# ON SLIPPAGE RANK TESTS-(II) : ASYMPTOTIC RELATIVE EFFICIENCIES

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# ON SLIPPAGE RANK TESTS-(II) ASYMPTOTIC RELATIVE EFFICIENCIES

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## §.0 Summary

Locally optimum rank tests for $k$-sample slippage problems of location and scale parameters in the nonparametric situations were derived by Kakiuchi and Kimura [5]. In this paper we shall consider a general class of slippage rank tests for those problems and investigate their asymptotic powers. The asymptotic relative efficiencies of the slippage rank tests with respect to classical optimum slippage tests are also investigated and it is shown that the efficiencies are the same as those in two sample problems.

## §1. Introduction

Let $F(x)$ be a continuous but unknown cummulative distribution function (cdf) on the real line, and let $X_{ij}$ $(i=1, 2, \cdots, k;\ j=1, 2, \cdots, n)$ be mutually independent random variables with $P(X_{ij}\leq x)=F((x-\theta_i)/\sigma_i)$ where $\theta_i$ and $\sigma_i$ are location and scale parameters. Let $\Delta$ be a positive real parameter which expresses the slip. Then the $k$-sample $(k\geq 2)$ slippage problems considered in this paper are given by [A] and [B] below.

[A] (location) Under the assumption of all $\sigma_i$ $(i=1, 2, \cdots, k)$ being equal to an unknown $\sigma$ $(>0)$, testing the hypothesis

(1.1) $$H_0:\ \theta_1=\theta_2=\cdots=\theta_k=\theta$$

against $k$ alternatives

(1.2) $$H_i(\Delta):\ \theta_1=\cdots=\theta_{i-1}=\theta_i-\Delta=\theta_{i+1}=\cdots=\theta_k=\theta\quad (i=1, 2, \cdots, k)$$

where $\theta$ is an unknown constant.

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[B] (scale) Under the assumption of all $\theta_i$ $(i=1, 2, \cdots, k)$ being equal to an unknown $\theta$, testing the hypothesis

$$H_0: \sigma_1=\sigma_2=\cdots=\sigma_k=\sigma \tag{1.3}$$

against $k$ alterntive

$$H_i(\Delta): \sigma_1=\cdots=\sigma_{i-1}=\sigma_i(1+\Delta)^{-1}=\sigma_{i+1}=\cdots=\sigma_k=\sigma \quad (i=1, 2, \cdots, k) \tag{1.4}$$

where $\sigma>0$ is an unknown constant.

The purpose of this paper is to consider a general class of slippage rank tests for problems [A] and [B], and to investigate their efficiencies with respect to the normal theory tests (classical slippage tests) studied by Paulson [7], Truax [10], Hall and Kudô [2], Hall, Kudô and Yeh [3], Kimura and Kudô [6], and others.

In section 2, slippage rank tests as well as classical slippage tests for problems [A] and [B] are introduced. In section 3, asymptotic powers of these tests for the problem [A] are investigated, and then their asymptotic efficiencies are considered in section 4. It is shown that the efficiencies are the same as those given by rank tests in the two sample problem. In section 5, the results corresponding to the previous sections are derived for the problem [B].

## § 2. Slippage tests

Let $F(x)$ be a continuous but unknown cummulative distribution function (cdf) on the real line, and let $X_{ij}$ $(i=1,2, \cdots, k;\ j=1, 2, \cdots, n)$ be mutually independent random variables with

$$P(X_{ij}\leq x)=F((x-\theta_i)/\sigma_i) \tag{2.1}$$

where $\theta_i$ and $\sigma_i$ are location and scale parameters. Let $d(x)=(d_0(x), d_1(x), \cdots, d_k(x))$ be a slippage test where $x$ is a point of $N(=nk)$ dimensional Euclidean space $R^N$ and $d_i(x)$ means the probability of accepting $H_i(\Delta)\,(H_0(\Delta)\equiv H_0)$. Then the slippage tests considered throughout this paper are given in the following form;

$$d_0(x)=\begin{cases} 1 & \text{if } \max_{1\leq j\leq k} S_j(x)<\lambda_{N,\alpha} \\ \xi_\alpha & \text{if } \quad = \\ 0 & \text{if } \quad > \end{cases} \tag{2.2}$$

$$d_i(x)=\begin{cases} 1/m(x) & \text{if } S_i(x)=\max_{1\leq j\leq k} S_j(x)>\lambda_{N,\alpha} \\ (1-\xi_\alpha)/m(x) & \text{if } \quad = \\ 0 & \text{if otherwise} \end{cases}, \quad (i=1, 2, \cdots, k)$$

where $m(x)$ is the number of times max $S_j(x)$ is attained, and $\lambda_{N,\alpha}$ and $\xi_\alpha$ are constants given by the size condition $E[d_0(X)\,|\,H_0]=1-\alpha$ $(0<\alpha<1)$.

We shall now give rank tests and classical tests for slippage problems [A] and [B] respectively.

