best population is significant, that means greater than 0.4.

Acknowledgement:

The author would like to thank H. Martin for computing the tables of section 5.
References


