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MINIMAX INVERSE THEOREMS IN DYNAMIC PROGRAMMING

By

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1. Introduction

This paper is a sequel to the two previous papers by the author [1, 2]. It may be regarded as a generalization from the viewpoint of the game theory.

In a main game player I (resp. II) can "control" (choose) objective (resp. constraint) functions so as to maximize (resp. minimize) the maximum value of the resulting mathematical programming problem. On the other hand, in an inverse game player I (resp. II) can control constraint (resp. objective) functions which are the objective (resp. constraint) functions in the main game. In this game player I (resp. II) wants to minimize (resp. maximize) the minimum value of the resulting mathematical programming problem.

We establish an inverse relation between the main and inverse games, provided that both the objective and constraint functions satisfy the dynamic programming structure, namely, the recursiveness with monotonicity.

In Section 2 we define mathematically the main and inverse games and give two minimax inverse theorems that the solution of one game characterizes the solution of the other game in an inverse sense. Some typical examples for which our main results are valid are illustrated in Section 3. The last section gives several comments on the minimax inverse theorems.

2. Minimax inverse theorems

Here are p functions f_i $(1\leqq i\leqq p)$ from E onto $\langle a, b\rangle$ and q functions g_j $(1\leqq j\leqq q)$ from E onto $\langle \alpha, \beta\rangle$, where $\langle r, s\rangle$ is an arbitrary interval in R^1 and E is an arbitrary interval in R^N, namely, for some $-\infty\leqq d_k<e_k\leqq\infty$

$$E=\langle d_1, e_1\rangle\times\langle d_2, e_2\rangle\times\cdots\times\langle d_N, e_N\rangle .$$

Then our *Main Game* is specified by a five-tuple $(E\,;\ \{f_i\}_{1\leqq i\leqq p},\ \langle a, b\rangle\,;\ \{g_j\}_{1\leqq j\leqq q},\ \langle\alpha, \beta\rangle)$ which is played as follows: If, first, player II chooses $j\in\{1, 2, \cdots, q\}$ and, second, player I chooses $i\in\{1, 2, \cdots, p\}$ on a state $c\in\langle\alpha, \beta\rangle$, then player I gets from II the possible "reward" gained by use of the objective function f_i subject to the constraint function g_j being less than or equal to c, that is, player II pays I the maximum-value

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of the following problem:

$$\begin{aligned} &\text{Maximize } f_i(x_1, x_2, \cdots, x_N) \\ &\text{subject to (1) } g_j(x_1, x_2, \cdots, x_N) \leqq c \\ &\qquad\qquad\;\; \text{(2) } (x_1, x_2, \cdots, x_N) \in E . \end{aligned} \tag{2.1}$$

Let $U_{ij}(c)$ denote the maximum-value of the problem (2.1). In Main Game player I will choose $i \in \{1, 2, \cdots, p\}$ so as to maximize $U_{ij}(c)$ and II will choose $j \in \{1, 2, \cdots, q\}$ so as to minimize $U_{ij}(c)$. However, note that in Main Game the choice by player II preceds to one by I. Hence Main Game started at initial state $c \in \langle \alpha, \beta \rangle$ yields the "value" $U(c)$ defined by

$$U(c) = \max_{1 \leqq i \leqq p} \min_{1 \leqq j \leqq q} U_{ij}(c) . \tag{2.2}$$

We call $U(c)$ and U the *value at c in Main Game* and the *value function in Main Game*, respectively.

On the other hand our *Inverse Game* is specified by a five-tuple $(E; \{g_j\}_{1 \leqq j \leqq q}, \langle \alpha, \beta \rangle; \{f_i\}_{1 \leqq i \leqq p}, \langle a, b \rangle)$ which is played as follows: If, first, player II chooses $j \in \{1, 2, \cdots, q\}$ and, second, player I chooses $i \in \{1, 2, \cdots, p\}$ on a state $c \in \langle a, b \rangle$, then player I pays II the possible reward gained by use of the objective function g_j subject to the constraint function f_i being greater than or equal to c, that is, player II gets from I the minimum-value of the following problem:

$$\begin{aligned} &\text{Minimize } g_j(y_1, y_2, \cdots, y_N) \\ &\text{subject to (1)}' \; f_i(y_1, y_2, \cdots, y_N) \geqq c \\ &\qquad\qquad\;\; \text{(2)}' \; (y_1, y_2, \cdots, y_N) \in E . \end{aligned} \tag{2.3}$$