**(i) Rank tests** Let $Z_{N,r}^{(i)}=1$, if the $r$-th smallest of $N=nk$ observations is from $i$-th samples, and otherwise let $Z_{N,r}^{(i)}=0$. Then we consider an important class of rank tests for both problems [A] and [B] given by taking $S_i(x)$ in (2.2) as

$$S_i(x)=T_{N,i}=\frac{1}{n}\sum_{r=1}^{N} E_{N,r}Z_{N,r}^{(i)},\ i=1,2,\cdots,k \tag{2.3}$$

where $E_{N,r}=J_N(r/(N+1))$, $r=1,2,\cdots,N$ are functions of the ranks $r(=1,2,\cdots,N)$ and are explicitly known.

Especially two such functions of particular interest are given as follows:

$$J_{1N}(r/(N+1))=E[-f'(X_N^{(r)})/f(X_N^{(r)})],\ r=1,2,\cdots,N \tag{2.4}$$

$$J_{2N}(r/(N+1))=E[-1-X_N^{(r)}(f'(X_N^{(r)})/f(X_N^{(r)}))],\ r=1,2,\cdots,N \tag{2.5}$$

where $X_N^{(1)}<X_N^{(2)}<\cdots<X_N^{(N)}$ are the order statistics of a sample of size $N$ from $F(x)$ with the density $f(x)$ and $E$ denotes the expectation. Kakiuch and Kimura [5] established that under general regularity conditions the rank tests corresponding to the functions (2.4) and (2.5) are the extended locally most powerful symmetric size $\alpha$ rank tests for problems [A] and [B] respectively (See Corollary 2B of [5]).

**(ii) Classical tests.** In the parametric theory, commonly used tests are respectively based on, for the problem [A],

$$S_i(x)=W_{N,i}=\frac{(\bar{x}_i-\bar{\bar{x}})}{\sqrt{\frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{\bar{x}})^2}},\ i=1,2,\cdots,k, \tag{2.6}$$

and for the problem [B],

$$S_i(x)=W'_{N,i}=\frac{\frac{1}{n}\sum_{j=1}^{n}(x_{ij}-\bar{\bar{x}})^2}{\frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{\bar{x}})^2},\ i=1,2,\cdots,k \tag{2.7}$$

where $\bar{x}_i=\frac{1}{n}\sum_{j=1}^{n}x_{ij}$ and $\bar{\bar{x}}=\frac{1}{k}\sum_{i=1}^{k}\bar{x}_i$. We shall call slippage tests (2.2) with statistics $S_i(x)$ given by (2.6) and (2.7) as the M test and the V test respectively.

Hall, Kudô and Yeh [3] proved that the M test is the uniformly most powerful symmetric similar size $\alpha$ test for the problem [A] when $F$ is a normal distribution. On the other hand, the optimum properties of the V test have not been established. However, it is conjectured that the V test will be the most powerful for the problem [B] when $F$ is a normal distribution.

Now a slippage test has three kinds of power, namely $P(D_{N,0}|H_i(\Delta))$, $P(D_{N,i}|H_i(\Delta))$ and $P(D_{N,j}|H_i(\Delta))(i\neq j)$, where $D_{N,i}$ means the decision based on samples of size $N$ to accept $H_i(\Delta)$.

Then we have the following lemma.

LEMMA 1. *For both problems* [A] *and* [B], *the rank tests, the* M *test and the* V *test satisfy*

$$P(D_{N,i}|H_i(\Delta))=P(D_{N,j}|H_j(\Delta)), \tag{2.8}$$

$$P(D_{N,i}|H_j(\Delta))=P(D_{N,l}|H_m(\Delta)) \tag{2.9}$$

*and*

$$P(D_{N,0}|H_i(\Delta))=P(D_{N,0}|H_j(\Delta)) \tag{2.10}$$

*for each n, and* $i\neq j$, $l\neq m$; $1, 2, \cdots, k$.

PROOF. We consider the same transformation group $G_1$ on $R^N$ as in the proof of Theorem 1A of [5], Then it follows that problems [A] and [B] are invariant under $G_1$. Further the rank tests, the M test and V test are invariant under $G_1$, and $G_1$ induces the symmetric group on $\{1, 2, \cdots, k\}$. Hence all the assumptions of Lemma 3 of [6] are satisfied and this completes the proof.

It should be remarked that for each n and j,

$$\sum_{i=0}^{k} P(D_{N,i}|H_j(\Delta))=1\,. \tag{2.11}$$

Then in order to get the powers of the slippage tests, it is enough for us to consider only two kinds among them.

