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# ASYMPTOTIC DISTRIBUTION OF EIGENVALUES AND DEGENERATION OF SPARSE RANDOM MATRICES

By

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## Abstract

This work is concerned with an asymptotical distribution of eigenvalues of sparse random matrices. It is shown that the semi-circle law which is known for random matrices is also valid for the sparse random matrices with sparsity  $n/N=o(1)$ , where  $n$  is the matrix size and  $2N$  the number of non-zero elements of the matrix. The degree of degeneration is also estimated for the matrices with  $2N \sim cn$  ( $c>0$ : const.) using knowledge of random graphs.

## I. Introduction

Random matrices have been studied concerning the excitation spectra of nuclei in the field of experimental physics. Energy levels of a system are considered to be described by the eigenvalues of an Hermitian operator, i.e., the Hamiltonian. And in general one must treat the operation in the infinite dimensional Hilbert space. However to avoid the difficulty in treating such operations, we make approximations by truncating the Hilbert space except a part that is relevant to the problem at hand. Thus, representing the Hamiltonian by a finite dimensional matrix and solving the eigenvalue equation:

$$H\Psi_i = E_i\Psi_i$$

we get all the eigenvalues  $E_i$  (energy levels) and the eigenfunctions  $\Psi_i$ , which yield any physical information about the system in principle. In actual situations, however, since we do not know the Hamiltonian exactly and since, even if we do it, it is too complicate to solve it, we make a statistical hypothesis on  $H$ ; the hypothesis that the elements of the matrix are random variables with appropriate distribution functions. The case of Gaussian distributions of the matrix elements has been treated analytically by Hsu, Mehta, Ginibre, and others. See L. M. Mehta [1] about the historical detail.

One may often face with various situations in which he treats random matrices

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in engineering or biology as well as in physics: They are relevant to solving problems concerning the behaviours of systems of a large number of interacting elements.

A biological organization is composed from an enormous number of elements such as protein. A collection of organella or cells composed from protein serves as a unit in a higher level system. It seems that there exists a key which unlocks a door through which one may reach comprehension of biological organization in some sense in the dynamical behavior of a system of many interacting elements of any level and in the hierarchical structures of such systems as one may appreciate from systems of proteins to society via those of cells, organs, individuals or troop of individuals, etc.

Since interaction between elements is in general nonlinear and the number of elements concerned is usually very huge, it is impossible to treat directly the dynamical equation describing the interacting system. In addition, we do not know the exact feature of interaction as we did not know the Hamiltonian. In biological system, it is more probable to consider that the interaction between units of a system at any level is determined in random manner except a small number of parameters which specify the randomness because of, for instance, the economy of information carriers such as DNA and that any unit or element of the system can not interact with all the others because of physical or geometrical restrictions on the units or the system. Thus, the first approximation of the system equations of the interacting elements gives random sparse matrices and we get information about the systems through the average property of the matrices.

The distribution function and the asymptotic distribution function (as the size  $n$  of the matrix tends to infinity) of the eigenvalues of random Hermitian matrices whose elements are subject to independent Gaussian distributions are already known. Especially, the latter is known to be a so called semi-circle distribution for a variety of distribution functions of the matrix elements.

In what follows, we will consider the asymptotic distribution of the eigenvalues of *sparse* random matrices and also the degree of the degeneration of the matrices in relevant to the sparsity of the matrices, i.e., the ratio the size  $n$  of the matrix to the number of non-zero elements  $2N(n)$  of it. The following discussion is partly parallel to that of W.H. Olson and V.R.R. Uppuluri [2] in the first half and in the latter half, we will make use of the results of the investigation on random graphs by P. Erdős and A. Rényi [3].

## II. Preliminaries

Let us consider a random graph  $\Gamma_{n,N}$  consisting of  $n$  labelled vertices,  $P_1, P_2, \dots, P_n$  and  $N$  unlabelled edges without multiplication. Let  $N$  edges be chosen among  $\binom{n}{2}$  possible edges at random. Thus the number of the ways of such choice is

$$\binom{\binom{n}{2}}{N}$$

and there are the same number of graphs with  $n$  vertices and  $N$  edges. Assume that each of the graphs occurs at equi-probability

$$\left(\frac{\binom{n}{2}}{N}\right)^{-1}.$$

Next let us define a random 0—1 matrix  $E_I = (\varepsilon_{ij})_{i,j=1}^n$  associated with a random graph  $\Gamma_{n,N}$  as follows:

$$\varepsilon_{ij} = \begin{cases} 1: & \text{if there exists an edge between } P_i \text{ and } P_j, \\ 0: & \text{otherwise.} \end{cases} \quad (1 \leq i, j \leq n)$$

We also define a random weighted matrix  $A_I = (a_{ij})_{i,j=1}^n$  associated with  $\Gamma_{n,N}$  as follows:

$$a_{ij} = \begin{cases} x_{ij}, & \text{for all } i \text{ and } j \text{ such that } \varepsilon_{ij}=1, \\ 0, & \text{for all } i \text{ and } j \text{ such that } \varepsilon_{ij}=0, \end{cases}$$

where  $x_{ij}$ 's are random variables defined on some probability space. Note that  $A_I$  is a symmetric matrix.

We will investigate the asymptotical property of the eigenvalues of random weighted matrices associated with random graphs. Let us denote the eigenvalues of an  $n \times n$  matrix  $A$  by  $\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)$ . We give the following lemmata.

LEMMA 1. *Let  $A$  be an  $n \times n$  matrix and suppose  $\varepsilon > 0$  is given. Then there exist a  $\delta > 0$  and a permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  such that for any matrix  $D = (d_{ij})_{i,j=1}^n$  such that*

$$\sum_{i,j=1}^n |d_{ij}| < \delta, \\ |\lambda_i(A) - \lambda_{\sigma(i)}(A + D)| < \varepsilon, \quad i = 1, 2, \dots, n.$$

A proof of this lemma may be found in A.M. Ostrowski [4].