Let $V_{ij}(c)$ denote the minimum-value of the problem (2.3). In Inverse Game player I will choose $i \in \{1, 2, \cdots, p\}$ so as to minimize $V_{ij}(c)$ and II will choose $j \in \{1, 2, \cdots, q\}$ so as to maximize $V_{ij}(c)$. However, note that in Inverse Game the choice by player II also precedes to one by I. Hence Inverse Game started at initial state $c \in \langle a, b \rangle$ yields the value $V(c)$ defined by

$$V(c) = \min_{1 \leqq i \leqq p} \max_{1 \leqq j \leqq q} V_{ij}(c) . \tag{2.4}$$

We call $V(c)$ and V the *value at c in Inverse Game* and the *value function in Inverse Game*, respectively. It should be noted that the meaning of the value in our games is different from one in the so-called zero-sum two-person game.

Now let us define the strategies for both players. *Strategies for players I, II in Main Game* are defined by mappings $m : \langle \alpha, \beta \rangle \to \{1, 2, \cdots, p\}$ and $n : \langle \alpha, \beta \rangle \to \{1, 2, \cdots, q\}$, respectively. *Strategies for them in Inverse Game* are defined by mappings $k : \langle a, b \rangle \to \{1, 2, \cdots, p\}$ and $l : \langle a, b \rangle \to \{1, 2, \cdots, q\}$, respectively. Note that from the viewpoint of the dynamic programming the strategies specify the behavior for both players at initial state, that is, they give the initial choices of both players. Therefore, we shall use "*initial choice*" in stead of "strategy" throughout the paper. Initial choices

(m^*, n^*) (resp. $(\hat{k}, \hat{l})$) are *optimal at state* $c \in \langle \alpha, \beta \rangle$ *(resp.* $\langle a, b \rangle$*) in Main (resp. Inverse) Game* if

$$U_{m^*(c)n^*(c)}(c) = U(c) \quad (\text{resp. } V_{\hat{k}(c)\hat{l}(c)}(c) = V(c)) .$$

Further they are *optimal in Main (resp. Inverse) Game* if they are optimal at all $c \in \langle \alpha, \beta \rangle$ (resp. $\langle a, b \rangle$).

Let us prepare a fundamental proposition which plays an important role in the following theorems.

PROPOSITION 1 (MINIMAX INVERSE PROPOSITION). *Let* $u_i : \langle a, b \rangle \to \langle \alpha, \beta \rangle$ *be an onto continuous and strictly increasing function to which* v_i *is the inverse function* $(1 \leqq i \leqq n)$. *Define* u, v *as follows:*

$$u(x) = \max_{1 \leqq i \leqq n} u_i(x) \qquad x \in \langle a, b \rangle ,$$

$$v(y) = \min_{1 \leqq i \leqq n} v_i(y) \qquad y \in \langle \alpha, \beta \rangle .$$

Then $u : \langle a, b \rangle \to \langle \alpha, \beta \rangle$ *is an onto continuous and strictly increasing function to which* v *is the inverse function.*

PROOF. The proof is straighforward.

Recall that E is the Cartesian product of the intervals $E_k = \langle d_k, e_k \rangle$ $1 \leqq k \leqq N$, namely,

$$E = E_1 \times E_2 \times \cdots \times E_N .$$

A continuous function $f : E \to R^1$ is called the *recursive function on* E if it is expressed as follows:

$$f(x_1, x_2, \cdots, x_N) = f_1(x_1 ; f_2(x_2 ; \cdots ; f_{N-1}(x_{N-1} ; f_N(x_N)) \cdots)) ,$$