## § 3. Asymptotic powers for the location problem.

Let us study asymptotic powers of the rank tests and the M test introduced in the previous section under the following sequences of alternatives $H_i(\Delta^{(n)})$, $i=1, 2, \cdots, k$:

$$H_i(\Delta^{(n)}):\ \theta_1=\cdots=\theta_{i-1}=\theta_i-\Delta^{(n)}=\theta_{i+1}=\cdots=\theta_k=\theta \tag{3.1}$$

where for some positive constant $\delta$,

$$\Delta^{(n)}=n^{-1/2}\delta\,. \tag{3.2}$$

Since the tests under consideration satisfy (2.8), (2.9) and (2.10), it is sufficient for our aim to study the limit of the powers $P(D_{N,0}|H_k(\Delta^{(n)}))$, $P(D_{N,k}|H_k(\Delta^{(n)}))$ and $P(D_{N,1}|H_k(\Delta^{(n)}))$ as $n\to\infty$.

**3.1. The case of the rank test.** Preliminarily, let us consider the size condition. We require the following lemma, the proof of which is immediately obtained from Theorem 7.1 of Puri [8].

LEMMA 2. *Suppose for each n the hypothesis $H_k(\Delta^{(n)})$ is valid and the assumptions of Theorem 7.1 of [8] are satisfied. Let $(U_1, U_2, \cdots, U_{k-1})$ be a $(k-1)$ dimensional random vector normally distributed with*

$$E(U_j)=0,\ \mathrm{Var}\ (U_j)=1 \text{ and } \mathrm{Cov}\ (U_j, U_{j'})=\frac{1}{2}\,,\ j\neq j';\ 1, 2, \cdots, k-1\,. \tag{3.3}$$

*Then the joint limiting distribution of the random vector $(V_{N,1}, V_{N,2}, \cdots, V_{N,k-1})$ defined by*

$$V_{N,j}=\left(\frac{n}{2A^2}\right)^{1/2}\{T_{N,j}-T_{N,k}-(\mu_{N,j}-\mu_{N,k})\} \tag{3.4}$$

*where*

(3.5) $$\mu_{N,i}=\int J[H(x)]\,dF^{(i)}(x),\ \ F^{(i)}(x)=F((x-\theta_i)/\sigma),$$

$$H(x)=\frac{1}{k}\sum_{i=1}^{k} F^{(i)}(x)\ \ (i=1,2,\cdots,k),$$

*and*

(3.6) $$A^2=\int_0^1 J^2(u)\,du-\left(\int_0^1 J(u)\,du\right)^2$$

*is the same as that of the random vector* $(U_1, U_2, \cdots, U_{k-1})$.

Note that Lemma 2 is also valid under $H_0$, where all $\mu_{N,i}$ $(i=1,2,\cdots,k)$ are equal.

Let $\lambda_\alpha$ be a constant given by

(3.7) $$P\left(U_j<\frac{1}{k}\sum_{l=1}^{k-1} U_l+\lambda_\alpha;\ j=1,2,\cdots,k-1,\ \ 0<\frac{1}{k}\sum_{l=1}^{k-1} U_l+\lambda_\alpha\right)=1-\alpha$$

Put

(3.8) $$C_N=\frac{1}{k}\sum_{i=1}^{k} T_{N,i}.$$

Since $\frac{1}{k}\sum_{i=1}^{k} T_{N,i}=\frac{1}{N}\sum_{i=1}^{N} E_{N,r}$, $C_N$ is a constant depending only on $N$. At this point the inequality

(3.9) $$\max_{1\le i\le k} T_{N,i}<\lambda_{N,\alpha}$$

is equivalent to

(3.10) $$T_{N,j}-T_{N,k}<\frac{1}{k}\sum_{l=1}^{k-1}(T_{N,l}-T_{N,k})+\lambda_{N,\alpha}-C_N;\ j=1,2,\cdots,k-1,$$

$$0<\frac{1}{k}\sum_{l=1}^{k-1}(T_{N,l}-T_{N,k})+\lambda_{N,\alpha}-C_N.$$

Thus, the size condition in the limit

(3.11) $$\lim_{n\to\infty} P(D_{N,0}\,|\,H_0)=1-\alpha$$

is satisfied by any sequences of the critical values $\lambda_{N,\alpha}$ such that

(3.12) $$\left(\frac{n}{2A^2}\right)^{1/2}(\lambda_{N,\alpha}-C_N)\to\lambda_\alpha \text{ as } n\to\infty$$

where $\lambda_\alpha$ is given by (3.7)

First, let us consider the limit of $P(D_{N,0}\,|\,H_k(\Delta^{(n)}))$ as $n\to\infty$. From (3.4), it follows that the inequality (3.9) implies

(3.13) $$V_{N,j}<\frac{1}{k}\sum_{l=1}^{k-1} V_{N,l}+\frac{1}{k}\sum_{i=1}^{k}\left(\frac{n}{2A^2}\right)^{1/2}(\mu_{N,i}-\mu_{N,j})$$

$$+\left(\frac{n}{2A^2}\right)^{1/2}(\lambda_{N,\alpha}-C_N);\ j=1,2,\cdots,k-1,$$

$$0<\frac{1}{k}\sum_{l=1}^{k-1} V_{N,l}+\frac{1}{k}\sum_{l=1}^{k-1}\left(\frac{n}{2A^2}\right)^{1/2}(\mu_{N,l}-\mu_{N,k})+\left(\frac{n}{2A^2}\right)^{1/2}(\lambda_{N,\alpha}-C_N)$$