LEMMA 2. *The ordered eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  are continuous functions of elements of a symmetric matrix  $A$ .*

See (2) for the proof of this lemma.

According to these two lemmata, denoting the eigenvalues of a random symmetric matrix  $A_I$  by  $\lambda_1(A_I) \leq \lambda_2(A_I) \leq \dots \leq \lambda_n(A_I)$ ,  $\lambda_i(A_I)$ ,  $i = 1, 2, \dots, n$  are also random variables because these are continuous functions of the elements of  $A_I$  which are themselves random variables.

Suppose that a random variable  $X$  and a sequence of random variables  $\{X_n\}$ ,  $n = 1, 2, \dots$ , are given and let  $F$  and  $F_n$  be distribution functions of  $X$  and  $X_n$ , respectively. Let the  $k$ -th order moments of the distribution functions  $F$  and  $F_n$ , if they exist, be

$$\alpha_k = \int_{R_1} x^k dF(x), \\ \alpha_{k,n} = \int_{R_1} x^k dF_n(x),$$

respectively. Then we have:

LEMMA 3. *If, for all  $k \geq k_0$  arbitrary but fixed, the sequence  $\alpha_{k,n} \rightarrow \alpha_k$  finite, then this sequence converges for every value of  $k$  and if the sequence  $\{\alpha_k\}_{k=1}^{\infty}$  uniquely determines  $F$ , then  $F_n(x) \rightarrow F(x)$  as  $n \rightarrow \infty$  at all points of continuity of  $F$ .*

A proof of this lemma may be found in M. Loève [5].

Let  $(\Omega, \mathcal{B}, P)$  be an arbitrary probability space, where  $\Omega$  is any space,  $\mathcal{B}$  is a Borel field of subsets of  $\Omega$  and  $P$  is a probability measure defined on  $\mathcal{B}$ . Let  $X = X(\omega)$  be a random variable defined on  $\Omega$ . An empirical distribution function of a set of random variables  $\{X_1, X_2, \dots, X_n\}$  is defined by a mapping  $W_n: R_1 \times \Omega \rightarrow [0, 1]$  such that for any  $\omega \in \Omega$

$$W_n(x)(\omega) = \frac{1}{n} \sum_{i=1}^n I_{[X_i \in (-\infty, x]]}(\omega),$$

where  $I_B$  denotes the indicator function of a set  $B$ .

Suppose that

$$M_{k,n}(\omega) = \int_{R_1} x^k dW_n(x)(\omega) = \frac{1}{n} \sum_{i=1}^n X_i^k(\omega), \quad \forall \omega \in \Omega.$$

Let  $W(x)$  be a distribution uniquely determined by the sequence of its moments  $\{\alpha_k\}_{k=1}^{\infty}$ . Then we have the following lemma about an asymptotic property of the empirical distribution function.

LEMMA 4. *If, for all  $k=1, 2, \dots, n$ ,  $M_{k,n} \xrightarrow{P} \alpha_k$  as  $n \rightarrow \infty$ , then  $W_n(x) \xrightarrow{P} W$  as  $n \rightarrow \infty$  at all points of continuity of  $W$ .*

See (2) for the proof.

### III. Asymptotic distribution of eigenvalues of random matrices

In this section, we will show the validity of so-called semi-circle law for some range of the ratio  $n$  to  $N(n)$ .

Let us consider a random weighted matrix  $A_F$  associated with a random graph  $\Gamma_{n,N}$ . Clearly

- (1)  $A_F$  is symmetric,
- (2)  $a_{ii} = 0, \quad i=1, 2, \dots, n$ ,
- (3) the number of non-zero elements of  $A_F$  is  $2N$ .

Suppose that for the non-zero elements  $a_{ij}$  of  $A_F$ ,

- (4)  $\{a_{ij}, i < j\}$  are independent and

$$P(a_{ij} = \sigma) = \frac{1}{2},$$

$$P(a_{ij} = -\sigma) = \frac{1}{2},$$

$N$  of  $\binom{n}{2}$  elements  $\{a_{12}, a_{13}, \dots, a_{1n}, a_{23}, \dots, a_{2n}, \dots, a_{n-1,n}\}$  of a random weighted matrix  $A_F$  are non-zero and the number of ways of choosing non-zero elements is

$$\binom{\binom{n}{2}}{N}.$$

Letting  $T(\Gamma)$  be a way of assigning  $\sigma$  or  $-\sigma$  to the non-zero elements, we have

$2^N$  different  $T(\Gamma)$ 's. Denote them by  $T_i(\Gamma)$ ,  $i=1,2,\dots,2^N$ . Then the probability of occurrence of a random weighted matrix is

$$\left\{2^N \binom{\binom{n}{2}}{N}\right\}^{-1}.$$

Let  $e_1(A_\Gamma), e_2(A_\Gamma), \dots, e_n(A_\Gamma)$  be the eigenvalues of a random weighted matrix  $A_\Gamma$  and denote the empirical distribution function of the set of random variables  $e_1(A_\Gamma), \dots, e_n(A_\Gamma)$  by  $W_n(x)(A_\Gamma)$ . Our aim is to show that by the aid of Lemma 4,

$$W_n(x)(A_\Gamma) \xrightarrow{P.} W(x),$$

where  $W(x)$  is a distribution function uniquely determined by the sequence of its moments  $\{\alpha_k\}$ . To do so, it is enough to show that the mean of the  $k$ -th order moment  $m_{k,n}$  and the variance converge to  $\alpha_k$  and zero, respectively.