where $f_N : E_N \to R^1$, $f_k : E_k \times \text{range}(f_{k+1}) \to R^1$ are continuous $(1 \leqq k \leqq N-1)$. Note that $\text{range}(f_k) = \{z : z = f_k(x ; y), (x, y) \in E_k \times \text{range}(f_{k+1})\}$ $(1 \leqq k \leqq N-1)$, and $\text{range}(f_N) = \{y ; y = f_N(x), x \in E_N\}$. In particular f is called the *recursive function with monotonicity on* E if each $f_k(x ; \cdot)$ $(1 \leqq k \leqq N-1, x \in E_k)$ is nondecreasing and f_N is strictly increasing. Moreover, f is called the *recursive function with strict increasingness on* E if each $f_k(x ; \cdot)$ $(1 \leqq k \leqq N-1, x \in E_k)$ is strictly increasing. A continuous function $f : E_k \times E_{k+1} \to R^1$ is called the *function with strict increasingness on* $E_k \times E_{k+1}$ if each $f(x ; \cdot)$ $(x \in E_k)$ is strictly increasing. A recursive function f with monotonicity on $E_k \times E_{k+1}$ is called the *maximum* (resp. *minimum*) *function on* $E_k \times E_{k+1}$ if it is expressed as follows:

$$f(x ; y) = \max(f_1(x), f_2(y)) \quad (\text{resp. } f(x ; y) = \min(f_1(x), f_2(y)))$$

where f_i $(i = 1, 2)$ is a continuous and strictly increasing function from E_{k+i-1} onto some interval.

Throughout the remainder of this section we assume that f_i $(1 \leqq i \leqq p) : E \to \langle \alpha, \beta \rangle$ and g_j $(1 \leqq j \leqq q) : E \to \langle a, b \rangle$ are onto recursive functions with monotonicity on E such that each f_{ik} $(1 \leqq i \leqq p, 1 \leqq k \leqq N-1)$ is either a function with strict increasingness on $E_k \times E_{k+1}$ or a minimum function on $E_k \times E_{k+1}$ and that each g_{jk} $(1 \leqq j \leqq q, 1 \leqq k \leqq N-1)$ is either a function with strict increasingness on $E_k \times E_{k+1}$ or a maximum function on $E_k \times E_{k+1}$. We consider Main Game $(E ; \{f_i\}_{1 \leqq i \leqq p}, \langle a, b \rangle ; \{g_j\}_{1 \leqq j \leqq q}, \langle \alpha, \beta \rangle)$ and

Inverse Game $(E; \{g_j\}_{1\leqq j\leqq q}, \langle \alpha, \beta\rangle; \{f_i\}_{1\leqq i\leqq p}, \langle a, b\rangle)$ provided that for each i, j either U_{ij} or V_{ij} is continuous and strictly increasing. Then we have

THEOREM 1 (MINIMAX INVERSE THEOREM I IN DYNAMIC PROGRAMMING). *One game has a continuous and strictly increasing value function W and optimal initial choices (t, u) if and only if the other game has a continuous and strictly increasing value function W^{-1} and optimal initial choices $(t\circ W^{-1}, u\circ W^{-1})$.*

PROOF. Let U and (m^*, n^*) be a continuous and strictly increasing value function and optimal initial choices in Main Game. That is, for $c\in\langle \alpha, \beta\rangle$

$$U(c)=U_{m^*(c)n^*(c)}(c)\,. \tag{2.5}$$

By Inverse Theorem II in Dynamic Programming [2] we have for $1\leqq i\leqq p$, $1\leqq j\leqq q$, $c\in\langle a, b\rangle$

$$U_{ij}^{-1}(c)=V_{ij}(c)\,. \tag{2.6}$$

Therefore, Minimax Inverse Proposition yields for $c\in\langle a, b\rangle$

$$\begin{aligned} U^{-1}(c)&=(\max_{1\leqq i\leqq p}\min_{1\leqq i\leqq q} U_{ij})^{-1}(c)\\ &=\min_{1\leqq i\leqq p}\max_{1\leqq j\leqq q} U_{ij}^{-1}(c)\\ &=\min_{1\leqq i\leqq p}\max_{1\leqq j\leqq q} V_{ij}(c)\,. \end{aligned} \tag{2.7}$$

By (2.5), (2.6) it holds that for $c\in\langle a, b\rangle$

$$\begin{aligned} U^{-1}(c)&=(U_{m^*(c)n^*(c)})^{-1}(c)\\ &=V_{m^*(U^{-1}(c))n^*(U^{-1}(c))}(c)\,. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) we have for $c\in\langle a, b\rangle$

$$\begin{aligned} U^{-1}(c)&=\min_{1\leqq i\leqq p}\max_{1\leqq j\leqq q} V_{ij}(c)\\ &=V_{m^*(U^{-1}(c))n^*(U^{-1}(c))}(c)\,. \end{aligned}$$

Hence U^{-1} and $(m^*\circ U^{-1}, n^*\circ U^{-1})$ are the continuous and strictly increasing value function and the optimal initial choices in Inverse Game. The converse is proved by the converse procedure to one stated above. Exchanging Main Game for Inverse, the proof is complete.