Now suppose the assumptions of Lemma 7.2 of [8] are satisfied. Then it is straightforward that under $H_k(\Delta^{(n)})$,

$$\lim_{n\to\infty} n^{1/2}(\mu_{N,i}-\mu_{N,k}) = -\frac{\delta}{\sigma} B\,,\ i=1,2,\cdots,k-1 \tag{3.14}$$

where

$$B=\int_{-\infty}^{+\infty} \frac{d}{dx} \{J[F(x)]\}\, dF(x)\,. \tag{3.15}$$

Hence, by (3.13), (3.14), (3.15) and Lemma 2, we obtain

$$\begin{aligned} &\lim_{n\to\infty} P(D_{N,0}|\, H_k(\Delta^{(n)}) \\ &=\lim_{n\to\infty} P(\max_{1\le i\le k} T_{N,i}<\lambda_{N,\alpha}|\, H_k(\Delta^{(n)})) \\ &=\lim_{n\to\infty} P(V_{N,j}<\frac{1}{k}\sum_{l=1}^{k-1} V_{N,l}+\frac{1}{k}\sum_{i=1}^{k}\Big(\frac{n}{2A^2}\Big)^{1/2}(\mu_{N,i}-\mu_{N,j})+\Big(\frac{n}{2A^2}\Big)^{1/2}(\lambda_{N,\alpha}-C_N)\,; \\ &\qquad j=1,2,\cdots,k-1, \\ &\qquad 0<\frac{1}{k}\sum_{l=1}^{k-1} V_{N,l}+\frac{1}{k}\sum_{l=1}^{k-1}\Big(\frac{n}{2A^2}\Big)^{1/2}(\mu_{N,l}-\mu_{N,k})+\Big(\frac{n}{2A^2}\Big)^{1/2}(\lambda_{N,\alpha}-C_N)) \\ &=\lim_{n\to\infty} P(U_j<\frac{1}{k}\sum_{l=1}^{k-1} U_l+\frac{1}{k}\Big(\frac{1}{2A^2}\Big)^{1/2}\frac{\delta}{\sigma}B+\lambda_\alpha\,;\ j=1,2,\cdots,k-1, \\ &\qquad 0<\frac{1}{k}\sum_{l=1}^{k-1} U_l-\frac{(k-1)}{k}\Big(\frac{1}{2A^2}\Big)^{1/2}\frac{\delta}{\sigma}B+\lambda_\alpha) \end{aligned} \tag{3.16}$$

Next, let us consider the limit of $P(D_{N,k}|\, H_k(\Delta^{(n)}))$ as $n\to\infty$. It is clear that

$$T_{N,k}=\max_{1\le i\le k} T_{N,i}>\lambda_{N,\alpha} \tag{3.17}$$

implies

$$T_{N,k}-T_{N,j}\geqq 0\,;\ j=1,2,\cdots,k-1,\ T_{N,k}>\lambda_{N,\alpha}\,, \tag{3.18}$$

and further the inequality (3.18) implies

$$\begin{aligned} &V_{N,j}\leqq -\Big(\frac{n}{2A^2}\Big)^{1/2}(\mu_{N,j}-\mu_{N,k})\,;\ j=1,2,\cdots,k-1\,, \\ &\frac{1}{k}\sum_{l=1}^{k-1} V_{N,l}<-\frac{1}{k}\sum_{l=1}^{k-1}\Big(\frac{n}{2A^2}\Big)^{1/2}(\mu_{N,l}-\mu_{N,k})-\Big(\frac{n}{2A^2}\Big)^{1/2}(\lambda_{N,\alpha}-C_N)\,. \end{aligned} \tag{3.19}$$

Hence, by (3.12), (3.14), (3.19) and Lemma 2, we obtain similary as before that

$$\begin{aligned} &\lim_{n\to\infty} P(D_{N,k}|\, H_k(\Delta^{(n)})) \\ &=P(U_j<\Big(\frac{1}{2A^2}\Big)^{1/2}\frac{\delta}{\sigma}B;\ j=1,2,\cdots,k-1,\ \frac{1}{k}\sum_{l=1}^{k-1} U_l<\frac{(k-1)}{k}\Big(\frac{1}{2A^2}\Big)^{1/2}\frac{\delta}{\sigma}B-\lambda_\alpha) \end{aligned} \tag{3.20}$$

Finally, since for each $n$

$$\sum_{i=0}^{k} P(D_{N,i}|\, H_k(\Delta^{(n)}))=1\,, \tag{3.21}$$

and since from (2.9), we have that

(3.22) $$\lim_{n\to\infty} P(D_{N,1}|H_k(\Delta^{(n)}))=\frac{1}{k-1}\{1-\lim_{n\to\infty} P(D_{N,0}|H_k(\Delta^{(n)}))-\lim_{n\to\infty} P(D_{N,k}|H_k(\Delta^{(n)}))\}.$$

Hence, substituting (3.16) and (3.20) in (3.22), we can obtain the limit of power $P(D_{N,1}|H_k(\Delta^{(n)}))$ as $n\to\infty$.