Let the expectation of  $W_n(x)(A_\Gamma)$  be  $EW_n(x)$ . Then

$$\begin{aligned} EW_n(x) &= \sum_{all A_\Gamma} W_n(x)(A_\Gamma) \times P_r(A_\Gamma) \\ &= \frac{1}{\left(\binom{\binom{n}{2}}{N}\right) 2^N} \sum_{all \Gamma_{n,N}} \sum_{T(\Gamma)} W_n(x)(A_\Gamma). \end{aligned}$$

This means that  $EW_n(x)$  is a discrete distribution function of  $n \binom{\binom{n}{2}}{N} 2^N$  eigenvalues (including multiplications) of all  $A_\Gamma$  which has jumps of height

$$\left\{n \binom{\binom{n}{2}}{N} 2^N\right\}^{-1}$$

(if there is an eigenvalue with multiplication  $s$ ,  $s$  times of the height). Thus, denoting the  $k$ -th order moment of  $EW_n(x)$  by  $Em_{k,n}$ , we have

$$\begin{aligned} Em_{k,n} &= \int_{R_1} x^k dEW_n(x) \\ &= \frac{1}{n \binom{\binom{n}{2}}{N} 2^N} \sum_{j=1}^{n \binom{\binom{n}{2}}{N} 2^N} e_j^k \\ &= E\left(\frac{1}{n} \sum_{i=1}^n e_i^k(A_\Gamma)\right). \end{aligned}$$

Since

$$\begin{aligned} \sum_{i=1}^n e_i^k(A_\Gamma) &= \text{tr}(A_\Gamma^k) \\ &= \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n \prod_{l=1}^k a_{j_l j_{l+1}}(A_\Gamma), \\ &\quad \text{with } j_1 = j_{k+1} \end{aligned}$$

we have

$$Em_{k,n} = E\left\{\frac{1}{n} \text{tr}(A_\Gamma^k)\right\}$$

$$= \frac{1}{n \binom{n}{N}} 2^{N \text{ all } \Gamma} \sum_{\text{all } T(\Gamma)} \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n \prod_{l=1}^k a_{j_l j_{l+1}}(A_\Gamma) .$$

( $j_{k+1}=j_1$ )

In what follows, we will evaluate the right hand side.

To begin with, we give:

LEMMA 5. For all  $i \neq j$

$$E(a_{ij}^m(A_\Gamma)) = \begin{cases} 0 & , \quad m: \text{ odd} , \\ \frac{N}{\binom{m}{2}} \sigma^m , & m: \text{ even} . \end{cases}$$

PROOF The number of graphs which have an edge between the vertices  $P_i$  and  $P_j$  is given by

$$\binom{\binom{n}{2}-1}{N-1}$$

For any  $E_\Gamma$  such that  $\varepsilon_{ij}=1$ , the number of associated weighted matrices with  $a_{ij}=\sigma$

is  $2^{N-1}$ ,

the number of those with  $a_{ij}=-\sigma$

is  $2^{N-1}$ .

Thus

$$E(a_{ij}^m(A_\Gamma)) = \frac{\binom{\binom{n}{2}-1}{N-1} (\sigma^m 2^{N-1} + (-\sigma)^m 2^{N-1})}{\binom{\binom{n}{2}}{N} 2^N} ,$$

which proves the above lemma. ◀

Let us consider a mapping  $f$  which makes a finite sequence  $f(1), f(2), \dots, f(k+1)$  such that  $f(l) \in \{1, 2, \dots, n\}$ ,  $l=1, \dots, k+1$ , and denote by  $A_{k,n}$  the class of all mappings:

$$A_{k,n} = \{f: i=1, 2, \dots, k+1, f(i) \in \{1, 2, \dots, n\}\} .$$

Denoting by  $\# A$  the cardinality of a set  $A$ , we have clearly

$$\# A_{k,n} = n^{k+1} .$$

Suppose that

$$B_{k,n} = \{f: f \in A_{k,n}, f(k+1) = f(1)\} .$$

Then we have

$$\begin{aligned} & \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n \prod_{l=1}^k a_{j_l j_{l+1}}(A_\Gamma) \\ &= \sum_{f \in B_{k,n}} \prod_{l=1}^k a_{f(l) f(l+1)}(A_\Gamma) . \end{aligned}$$

Since  $a_{ii}(A_\Gamma)=0$  for all  $i$ , putting

$$C_{k,n} = \{f: f \in B_{k,n}, f(l) \neq f(l+1), l=1, 2, \dots, k\} ,$$

we can equate the right hand side to

$$\sum_{f \in \mathcal{C}_{k,n}} \prod_{l=1}^k a_{f(l)f(l+1)}(A_T).$$

Clearly  $\#C_{k,n} = n(n-1)^{k-1}$ .

For any  $f \in C_{k,n}$ , define

$$g_f(l) = (f(l), f(l+1)), \quad l=1, 2, \dots, k.$$

Let  $\#(i, j)_f$  denote

$$\# \{g_f(l) = (f(l), f(l+1)), \quad l=1, 2, \dots, k, \quad f(l)=i, \quad f(l+1)=j\}$$

By definition,  $\#(i, j)_f$  is the number of the ordered pair  $(i, j)$  which, for a given  $f \in C_{k,n}$ , the set of  $k$  ordered pairs  $\{g_f(l), \quad l=1, 2, \dots, k\}$  contains.