The following theorem gives an inverse relation between the pairs of the value function and the *optimal-point* (the point which attains the maximin-value in Main Game or the minimax-value in Inverse Game) *function.*

THEOREM 2 (MINIMAX INVERSE THEOREM II IN DYNAMIC PROGRAMMING). *One game has a continuous and strictly increasing value function W and an optimal-point function $(z_1, z_2, \cdots, z_N)$ if and only if the other game has a continuous and strictly increasing value function W^{-1} and an optimal-point function $(z_1\circ W^{-1}, z_2\circ W^{-1}, \cdots, z_N\circ W^{-1})$.*

PROOF. Let U and $(x_1^*, x_2^*, \cdots, x_N^*)$ be a value function and a maximin-point function in Main Game, that is,

$$\begin{aligned} U(c) &= \max_{1 \leqq i \leqq p} \min_{1 \leqq j \leqq q} U_{ij}(c) \\ &= U_{m^*(c)n^*(c)}(c) \\ &= f_{m^*(c)}(x_1^*(c), x_2^*(c), \cdots, x_N^*(c)), \end{aligned} \tag{2.9}$$

and

$$g_{n^*(c)}(x_1^*(c), x_2^*(c), \cdots, x_N^*(c)) \leqq c, \tag{2.10}$$

where (m^*, n^*) are the optimal initial choices. Note that the optimal initial choices always exist. By the properties of the functions f_i and g_j we may assume without loss of generality that

$$g_{n^*(c)}(x_1^*(c), x_2^*(c), \cdots, x_N^*(c)) = c. \tag{2.11}$$

Hence (2.9), (2.11) yields an optimal solution $(x_1^*(c), x_2^*(c), \cdots, x_N^*(c))$ of the following problem:

$$\begin{aligned} &\text{Minimize} \quad g_{n^*(c)}(y_1, y_2, \cdots, y_N) \\ &\text{subject to (1)} \quad f_{m^*(c)}(y_1, y_2, \cdots, y_N) \geqq U(c) \\ &\qquad\qquad\quad \text{(2)} \quad (y_1, y_2, \cdots, y_N) \in E. \end{aligned}$$

Therefore we have

$$V_{m^*(c)n^*(c)}(d) = g_{n^*(c)}(x_1^*(c), x_2^*(c), \cdots, x_N^*(c)), \tag{2.12}$$

where $d = f_{m^*(c)}(x_1^*(c), x_2^*(c), \cdots, x_N^*(c))$.

On the other hand, since $U_{ij}^{-1} = V_{ij}$, Minimax Inverse Proposition, and (2.9) we have for $d \in \langle a, b \rangle$

$$\begin{aligned} U^{-1}(d) &= \min_{1 \leqq i \leqq p} \max_{1 \leqq i \leqq q} V_{ij}(d) \\ &= V_{m^*(U^{-1}(d))n^*(U^{-1}(d))}(d). \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13) we have for $c \in \langle a, b \rangle$

$$\begin{aligned} U^{-1}(c) &= \min_{1 \leqq i \leqq p} \max_{1 \leqq j \leqq q} V_{ij}(c) \\ &= V_{m^*(U^{-1}(c))n^*(U^{-1}(c))}(c) \\ &= g_{n^*(U^{-1}(c))}(x_1^*(U^{-1}(c)), x_2^*(U^{-1}(c)), \cdots, x_N^*(U^{-1}(c))). \end{aligned}$$

Hence U^{-1} and $(x_1^* \circ U^{-1}, x_2^* \circ U^{-1}, \cdots, x_N^* \circ U^{-1})$ are the value function and the minimax-point function in Inverse Game. The converse is proved by the converse procedure to one stated above. Exchanging Main Problem for Inverse, the proof is complete.

COROLLARY. *One game has a continuous and strictly increasing value function W, optimal initial choices (t, u), and an optimal-point function $(z_1, z_2, \cdots, z_N)$ if and only if the other game has a continuous and strictly increasing value function W^{-1}, optimal initial choices $(t \circ W^{-1}, u \circ W^{-1})$, and an optimal-point function $(z_1 \circ W^{-1}, z_2 \circ W^{-1}, \cdots, z_N \circ W^{-1})$.*

PROOF. This corollary is a combination of Theorems 1, 2.