**3.2. The case of the M test.** The following lemma is an analogue of Lemma 2.

LEMMA 3. *Suppose for each n the hypothesis $H_k(\Delta^{(n)})$ is valid and $F(x)$ possesses a finite second order moment. Then the joint limiting distribution of the random vector $(Z_{N,1}, Z_{N,2}, \cdots, Z_{N,k-1})$ defined by*

(3.23) $$Z_{N,j}=\left(\frac{1}{n}\right)^{1/2}\{W_{N,j}-W_{N,k}-(\eta_{N,j}-\eta_{N,k})\}$$

*where $\eta_{N,i}$ is the expectation of $W_{N,i}$ is the same as that of the random vector $(U_1, U_2, \cdots, U_{k-1})$ given by (3.3).*

PROOF. Noting that the denominator of $W_{N,i}$ converges to the variance of $F((x-\theta)/\sigma)$ in probability as $n\to\infty$, the proof of this lemma follows immediately.

Note that Lemma 3 is also valid under $H_0$, where **all** $\eta_{N,i}$ $(i=1, 2, \cdots, k)$ are equal.

First, we consider the size condition. Since

(3.24) $$\sum_{i=1}^{k} W_{N,i}=0,$$

(3.25) $$\max_{1\le i\le k} W_{N,i}<\lambda_{N,\alpha}$$

implies

(3.26) $$W_{N,j}-W_{N,k}<\frac{1}{k}\sum_{l=1}^{k-1}(W_{N,l}-W_{N,k})+\lambda_{N,\alpha};\ j=1, 2, \cdots, k-1,$$
$$0<\frac{1}{k}\sum_{l=1}^{k-1}(W_{N,l}-W_{N,k})+\lambda_{N,\alpha}.$$

Thus, the size condition in the limit given by (3.11) is satisfied by any sequences of the critical values $\lambda_{N,\alpha}$ such that

(3.27) $$\left(\frac{n}{2}\right)^{1/2}\lambda_{N,\alpha}\to\lambda_\alpha$$

where $\lambda_\alpha$ is given by (3.7).

Now, from (3.23) it follows that the inequality (3.25) implies

(3.28) $$Z_{N,j}<\frac{1}{k}\sum_{l=1}^{k-1} Z_{N,l}+\frac{1}{k}\sum_{i=1}^{k}\left(\frac{n}{2}\right)^{1/2}(\eta_{N,i}-\eta_{N,j})+\left(\frac{n}{2}\right)^{1/2}\lambda_{N,\alpha};\ j=1, 2, \cdots, k-1,$$
$$0<\frac{1}{k}\sum_{l=1}^{k-1} Z_{N,l}+\frac{1}{k}\sum_{l=1}^{k-1}\left(\frac{n}{2}\right)^{1/2}(\eta_{N,l}-\eta_{N,k})+\left(\frac{n}{2}\right)^{1/2}\lambda_{N,\alpha}.$$

Further under $H_k(\Delta^{(n)})$,

(3.29) $$\lim_{n\to\infty} n^{1/2}(\eta_{N,i}-\eta_{N,k}) = -\frac{\delta}{\sigma\sigma_F},\ i=1,2,\cdots,k-1$$

where

(3.30) $$\sigma_F^2=\int_{-\infty}^{+\infty} x^2 dF(x)-\left(\int_{-\infty}^{+\infty} x dF(x)\right)^2.$$

Hence, by (3.27), (3.28), (3.29) and Lemma 3, we obtain that

(3.31) $$\lim_{n\to\infty} P(D_{N,0}|H_k(\Delta^{(n)}))$$
$$=P\Big(U_j<\frac{1}{k}\sum_{l=1}^{k-1} U_l+\frac{1}{k}\Big(\frac{1}{2}\Big)^{1/2}\frac{\delta}{\sigma\sigma_F}+\lambda_\alpha;\ j=1,2,\cdots,k-1,$$
$$0<\frac{1}{k}\sum_{l=1}^{k-1} U_l-\frac{(k-1)}{k}\Big(\frac{1}{2}\Big)^{1/2}\frac{\delta}{\sigma\sigma_F}+\lambda_\alpha\Big).$$

Next, similary as before

(3.32) $$W_{N,k}=\max_{1\leq i\leq k} W_{N,i}>\lambda_{N,\alpha}$$

implies

(3.33) $$Z_{N,j}\leqq -\Big(\frac{n}{2}\Big)^{1/2}(\eta_{N,j}-\eta_{N,k})\ ;\ j=1,2,\cdots,k-1,$$
$$\frac{1}{k}\sum_{l=1}^{k-1} Z_{N,l}<-\frac{1}{k}\sum_{l=1}^{k-1}\Big(\frac{n}{2}\Big)^{1/2}(\eta_{N,l}-\eta_{N,k})-\Big(\frac{n}{2}\Big)^{1/2}\lambda_{N,\alpha}.$$