Let  $d_{ij}^f = \#(i, j)_f + \#(j, i)_f$ , and let

$$D_{k,n} = \{f: f \in C_{k,n}, \quad d_{ij}^f = \text{even for all } (i, j) \in \{g_f(l), \quad l=1, \dots, k\}\},$$

$$E_{k,n} = \{f: f \in C_{k,n}, \quad d_{ij}^f = \text{odd for some } (i, j) \in \{g_f(l), \quad l=1, \dots, k\}\}.$$

Then

$$C_{k,n} = D_{k,n} \cup E_{k,n}, \quad D_{k,n} \cap E_{k,n} = \emptyset.$$

If  $k$  is odd, then

$$D_{k,n} = \emptyset, \quad E_{k,n} = C_{k,n}.$$

Let us evaluate

$$\begin{aligned} & E \left\{ \sum_{f \in E_{k,n}} \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\} \\ &= E \left\{ \sum_{f \in E_{k,n}} \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\} \\ &= \sum_{f \in E_{k,n}} E \left\{ \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\} \\ &= \sum_{f \in E_{k,n}} E \left\{ a_{ij}^{g_f} \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\} \\ &= \sum_{f \in E_{k,n}} E \left\{ a_{ij}^{g_f} \right\} E \left\{ \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\} = 0, \end{aligned}$$

where  $\prod_{l=1}^k$  is taken for all  $l$  except those that give  $g_f(l) = (i, j)$  or  $(j, i)$ . The last equality is from Lemma 5. It is immediate that

$$E \left\{ \sum_{f \in \mathcal{C}_{k,n}} \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\} = E \left\{ \sum_{f \in D_{k,n}} \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\}.$$

Thus for odd  $k$ ,  $m_{k,n} = 0$

and for even  $k$ ,  $m_{k,n} = \frac{1}{n} E \left\{ \sum_{f \in D_{k,n}} \prod_{l=1}^k a_{f(l)f(l+1)}(A_T) \right\}.$

For any  $f \in D_{2\nu,n}$ , let  $p_f$  be the number of different values of  $f(l)$ ,  $l=1, 2, \dots, 2\nu$ :

$$p_f = \# \{f(l): f(l) \notin \{f(1), \dots, f(l-1)\}, \quad l=2, \dots, 2\nu+1\} + 1.$$

It is clear that for any  $f \in D_{2\nu,n}$



$$2 \leq p_f \leq \nu + 1.$$

Let  $q_f(p)$  denote the number of different pairs in the set

$$\{g_f(l), l=1, 2, \dots, 2\nu\} \quad \text{for } f \text{ with } p_f=p$$

without distinguishing a pair  $(i, j)$  from  $(j, i)$  if they are contained in the set:

$$p-1 \leq q_f(p) \leq q_p^* = \min \{\nu, \binom{\nu}{2}\}.$$

Let us denote by  $D_{2\nu, n}(p, q)$  a subset of  $D_{2\nu, n}$  such that

$$p_f=p, \quad q_f=q \quad (p=2, 3, \dots, \nu+1, \quad q=p-1, \dots, q^*)$$

and by  $Z_\nu(p, q, n) \# D_{2\nu, n}(p, q)$ .

It immediately follows that

$$\sum_{p=2}^{\nu+1} \sum_{q=p-1}^{q_p^*} Z_\nu(p, q, n) = \# D_{2\nu, n}$$

For the sake of evaluation of  $Z_\nu(p, q, n)$ , it is convenient to make use of the concepts of graph: Let us consider connected graphs consisting of  $p$  vertices unlabelled except one and  $\nu$  unlabelled edges, such that the number of adjacent pairs of vertices is exactly  $q (\geq p-1)$ . Thus there may be more than one edge between adjacent vertices. Denote the class of non-isomorphic graphs by  $\mathcal{G}_{p, \nu}^q$ . Let us consider, for any  $G \in \mathcal{G}_{p, \nu}^q$ , a double stroke Eulerian cycle (abbreviated by d.s. Eulerian cycle) which starts from the labelled vertex, passes through every adjacent pairs of vertices of  $G$  exactly twice the number of edges between them and returns to the original vertex. We write the number of different d.s. Eulerian cycles for any  $G \in \mathcal{G}_{p, \nu}^q$  by  $S(G)$ .

Letting

$$Z_\nu(p, q) = \sum_{G \in \mathcal{G}_{p, \nu}^q} S(G)$$

we have the following lemma:

LEMMA 6.

$$Z_\nu(p, q, n) = Z_\nu(p, q) n(n-1) \cdots (n-p+1).$$

PROOF. For any one of  $S(G)$  d.s. Eulerian cycles for any  $G \in \mathcal{G}_{p, \nu}^q$  when it reaches a new vertex after the labelled vertex, assign a number among  $\{1, 2, \dots, n\}$  to the vertex without duplication. Clearly there are  $n(n-1) \cdots (n-p+1)$  ways of such assignment for any d.s. Eulerian cycle. We will show that to every assignment for a given d.s. Eulerian cycle, there corresponds an  $f \in D_{2\nu, n}(p, q)$  uniquely. This is so because: For the number which is assigned to the starting vertex of the d.s. Eulerian cycle, say  $t_1$ , let  $f(1) = t_1$ . Let the number assigned to the  $l$ -th vertex ( $l=1, 2, \dots, 2\nu+1$ ) along the d.s. Eulerian cycle be  $t_l$ . When it reaches through an edge from the  $l$ -th vertex to the  $l+1$ -th, make a pair  $(f(l), f(l+1))$  such that  $f(l) = t_l$  and  $f(l+1) = t_{l+1}$ . Then for the d.s. Eulerian cycle, a sequence of  $2\nu$  pairs  $(f(1), f(2)), (f(2), f(3)), \dots, (f(2\nu), f(2\nu+1))$  is constructed. Since the d.s. Eulerian cycle ends at its starting vertex,

$$f(2\nu+1) = t_1 = f(1).$$

Since the d.s. Eulerian cycle covers  $p$  different vertices, the number of different values of  $f(l)$ ,  $l=1,2,\dots,2\nu+1$  is  $p$ , i.e.,

$$p_f = p.$$

Since the number of different adjacent pairs of vertices is  $q$ , the number of different pairs in the above sequence of  $2\nu$  pairs is  $q$  without distinguishing a pair  $(i,j)$  from  $(j,i)$  if they are contained in the sequence, i.e.,  $q_f(p)=p$ . Since the d.s. Eulerian cycle passes any adjacent pair of vertices of a graph  $G$  twice the number of edges between them, for any  $(i,j) \in \{(f(l), f(l+1)), l=1,2,\dots,2\nu\}$

$$\#(i,j) + \#(j,i) = \text{even},$$

i.e.,  $d_{ij}^f$  is even. Thus  $f \in D_{2\nu,n}(p,q)$ .