3. Examples

EXAMPLE 1. Let $E=R^N$, $\langle a, b\rangle=\langle\alpha, \beta\rangle=R^1$, and

$$f_i(x_1, x_2, \cdots, x_N)=\min_{1\leq k\leq N} b_{ik}x_k^{q_{ik}} \qquad 1\leqq i\leqq p,$$

$$g_j(x_1, x_2, \cdots, x_N)=\max_{1\leq k\leq N} a_{jk}x_k^{p_{jk}} \qquad 1\leqq j\leqq q,$$

where $a_{jk}>0$, $b_{ik}>0$, and p_{jk}, q_{ik} are positive odd integers. Then $f_i, g_j: R^N\to R^1$ are recursive functions with monotonicity on R^N. Strictly speaking, f_i (resp. g_j) is minimum (resp. maximum) on R^N [1, 2]. It is easily shown that

$$U(c)=\max_{1\leq i\leq p}\min_{1\leq j\leq q}\min_{1\leq k\leq N} b_{ik}\left(\frac{c}{a_{jk}}\right)^{\frac{q_{ik}}{p_{jk}}}, \tag{3.1}$$

$$V(c)=\min_{1\leq i\leq p}\max_{1\leq j\leq q}\max_{1\leq k\leq N} a_{jk}\left(\frac{c}{b_{ik}}\right)^{\frac{p_{jk}}{q_{ik}}}, \tag{3.2}$$

$$U^{-1}=V, \qquad V^{-1}=U.$$

Let (m^*, n^*) (resp. $(\hat{k}, \hat{l})$) be the optimal initial choices in this main (resp. inverse) game. Note that they are specified by the (i, j) at which (3.1) (resp. (3.2)) is attained. Then we have by Theorem 1

$$\hat{k}=m^*\circ V, \qquad \hat{l}=n^*\circ V.$$

The main (resp. inverse) game has the optimal-point function $(x_1^*, x_2^*, \cdots, x_N^*)$ (resp. $(\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_N)$) defined by

$$x_k^*(c)=\left(\frac{c}{a_{n^*(c)k^*(c)}}\right)^{\frac{1}{p_{n^*(c)k^*(c)}}}\left(\text{resp. } \hat{y}_k(c)=\left(\frac{c}{b_{\hat{k}(c)\hat{\hat{k}}(c)}}\right)^{\frac{1}{q_{\hat{k}(c)\hat{\hat{k}}(c)}}}\right)$$

where $k^*(c)$ (resp. $\hat{\hat{k}}(c)$) is the k such that

$$\min_{1\leq k\leq N} b_{m^*(c)k}\left(\frac{c}{a_{n^*(c)k}}\right)^{\frac{q_{m^*(c)k}}{p_{n^*(c)k}}}=b_{m^*(c)k^*(c)}\left(\frac{c}{a_{n^*(c)k^*(c)}}\right)^{\frac{q_{m^*(c)k^*(c)}}{p_{n^*(c)k^*(c)}}}$$

$$\left(\text{resp. } \max_{1\leq k\leq N} a_{\hat{l}(c)k}\left(\frac{c}{b_{\hat{k}(c)k}}\right)^{\frac{p_{\hat{l}(c)k}}{q_{\hat{k}(c)k}}}=a_{\hat{l}(c)\hat{\hat{k}}(c)}\left(\frac{c}{b_{\hat{k}(c)\hat{\hat{k}}(c)}}\right)^{\frac{p_{\hat{l}(c)\hat{\hat{k}}(c)}}{q_{\hat{k}(c)\hat{\hat{k}}(c)}}}\right).$$

It is easily verified that

$$\hat{y}_k=x_k^*\circ V, \qquad x_k^*=\hat{y}_k\circ U \qquad 1\leqq k\leqq N.$$