Hence, by (3.27), (3.29), (3.33) and Lemma 3,

(3.34) $$\lim_{n\to\infty} P(D_{N,k}|H_k(\Delta^{(n)}))$$
$$=P\Big(U_i\leqq\Big(\frac{1}{2}\Big)^{1/2}\frac{\delta}{\sigma\sigma_F};\ j=1,2,\cdots,k-1,\frac{1}{k}\sum_{l=1}^{k-1} U_l<\frac{(k-1)}{k}\Big(\frac{1}{2}\Big)^{1/2}\frac{\delta}{\sigma\sigma_F}-\lambda_\alpha\Big).$$

Finally substituting (3.31) and (3.34) in (3.22), we can obtain the limit of power $P(D_{N,1}|H_k(\Delta^{(n)}))$ as $n\to\infty$.

## §4. Asymptotic relative efficiencies for the location problem.

We are now in a position to make large sample comparison between the rank tests and the M test.

Let $D_{N,i}^R$ and $D_{N,i}^M$ denote the decisions under a rank test and the M test, respectively, based on $N(=nk)$ observations. We assume that the two tests are at the same size $1-\alpha$. Consider the sequences of alternatives $H_i(\Delta^{(n)})$, $i=1,2,\cdots,k$ given by (3.1) and the sequence $N'=h(N)$ such that

(4.1) $$\lim_{N\to\infty}\sum_{i=1}^{k} P(D_{N',i}^R|H_i(\Delta^{(n)}))=\lim_{N\to\infty}\sum_{i=1}^{k} P(D_{N,i}^M|H_i(\Delta^{(n)})).$$

Then the asymptotic relative efficiency of the rank test with respect to the M test for testing the hypothesis $H_0$ given by (1.1) against alternatives $H_i(\Delta^{(n)})$, $i=1,2,\cdots,k$ is defined as

(4.2) $$e_{R,M}=\lim_{N\to\infty} N/N'=\lim_{n\to\infty} n/n'$$

where $n'$ is given by $N'=n'k$.

This will be the direct generalization of the idea of Pitman to the slippage problems, since $\sum_{i=1}^{k} P(D_{N,i}|H_i(\Delta))$ is the total sum of the probabilities of making the correct decisions under alternatives for our problems.

Now from Lemma 1, (4.1) is replaced by

(4.3) $$\lim_{N\to\infty} P(D^{R}_{N',k}|H_k(\Delta^{(n)}))=\lim_{N\to\infty} P(D^{M}_{N,k}|H_k(\Delta^{(n)}))\,.$$

We readily get the following theorem.

THEOREM 1. *Suppose the assumptions of Theorem 7.1 and Lemma 7.2 of Puri* [8] *are satisfied and $F(x)$ possesses a finit second order moment. Then the asymptotic relative efficiencies of the rank tests with respect to the* M *test are*

(4.4) $$e_{R,M}=\frac{\sigma_F^2\Big(\int_{-\infty}^{+\infty}\frac{d}{dx}\{J[F(x)]\}\,dF(x)\Big)^2}{\int_0^1 J^2(u)\,du-\Big(\int_0^1 J(u)\,du\Big)^2}$$

*where $\sigma_F^2$ is given by (3.30).*

PROOF. The rank tests and the M test have the asymptotic powers (3.20) and (3.34) respectively. Hence, in order to have

$$\lim_{N\to\infty} P(D^{R}_{N',k}|H_k(\Delta'^{(n')}))=\lim_{N\to\infty} P(D^{M}_{N,k}|H_k(\Delta^{(n)}))$$

where $\Delta'^{(n')}=(n')^{-1/2}\delta'$, we must take

$$\Big(\frac{1}{2A^2}\Big)^{1/2}\frac{B\delta'}{\sigma}=\Big(\frac{1}{2}\Big)^{1/2}\frac{\delta}{\sigma\sigma_F}\,.$$

Further to have the same sequences of alternative hypothesis for each tests, we must take $(n')^{-1/2}\delta'=n^{-1/2}\delta$. The substitution of $\delta/\delta'=(n/n')^{1/2}$ in (4.2) with requirement $\Big(\frac{B^2}{A^2}\Big)^{1/2}\delta'=\Big(\frac{1}{\sigma_F}\Big)\delta$ yields the formula (4.4), which proves the theorem.

Now we should remark a stronger result. If the limiting ratio (4.2) is taken to be equal to (4.4), then comparing (3.16) with (3.31), we get

(4.5) $$\lim_{N\to\infty} P(D^{R}_{N',0}|H_k(\Delta^{(n)}))=\lim_{N\to\infty} P(D^{M}_{N,0}|H_k(\Delta^{(n)}))$$

and from (3.22)

(4.6) $$\lim_{N\to\infty} P(D^{R}_{N',1}|H_k(\Delta^{(n)}))=\lim_{N\to\infty} P(D^{M}_{N,1}|H_k(\Delta^{(n)}))\,.$$

Thus our efficiency $e_{R,M}$ is quite sufficient for our purpose.