Conversely, it can be shown that for any  $f \in D_{2\nu,n}(p,q)$ , there exists only one d.s. Eulerian cycle on a graph  $G \in \mathcal{G}_{p,\nu}^q$ . ◀

In consequence, in order to evaluate  $Z_\nu(p,q,n)$ , it is necessary to do  $Z_\nu(p,q)$ . However it is very complicate to evaluate  $Z_\nu(p,q)$  except the case;  $p=\nu+1$  and  $q=\nu$ .

LEMMA 7. Any graph which belongs to  $\mathcal{G}_{p,\nu}^{p-1}$  is a rooted tree whose root is the labelled vertex if one ignores the multiplicity of edges.

PROOF. This accords to the fact that a connected graph with  $p$  vertices and  $p-1$  edges is a tree. ◀

LEMMA 8. If, for any  $f \in D_{2\nu,n}(p,p-1)$ ,

$$(i,j) \in \{(f(l), f(l+1)), l=1,2,\dots,2\nu\}$$

then if  $\#(i,j)=d$ ,  $\#(j,i)=d$ .

PROOF. The graph which corresponds to the  $f$  belongs to  $\mathcal{G}_{p,\nu}^{p-1}$ . This lemma follows from the fact that any d.s. Eulerian cycle which starts from the root of a tree with labelled vertices, if it passes the edge from the vertex  $i$  to  $j$ , can not be at the vertex  $i$  again unless it passes from  $j$  to  $i$  reversely.

REMARK: If  $p_f=\nu+1$ ,  $q_f=\nu$ , then for any  $(i,j) \in \{g_f(l), l=1,2,\dots,2\nu\}$

$$\#(i,j) = \#(j,i) = 1.$$

Table 1 illustrates the values of  $Z_\nu(p,p-1)$  for small  $\nu$ ,  $p$ .

Table 1. Numbers of different double stroke Eulerian cycles for graphs  $\in \mathcal{G}_{p,\nu}^{p-1}$ . The numbers in the diagonal are identical with those by Wigner' theorem.

$\nu \backslash p$	2	3	4	5	6
1	1	*	*	*	*
2	1	2	*	*	*
3	1	6	5	*	*
4	1	14	28	14	*
5	1	30	110	107	42

Now we will show a theorem by Wigner.

WIGNER'S THEOREM.

$$Z_\nu(\nu+1, \nu) = \frac{(2\nu)!}{\nu!(\nu+1)!}.$$

PROOF. We will follow the proof in [2]. Let  $f \in D_{2\nu, n}(\nu+1, \nu)$ . If a pair  $(i, j)$  is contained in a sequence of  $2\nu$  pairs  $\{g_f(l) = (f(l+1)), l=1, 2, \dots, 2\nu\}$ , so is  $(j, i)$ . If a pair  $(i, j)$  appears before  $(j, i)$  in the sequence, assign  $+1$  to  $(i, j)$  and  $-1$  to  $(j, i)$ . Thus there corresponds a sequence of  $+1$ 's and  $-1$ 's of length  $2\nu$  uniquely to the sequence of  $2\nu$  pairs. Since a  $+1$  is assigned before the corresponding  $-1$ , the number of  $+1$ 's assigned to  $g_f(1), \dots, g_f(l)$ ,  $1 \leq l \leq 2\nu-1$  is greater than or equal to that of  $-1$ 's assigned to them. Thus the number of  $f$ 's which belong to  $D_{2\nu, n}(\nu+1, \nu)$  is equal to, among the paths which start from the origin, jump  $+1$  or  $-1$  on the  $x$ -axis at every step of time and return to the origin again at the  $2\nu$ -th step, the number of those paths which run only the plus side of the  $x$ -axis. Let us write the number by  $S_\nu$  simply. Suppose that  $S'_\nu$  is the number of paths which make the first return to the origin at the  $2\nu$ -th step. Such paths must exist in the positive side of the  $x$ -axis except at the final step. Thus  $S'_\nu = S_{\nu-1}$ . Consequently we have

$$S_\nu = \sum_{k=1}^{\nu} S'_k S_{\nu-k} = \sum_{k=1}^{\nu} S_{k-1} S_{\nu-k}, \quad S'_1 = S_0 = 1.$$

Letting  $t(x) = \sum_{k=0}^{\infty} S_k x^k$ , we have

$$t(x) = 1 + xt^2(x),$$

which gives us the solutions

$$t(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Since  $t(x)$  should be a monotone increasing function of  $x$  because  $S_k > 0$ , we discard the solution with a plus sign. The expansion with respect to  $x$  gives:

$$S_\nu = \frac{(2\nu)!}{\nu!(\nu+1)!} \quad \nu=1, 2, \dots$$

Thus

$$Z_\nu(\nu+1, \nu) = \frac{(2\nu)!}{\nu!(\nu+1)!}.$$

Now let us return to compute

$$E\left\{\sum_{f \in D_{2\nu, n}(p, q)} \prod_{l=1}^{2\nu} a_{f(l)f(l+1)}(A_f)\right\}.$$