EXAMPLE 2. Let $E=R^N$, $\langle a, b\rangle=\langle\alpha, \beta\rangle=R^1$, and

$$f_i(x_1, x_2, \cdots, x_N)=\sum_{k=1}^{N} b_{ik}x_k^{q_{ik}} \qquad 1\leqq i\leqq p,$$

$$g_j(x_1, x_2, \cdots, x_N)=\sum_{k=1}^{N} a_{jk}x_k^{p_{jk}} \qquad 1\leqq j\leqq q,$$

where $a_{jk}>0$, $b_{ik}>0$, and p_{jk}, q_{ik} are positive odd integers with $\frac{q_{ik}}{p_{jk}}\geqq 1$. Then $f_i, g_j: R^N\to R^1$ are recursive functions with strict increasingness on R^N. More exactly, they are additive on R^N [1, 2]. These games have the same value functions, optimal choices, and optimal-point functions as Example 1.

EXAMPLE 3. Let $E=R^N$, $\langle a, b\rangle=\langle\alpha, \beta\rangle=R^1$, and

$$f_i(x_1, x_2, \cdots, x_N)=\sum_{k=1}^{N} b_{ik}x_k^{q_{ik}} \qquad 1\leqq i\leqq p\,,$$

$$g_j(x_1, x_2, \cdots, x_N)=\max_{1\leqq k\leqq N} a_{jk}x_k^{p_{jk}} \qquad 1\leqq j\leqq q\,,$$

where $a_{jk}>0$, $b_{ik}>0$, and p_{jk}, q_{ik} are positive odd integers. Then this main game has the value function

$$U(c)=\max_{1\leqq i\leqq p}\min_{1\leqq j\leqq q}\sum_{k=1}^{N} b_{ik}\Big(\frac{c}{a_{jk}}\Big)^{\frac{q_{ik}}{p_{jk}}}. \tag{3.3}$$

Further the optimal-point function $(x_1^*, x_2^*, \cdots, x_N^*)$ is given as follows:

$$x_k^*(c)=\Big(\frac{c}{a_{n^*(c)k}}\Big)^{\frac{1}{p_{n^*(c)k}}},$$

where (m^*, n^*) are the optimal choices specified by the (i, j) at which the maximin-value of (3.3) is attained. By applying Theorems 1, 2 the value function, the optimal choices, and the optimal-point function in the inverse game are given as follows:

$$U^{-1},\ (m^*\circ U^{-1}, n^*\circ U^{-1}),\ (x_1^*\circ U^{-1}, x_2^*\circ U^{-1}, \cdots, x_N^*\circ U^{-1}).$$

EXAMPLE 4. Let $E=R^N$, $\langle a, b\rangle=\langle\alpha, \beta\rangle=R^1$, and

$$f_i(x_1, x_2, \cdots, x_N)=\min_{1\leqq k\leqq N} b_{ik}x_k^{q_{ik}} \qquad 1\leqq i\leqq p\,,$$

$$g_j(x_1, x_2, \cdots, x_N)=\sum_{k=1}^{N} a_{jk}x_k^{p_{jk}} \qquad 1\leqq j\leqq q\,,$$

where $a_{jk}>0$, $b_{ik}>0$, and p_{jk}, q_{ik} are positive odd integers. Then this inverse game has the value function

$$V(c)=\min_{1\leqq i\leqq p}\max_{1\leqq j\leqq q}\sum_{k=1}^{N} a_{jk}\Big(\frac{c}{b_{ik}}\Big)^{\frac{p_{jk}}{q_{ik}}}. \tag{3.4}$$

Further the optimal-point function $(\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_N)$ is given as follows:

$$\hat{y}_k(c)=\Big(\frac{c}{b_{\hat{k}(c)k}}\Big)^{\frac{1}{q_{\hat{k}(c)k}}},$$

where $(\hat{k}, \hat{l})$ are the optimal choices specified by the (i, j) at which the minimax-value of (3.4) is attained. Our theorems yield the solutions of the main game as follows:

$$V^{-1},\ (\hat{k}\circ V^{-1}, \hat{l}\circ V^{-1}),\ (\hat{y}_1\circ V^{-1}, \hat{y}_2\circ V^{-1}, \cdots, \hat{y}_N\circ V^{-1}).$$