Note that $e_{R,M}$ given by (4.4) is the same as the one given by Chernoff and Savage [1] for the rank tests corresponding to the two sample lacation problem and the one given by Puri [8] to the $c$-sample problem. For examples, we consider two special cases:

(i) If $F$ is a logistic distribution, the rank test corresponding to (2.4) reduces to the rank-sum test, which takes $T_{N,i}=\sum_{j=1}^{n} R_{ij}$ where $R_{ij}$ is the rank of $X_{ij}$ in the pooled samples. The efficiency (4.4), then is known to satisfy $e_{R,M}\geq .864$ for all $F$; $e_{R,M}=3/\pi\sim .955$ when $F$ is normal; and $e_{R,M}>1$ for many non-normal $F$.

(ii) If $F$ is a normal distribution, the efficiency (4.4) of the normal scores test corresponding to (2.4), is known to satisfy $e_{R,M}\geq 1$ for all $F$, and $e_{R,M}=1$ if and only if $F$ is normal (See [4]).

Thus from the efficiency point of view the above rank tests appear to be advantagous compared with the M test, unless one can be reasonably sure of the absence of gross errors and the other departures of the normality.

## § 5. Asymptotic relative efficiencies for the scale problem.

In this section, we shall study the asymptotic relative efficiencies of the rank tests with respect to the V test for the scale paoblem [B] under the following sequences of alternatives $H_i(\Delta^{(n)})$, $i=1, 2, \cdots, k$;

$$H_i(\Delta^{(n)}): \sigma_1=\cdots=\sigma_{i-1}=\sigma_i(1+\Delta^{(n)})^{-1}=\sigma_{i+1}=\cdots=\sigma_k=\sigma \tag{5.1}$$

where $\Delta^{(n)}$ is given by (3.2). The argument runs parallel to those in the previous sections, and is therefore indicated briefly.

We preliminarily consider the asymptotic powers of the two tests.

**5.1. Asymptotic powers of the rank test.** Under the same condition as in Lemma 2, it is obtained that the joint limiting distribution of the $(k-1)$ dimensional random vector

$$\left(\frac{n}{2A^2}\right)^{1/2}\{T_{N,j}-T_{N,k}-(\mu'_{N,j}-\mu'_{N,k})\,;\ j=1, 2, \cdots, k-1\} \tag{5.2}$$

is the same as that of the random vector $(U_1, U_2, \cdots, U_{k-1})$ given by (3.3), where

$$\mu'_{N,i}=\int_{-\infty}^{+\infty} J[H(x)]dF^{(i)}(x),\ F^{(i)}(x)=F((x-\theta)/\sigma_i), \tag{5.3}$$

$$\text{and } H(x)=\sum_{i=1}^{k} F^{(i)}(x),\ i=1, 2, \cdots, k,$$

and where $A^2$ is given by (3.6). If the assumptions of Lemma 7.2 of [8] are satisfied, we obtain that under $H_k(\Delta^{(n)})$,

$$\lim_{n\to\infty} n^{1/2}(\mu'_{N,j}-\mu'_{N,k})=-\delta B',\ j=1, 2, \cdots, k-1. \tag{5.4}$$

where

$$B'=\int_{-\infty}^{+\infty} x\frac{d}{dx}\{J[F(x)]\}\,dF(x). \tag{5.5}$$

Then, replacing $\mu_{N,i}$ by $\mu'_{N,i}$ in (3.13) and (3.19), it can be easily shown that

$$\lim_{n\to\infty} P(D_{N,0}|\,H_k(\Delta^{(n)})) \tag{5.6}$$

$$=P\Big(U_j<\frac{1}{k}\sum_{l=1}^{k-1}U_l+\frac{1}{k}\Big(\frac{1}{2A^2}\Big)^{1/2}\delta B'+\lambda_\alpha;\ j=1,2,\cdots,k-1,$$
$$0<\frac{1}{k}\sum_{l=1}^{k-1}U_l-\frac{(k-1)}{k}\Big(\frac{1}{2A^2}\Big)^{1/2}\delta B'+\lambda_\alpha\Big),$$

and

(5.7)
$$\lim_{n\to\infty}P(D_{N,k}|H_k(\Delta^{(n)}))$$
$$=P\Big(U_j\leqq\Big(\frac{1}{2A^2}\Big)^{1/2}\delta B';\ j=1,2,\cdots,k-1,$$
$$\frac{1}{k}\sum_{l=1}^{k-1}U_l<\frac{(k-1)}{k}\Big(\frac{1}{2A^2}\Big)^{1/2}\delta B'-\lambda_\alpha\Big)$$

where $\lambda_\alpha$ is given by (3.7).