Noting that, for any  $f \in D_{2\nu, n}(p, q)$ ,  $a_{ij}(A_f) = a_{ji}(A_f)$ ,

$$\begin{aligned} & \prod_{l=1}^{2\nu} a_{f(l)f(l+1)}(A_f) \\ &= \prod_{l=1}^{2\nu} a_{j_l i_{l+1}}^{h_l}(A_f), \end{aligned}$$

where  $j_l < j_{l+1}$ ,

$$(j_l, j_{l+1}) \in \{g_f(\mu), \mu=1, 2, \dots, 2\nu\}$$

$$h_l = d_{j_l j_{l+1}}^f \quad \text{with} \quad \sum_{l=1}^q h_l = 2\nu,$$

and the number of different  $j_l$ ,  $l=1, 2, \dots, q+1$  is  $p$ .

Thus

$$\begin{aligned} & E\left\{ \sum_{f \in D_{2\nu, n}(p, q)} \prod_{l=1}^{2\nu} a_{f(l)f(l+1)}(A_f) \right\} \\ &= \sum_{f \in D_{2\nu, n}(p, q)} E\left\{ \prod_{l=1}^q a_{j_l j_{l+1}}^{h_l}(A_f) \right\} \\ &= \sum_{f \in D_{2\nu, n}(p, q)} \prod_{l=1}^q \frac{N}{\binom{n}{2}} \sigma^{h_l} \\ &= Z_\nu(p, q, n) \left( \frac{N}{\binom{n}{2}} \right)^q \sigma^{2\nu}. \end{aligned}$$

It immediately follows that

$$\begin{aligned} & E\left\{ \sum_{f \in D_{2\nu, n}} \prod_{l=1}^{2\nu} a_{f(l)f(l+1)}(A_f) \right\} \\ &= \sigma^{2\nu} \sum_{p=2}^{\nu+1} n(n-1)\dots(n-p+1) \sum_{q=p-1}^{q_p^*} \left( \frac{N}{\binom{n}{2}} \right)^q Z_\nu(p, q) \\ & \quad \text{with} \quad q_p^* = \min \left\{ \nu, \binom{n}{2} \right\}. \end{aligned}$$

For a large  $n$ , it holds that

$$\begin{aligned} & E\left\{ \sum_{f \in D_{2\nu, n}} \prod_{l=1}^{2\nu} a_{f(l)f(l+1)}(A_f) \right\} \\ & \sim \sigma^{2\nu} n \left\{ Z_\nu(\nu+1, \nu) \left( \frac{2N}{n} \right)^\nu Z_\nu(\nu, \nu-1) \left( \frac{2N}{n} \right)^{\nu-1} + \dots + Z_\nu(2, 1) \left( \frac{2N}{n} \right) \right\}. \end{aligned}$$

We have the following theorem:

**THEOREM 1.** *Let us denote by  $W_n(x)$  the empirical distribution function of the eigenvalues of a normalized matrix  $B_n$  of an  $n \times n$  random symmetric matrix  $A_f$ :*

$$B_n = \frac{1}{2\sigma\sqrt{\frac{2N}{n}}} A_f.$$

If  $\frac{n}{N} = o(1)$ , then

$$\lim_{n \rightarrow \infty} E\{W_n(x)\} = W(x),$$

where  $W(x)$  is a continuous distribution function with the density  $w(x)$ :

$$w(x) = \begin{cases} \frac{2}{\pi} \sqrt{1-x^2} & : |x| \leq 1, \\ 0 & : |x| > 1. \end{cases}$$

PROOF. Write by  $m_{k,n}^*$  the  $k$ -th order moment of the eigenvalues of the matrix  $B_n$ . From the above discussion, if  $\frac{n}{N} = o(1)$ , then

$$\lim_{n \rightarrow \infty} m_{k,n}^* = m_k = \begin{cases} 0 & : k: \text{ odd}, \\ \frac{k!}{2^k \left(\frac{k}{2}\right)! \left(\frac{k}{2} + 1\right)!} & : k: \text{ even}. \end{cases}$$

On the other hand, the distribution function  $W(x)$  with the density;

$$w(x) = \begin{cases} \frac{2}{\pi} \sqrt{1-x^2} & : |x| \leq 1 \\ 0 & : |x| > 1 \end{cases}$$

is uniquely determined by its moment sequence  $\{m_k\}$ . Thus, according to Lemma 3, the distribution function of the eigenvalues of the matrix  $B_n$  has the semi-circle density for a large  $n$ . ◀

In order to show that the empirical distribution  $W_n(x)$  converges to  $W(x)$  in probability, it is necessary to do that the variances of the  $k$ -th order moment,  $k=1, 2, \dots$ :

$$E(m_{k,n}^* - E(m_{k,n}^*))^2$$

vanish as  $n$  tends to infinity. This can be verified by the above discussion and discussions similar to those described in [2]. Thus the above theorem can be rewritten as follows:

“Under the same conditions,

$$W_n(x) \xrightarrow{P} W(x)”$$

REMARK: We have discussed only the case that each of the non-zero elements takes its value  $+\sigma$  or  $-\sigma$  in equal probability  $1/2$ . However, it can be shown that the above theorem is also true in the case that each of the non-zero elements of  $A_T$  subjects to any one of independent symmetric distributions with zero mean and variance  $\sigma^2$  of various types including the normal distribution with zero mean and variance  $\sigma^2$ .