EXAMPLE 5. Let $E=R_+^2$, $\langle a, b\rangle=\langle\alpha, \beta\rangle=R_+^1$, and

$$f_1(x, y)=x^2+2y\,, \qquad f_2(x, y)=xy\,,$$

$$g_1(x, y)=x+y\,, \qquad g_2(x, y)=\max(2x, y)\,.$$

Then f_i, g_j $(i, j=1, 2)$: $R_+^2\to R_+^1$ are recursive functions with monotonicity on R_+^2. More exactly, f_1 and g_1 are additive on R^2_+, f_2 is multiplicative, and g_2 is maximum on R^2_+ [1, 2]. This main game has the following solutions:

$$U(c)=\begin{cases} 2c \\ c^2 \\ \frac{c^2}{4}+2c \end{cases}, \quad (x^*(c), y^*(c))=\begin{cases} (0, c) & \text{on } [0, 2) \\ (c, 0) & \text{on } \left[2, \frac{8}{3}\right) \\ \left(\frac{c}{2}, c\right) & \text{on } \left[\frac{8}{3}, \infty\right) \end{cases}$$

$$m^*(c)\equiv 1, \quad n^*(c)=\begin{cases} 1 & \text{on } \left[0, \frac{8}{3}\right) \\ 2 & \text{on } \left[\frac{8}{3}, \infty\right). \end{cases}$$

On the other hand the inverse game has the following solutions:

$$V(c)=\begin{cases} \frac{1}{2}c \\ \sqrt{c} \\ -4+2\sqrt{c+4} \end{cases}, \quad (\hat{x}(c), \hat{y}(c))=\begin{cases} \left(0, \frac{1}{2}c\right) & \text{on } [0, 4) \\ (\sqrt{c}, 0) & \text{on } \left[4, \frac{64}{9}\right) \\ (-2+\sqrt{c+4}, -4+2\sqrt{c+4}) & \text{on } \left[\frac{64}{9}, \infty\right) \end{cases}$$

$$\hat{k}(c)\equiv 1, \quad \hat{l}(c)=\begin{cases} 1 & \text{on } \left[0, \frac{64}{9}\right) \\ 2 & \text{on } \left[\frac{64}{9}, \infty\right). \end{cases}$$

It is easily verified that

$$U^{-1}=V, \qquad V^{-1}=U,$$

$$\hat{k}=m^*\circ V, \quad \hat{l}=n^*\circ V, \quad \hat{x}=x^*\circ V, \quad \hat{y}=y^*\circ V.$$

4. Further comments

Throughout the following Remarks 1, 2 and 3 we assume that the functions f_i $(1\leqq i\leqq p)$, g_j $(1\leqq j\leqq q)$ satisfy the condition stated in Section 2. Note that Theorems 1, 2 and Corollary hold for games defined in Remarks 1, 2 and 3.

REMARK 1. Consider another main game and its inverse game where, however, the order of the choices of moves by both players is reversed in the original games. Then the resulting main (resp. inverse) game has the value function

$$U'=\min_{1\leqq j\leqq q}\max_{1\leqq i\leqq p} U_{ij} \text{ (resp. } V'=\max_{1\leqq j\leqq q}\min_{1\leqq i\leqq p} V_{ij}).$$

REMARK 2. Consider another main game and its inverse game where, however, the payoff relation between both players is exchanged in the original games. Then the resulting main (resp. inverse) game has the value function

$$U''=\min_{1\leqq i\leqq p}\max_{1\leqq j\leqq q} U_{ij} \text{ (resp. } V''=\max_{1\leqq i\leqq p}\min_{1\leqq j\leqq q} V_{ij}).$$

REMARK 3. Consider another main game and its inverse game where, however, not only the order of choices is reversed but also the payoff relation is exchanged in the original games. Then the resulting main (resp. inverse) game has the value function

$$U''' = \max_{1 \leqq j \leqq q} \min_{1 \leqq i \leqq p} U_{ij} \text{ (resp. } V''' = \min_{1 \leqq j \leqq q} \max_{1 \leqq i \leqq p} V_{ij}).$$

REMARK 4. Let $p=q=1$ in the original games. Then Minimax Inverse Theorem II in Dynamic Programming reduces to Inverse Theorem II in Dynamic Programming [2]. Hence the former is viewed as a game-theoretic generalization of the latter.

REMARK 5. Under some regularity conditions, Proposition 1 and Theorems 1, 2 can be easily generalized to the case where there exist infinitely many moves, namely, $\{f_\lambda\}_{\lambda \in A}$ and $\{g_\mu\}_{\mu \in B}$, A, B infinite.

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