**5.2. Asymptotic powers of the V test.** Suppose for each n the hypothesis $H_k(\Delta^{(n)})$ is valid and $F(x)$ possesses a finite fourth order moment. Let $\nu_2$ be given by

(5.8)
$$\nu_2=E(X-E(X))^4/\{[E(X-E(X))^2]^2\}-3$$

where $X$ is the random variable with cdf $F(x)$. Then, it is obtained by the same fashion as in Lemma 3 that the joint limiting distribution of the $(k-1)$ dimensional random vector

(5.9)
$$\Big(\frac{n}{2(\nu_2+2)}\Big)^{1/2}\{W'_{N,j}-W'_{N,k}-(\eta'_{N,}-\eta'_{N,k});\ j=1,2,\cdots,k-1\},$$

where $\eta'_{N,i}$ is the expectation of $W'_{N,i}$, is the same as that of the random vector $(U_1, U_2, \cdots, U_{k-1})$ given by (3.3).

It should be noted that

(5.10)
$$\frac{1}{k}\sum_{i=1}^{k}W'_{N,i}=\frac{N-1}{N}.$$

Simiraly as in the M test, the size condition in the limit (3.11) is satisfied by any sequences of the critical values $\lambda_{N,\alpha}$ such that

(5.11)
$$\Big(\frac{n}{2(\nu_2+2)}\Big)^{1/2}\Big(\lambda_{N,\alpha}-\frac{N-1}{N}\Big)\longrightarrow\lambda_\alpha \text{ as } n\to\infty$$

where $\lambda_\alpha$ is given by (3.7). Now it follows after some simple manupulation that under $H_k(\Delta^{(n)})$,

(5.12)
$$\lim_{n\to\infty}n^{1/2}(\eta'_{N,j}-\eta'_{N,k})=-2\delta,\ j=1,2,\cdots,k-1.$$

Then we easily obtain that

(5.13)
$$\lim_{n\to\infty}P(D_{N,0}|H_k(\Delta^{(n)}))$$
$$=P\Big(U_j<\frac{1}{k}\sum_{l=1}^{k-1}U_l+\frac{2}{k}\Big(\frac{1}{2(\nu_2+2)}\Big)^{1/2}\delta+\lambda_\alpha;\ j=1,2,\cdots,k-1,$$
$$0<\frac{1}{k}\sum_{l=1}^{k-1}U_l-\frac{2(k-1)}{k}\Big(\frac{1}{2(\nu_2+2)}\Big)^{1/2}\delta+\lambda_\alpha\Big)$$

and

(5.14)
$$\lim_{n\to\infty} P(D_{N,k}|H_k(\Delta^{(n)}))$$
$$=P\Big(U_j\leqq 2\Big(\frac{1}{2(\nu_2+2)}\Big)^{1/2}\delta;\ j=1,2,\cdots,k-1,$$
$$\frac{1}{k}\sum_{l=1}^{k-1}U_l<\frac{2(k-1)}{k}\Big(\frac{1}{2(\nu_2+2)}\Big)^{1/2}\delta-\lambda_\alpha\Big).$$

**5. 3. Asymptotic relative efficiencies.** Let us adopt the same definition of efficiency as in section 6. Then, from (5.7) and (5.14), the asymptotic relative efficiencies of the rank tests with respect to the V test for testing the hypothesis $H_0$ given by (1.3) against alternatives $H_i(\Delta^{(n)})$, $i=1,2,\cdots,k$ are obtained in the following theorem.

THEOREM 2. *Suppose the assumptions of Theorem 7.1 and Lemma 7.2 of Puri* [8] *are satisfied and $F(x)$ possesses a finite fourth order moment. Then the asymptotic relative efficiencies of the rank tests with respect to the* V *test are*

(5.15)
$$e_{R,V}=\frac{(\nu_2+2)\Big(\int_{-\infty}^{+\infty}x\frac{d}{dx}\{J[F(x)]\}\,dF(x)\Big)^2}{4\Big(\int_0^1 J^2(u)\,du-\Big(\int_0^1 J(u)\,du\Big)^2\Big)}$$

*where $\nu_2$ is given by (5.8).*

The proof of this theorem is similary as the one of Theorem 1 and is therefore omitted.

If the limiting ratio of two sample sizes $N$, $N'$ for each tests is taken to be equal to (5.15), then by (5.6), (5.13) and (3.22), we get

(5.16)
$$\lim_{N\to\infty} P(D^{R}_{N',0}|H_k(\Delta^{(n)}))=\lim_{N\to\infty} P(D^{V}_{N,0}|H_k(\Delta^{(n)}))$$

and

(5.17)
$$\lim_{N\to\infty} P(D^{R}_{N',1}|H_k(\Delta^{(n)}))=\lim_{N\to\infty} P(D^{V}_{N,1}|H_k(\Delta^{(n)}))\,.$$

Thus our efficiencies $e_{R,V}$ have the stronger meaning than the appearence. Note that $e_{R,V}$ given by (5.15) is the same as the one of the rank tests corresponding to the two sample and $c$-sample problems (For example, see [9]).

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