Now, when  $N \sim \frac{c}{2}n$ , the expectations of the  $k$ -th order moment of the eigenvalues of  $B_n (k=1, 2, \dots)$  are not identical with those defined from the semi-circle density and are obtained if  $Z_r(p, p-1)$  are evaluated. We have not succeeded to obtain them so far. However, as we mentioned previously, since the degree of sparsity of a random matrix with  $N \sim \frac{c}{2}n$  is large, so is the degree of the degeneration.

In the next section, we will discuss the degree of the degeneration of a random matrix with  $N \sim \frac{c}{2}n$  using the beautiful results on random graphs by Erdős and Rényi [3].

#### IV. Degeneration of a random matrix

To begin with, we shall list several properties of random graphs defined in II. They concern the structure of random graphs with  $n$  vertices and  $N$  edges. About the notions of graphs, see a text book on graph theory [6].

The following lemma holds on the connectivity of a random graph  $\Gamma_{n,N}$ .

LEMMA 7. *The probability  $P_{n,N}(C)$  that a random graph  $\Gamma_{n,N}$  has a property  $C$  that it is connected is given by*

$$\lim_{n \rightarrow \infty} P_{n,N}(C) = e^{-e^{-2y}}, \quad -\infty < y < \infty,$$

where we put

$$\frac{N(n) - \frac{1}{2}n \log n}{n} = y.$$

Since if, for instance,  $N \sim \frac{c}{2}n^2$  ( $0 < c < 1$ ), then  $y \rightarrow \infty$  as  $n \rightarrow \infty$ , every vertex of a random graph is connected. On the other hand, when  $N \sim \frac{c}{2}n \log n + o(n)$ , if  $c > 1$ , then a random graph is certainly connected and if  $c < 1$ , then all vertices are not connected and a new structure (eg. a tree or a cycle) appears. In what follows, we represent by  $P_{n,N}(\cdot)$  the probability of an event  $\cdot$  on a random graph.

LEMMA 8. *Let  $\tau_k$  be the number of isolated trees of order  $k$  contained in  $\Gamma_{n,N}$ . If*

$$\lim_{n \rightarrow \infty} \frac{N(n)}{n^{\frac{k-2}{k-1}}} = \rho > 0,$$

$$\lim_{n \rightarrow \infty} P_{n,N}(\tau_k = j) = \frac{\lambda^j e^{-j}}{j!}, \quad j = 0, 1, \dots; \quad \lambda = \frac{(2\rho)^{k-1} k^{k-2}}{k!}.$$

LEMMA 9. *Let  $\tau_k$  be the number of isolated trees of order  $k$  contained in  $\Gamma_{n,N}$  ( $k=1, 2, \dots$ ). If*

$$\lim_{n \rightarrow \infty} \frac{N(n)}{n^{\frac{k-2}{k-1}}} = \infty$$

$$\lim_{n \rightarrow \infty} \frac{N(n) - \frac{1}{2k}n \log n - \frac{k-1}{2k}n \log \log n}{n} = -\infty$$

then for  $-\infty < x < \infty$

$$\lim_{n \rightarrow \infty} P_{n,N}\left(\frac{\tau_k - M_{n,N}}{\sqrt{M_{n,N}}} < x\right) = \Phi(x),$$

$$M_{n,N} = n^{\frac{k-2}{k-1}} \left(\frac{2N}{n}\right)^{k-1} e^{-\frac{2kN}{n}},$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du.$$

LEMMA 10. *Let  $\tau_k$  be the number of isolated trees of order  $k$  contained in  $\Gamma_{n,N}$  ( $k=1, 2, \dots$ ). If*

$$N(n) \sim \frac{1}{2k} n \log n + \frac{k-1}{2k} n \log \log n + y n + o(n),$$

$$(-\infty < y < \infty)$$

$$\lim_{n \rightarrow \infty} P_{n,N}(\tau_k = j) = \frac{\lambda^j e^{-\lambda}}{j!}, \quad (j=0, 1, 2, \dots)$$

$$\lambda = \frac{e^{-2ky}}{k k!}.$$

We will discuss the structure of random graphs with  $N \sim \frac{c}{2} n$ .

LEMMA 11. If  $N(n) = o(n)$ , then  $\Gamma_{n,N}$  consists of only isolated trees as  $n \rightarrow \infty$ .

LEMMA 12. Let  $\tau_k$  be the number of isolated trees of order  $k$  contained in  $\Gamma_{n,N}$  and let the mean of  $\tau_k$  be  $M(\tau_k)$  ( $k=1, 2, \dots$ ). If  $N \sim \frac{c}{2} n$  ( $c > 0$ ), then

$$\lim_{n \rightarrow \infty} \frac{M(\tau_k)}{n} = \frac{1}{c} \frac{k^{k-2}}{k!} (ce^{-c})^k.$$

LEMMA 13. Let  $V_{n,N}$  denote the number of all vertices belonging to all isolated trees of  $\Gamma_{n,N}$  and let the mean of  $V_{n,N}$  be  $M(V_{n,N})$ . If  $N \sim \frac{c}{2} n$

$$\lim_{n \rightarrow \infty} \frac{M(V_{n,N})}{n} = \begin{cases} 1 & ; \quad 0 < c \leq 1, \\ \frac{x(c)}{c} & ; \quad 1 < c \end{cases},$$

where  $x(c)$  is the solution of  $xe^{-x} = ce^{-c}$  ( $0 < x < 1$ ) and can be represented by

$$x(c) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k.$$

It is immediate from the above lemma that if  $0 < c \leq 1$ , almost all of vertices belong to trees and if  $c > 1$ ,  $x(c)/c$  of all vertices belong to trees. The structure of subgraphs which the remaining vertices construct when  $c > 1$  are known from the following lemma.

LEMMA 14. Let the size of the largest subgraph of  $\Gamma_{n,N}$  be  $\rho_{n,N}$ . If  $N \sim \frac{c}{2} n$  ( $c > 1$ ), then for any  $\eta > 0$

$$\lim_{n \rightarrow \infty} P_{n,N} \left( \left| \frac{\rho_{n,N}}{n} - G(c) \right| < \eta \right) = 1,$$

where

$$G(c) = 1 - \frac{x(c)}{c}, \quad x(c) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k.$$

See [3] on the proofs of Lemmata 7-14.

REMARK: The above lemmata hold on balanced graphs. So far as the balanced graph concerns, any subgraph containing an isolated cycle does not exist in a finite rate.

Since the number of edges of a tree of order  $k$  is  $k-1$ , the mean of the number of edges contained in all the isolated trees, represented by  $M(E_{n,N})$ ,

$$\frac{M(E_{n,N})}{n} = \sum_{k=1}^{\infty} (k-1) \frac{M(\tau_k)}{n} = \frac{x^2(c)}{2c}.$$

Thus the number  $N_G$  of edges contained in the largest subgraph is given by

$$N_G \sim n \left( 1 - \frac{x^2(c)}{2c} \right).$$

The number  $n_g$  of vertices in the largest subgraph is given by

$$n_g \sim n \left( 1 - \frac{x(c)}{c} \right).$$

It follows that

$$N_G \sim c' n_g \quad (c' > 0).$$

Now let us denote by  $E_k$  the matrix associated with a tree of order  $k$ . When  $N \sim \frac{c}{2}n$ ,  $0 < c \leq 1$ , letting

$$\mathcal{E}_T = \begin{bmatrix} \overbrace{E_1 \cdots E_1}^{\tau_1} & & 0 \\ & \overbrace{E_2 \cdots E_2}^{\tau_2} & \\ & & \ddots & \\ 0 & & & \overbrace{E_k \cdots E_k}^{\tau_k} \end{bmatrix},$$

we have

$$\lim_{n \rightarrow \infty} E_T \sim \mathcal{E}_T \quad \text{with P.1,}$$

where  $A \sim B$  implies the matrix  $A$  is similar to a matrix  $B$ . When  $N \sim \frac{c}{2}n$ ,  $c > 1$ , letting

$$\mathcal{E}_T = \begin{bmatrix} G & & 0 \\ & \overbrace{E_1 \cdots E_1}^{\tau_1} & \\ & & \ddots & \\ 0 & & & \overbrace{E_k \cdots E_k}^{\tau_k} \end{bmatrix},$$



where  $G$  is a matrix associated with the largest subgraph, we have

$$\lim_{n \rightarrow \infty} E_{\mathcal{F}} \sim \mathcal{E}_{\mathcal{F}} \quad \text{with P.1.}$$

Now we will estimate the degree of degeneration of a random matrix  $A_{\mathcal{F}}$  associated with a random graph  $\Gamma_{n,N}$  with  $N \sim \frac{c}{2}n$ . Firstly we give the following lemma on the determinant of a matrix  $E_k$  associated with a tree of order  $k$ .

LEMMA 15. *Let us denote by  $E_k = (\varepsilon_{ij}^k)$  a matrix associated with a tree of order  $k$ ; i.e., for*

$$\varepsilon_{ij}^k = \begin{cases} 1: & \text{if there is an edge between the vertex } i \text{ and the vertex } j, \\ 0: & \text{otherwise.} \end{cases}$$

Then

$$|E_k| = 0 \quad \text{for odd } k.$$

A proof of this lemma may be found in A.M. Mowshowitz [7] and the property subjects to a structure of a tree.

According to this lemma, a matrix associated with a tree of order  $k$  odd always degenerates and it follows that a weighted matrix  $A_k$  associated with a tree of order  $k$  odd degenerate also. Thus we have:

THEOREM 2. *Let us denote by  $d_g$  the degree of degeneration of a random weighted matrix  $A_{\mathcal{F}}$  associated with a random graph. If  $N \sim \frac{c}{2}n$  ( $c > 0$ ),*

$$\lim_{n \rightarrow \infty} \frac{d_g}{n} > \frac{1}{c} \sum_{k: \text{odd}} \frac{k^{k-2}}{k!} (ce^{-c})^k \quad (\text{P.1}).$$

REMARK: The above estimation of the degeneration is too under since the degree of the degeneration of  $E_k$  for odd  $k$  may be more than one and  $E_k$  for even  $k$  may degenerate also, depending on the structure of a tree constructed as well as on its order. As one might know, the degeneration could be evaluated by means of knowledge either on  $Z_{\nu}(p, p-1)$ ,  $p=2, 3, \dots, \nu+1$  or on the *fine* structure of random graphs. It is complicate to obtain either of them and we have succeeded in obtaining none so far.

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### Referencer

- [ 1 ] MEHTA, M.L.: Random matrices, New York, Academic Press (1967).
- [ 2 ] OLSON, W.H. and UPPULURI, V.R.R.: *Asymptotic distribution of eigenvalues of random matrices*, Sixth Berkeley Symposium, 3 (1970), 615-644.

- [ 3 ] ERDÖS, P. and RÉNYI, A.: *On the evolution of random graphs*, Publ. Math. Inst. Hung. Acad. Sci. 5 (1960), 17-61.
- [ 4 ] OSTROWSKI, A.M.: *Solution of Equations in Euclidean and Banach Spaces*, New York, Academic Press (1973 3rd ed.).
- [ 5 ] LOEVE, M.: *Probability Theory*, Princeton, Van Nostrand, (1963 3rd ed.).
- [ 6 ] BERGE, C.: *Graphes et hypergraphes*, Paris, Dunod (1970).
- [ 7 ] MOWSHOWITZ, A: *The characteristic polynomial of a graph*, Journal of Combinatorial Theory 12(B) (1972), 177-193.