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UNIFORM STRONG CONVERGENCE OF A GENERALIZED FAILURE RATE ESTIMATE

By

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Abstract

Under certain conditions it is shown that uniform continuity of the generalized failure rate function is necessary and sufficient for strong uniform consistency of a class of estimators based on the kernel estimates of the probability density and distribution functions due to Parzen (1962) and Nadaraya (1970). The result is proved for the univariate generalized failure rate function due to Barlow and Van Zwet (1970) and for its bivariate extension. Our results contain as special cases the work of Schuster (1969) for the univariate densities and Samanta (1973) for the bivariate densities.

1. Introduction

Let X be a random variable (r. v.), with probability density function (p. d. f.) f, and distribution function (d. f.) F. Let $\bar{F}(x)=1-F(x)$, and define the failure rate function $r(x)$ for all x in the support of $\bar{F}$ by

$$r(x)=f(x)/\bar{F}(x)\,. \tag{1}$$

Barlow and Van Zwet (1970) extended the above definition as follows; let $g(G)$ be a known p. d. f. (d. f.) and define the generalized failure rate function (GFRF) by

$$r_{FG}(x)=f(x)/g\bar{G}^{-1}\bar{F}(x)\,, \tag{2}$$

for all x in the support S of $g\bar{G}^{-1}\bar{F}$.

Let $X_1, \cdots, X_n$ be a random sample from F. Let $k(u)$ be a known p. d. f. satisfying the conditions:

$$\sup k(u)<\infty \quad \text{and} \quad |u|k(u)\to 0 \quad \text{as } |u|\to\infty\,. \tag{3}$$

Further, let $\{a_n\}$ be a sequence of real numbers such that

$$a_n\to 0 \quad \text{as} \quad n\to\infty\,. \tag{4}$$

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The kernel estimate of $f(x)$ using $k(u)$ is given by

$$\hat{f}(x)=\hat{f}(x; X_1, \cdots, X_n)=a_n^{-1}\int k[(x-u)/a_n]dF_n(u)$$

$$=(na_n)^{-1}\sum_{i=1}^{n} k[(x-X_i)/a_n], \tag{5}$$

where $F_n(u)$ denotes the empirical d. f. From (5), the kernel estimate of $\bar{F}(x)$ may be given as

$$\hat{\bar{F}}(x)=\hat{\bar{F}}(x; X_1, \cdots, X_n)=\int_x^{\infty}\hat{f}(u)du=(na_n)^{-1}\sum_{i=1}^{n}\bar{K}[(x-X_i)/a_n],$$

where

$$\bar{K}[(x-X_i)/a_n]=\int_x^{\infty}k[(u-X_i)/a_n]du.$$

Note that

$$a_n^{-1}K(x/a_n)\rightarrow I_{(0,\infty)}(x), \quad \text{as} \quad n\rightarrow\infty, \tag{7}$$

where $I_{(0,\infty)}(x)=1$ if $x>0$, and $=0$ for $x\leqq 0$. From (5) and (6) the kernel estimate of the GFRF is given by

$$\hat{r}_{FG}(x)=\hat{r}_{FG}(x, X_1, \cdots, X_n)=\hat{f}(x)/g\bar{G}^{-1}\hat{\bar{F}}(x), \tag{8}$$

for all $x\in S$. Assume that $g\bar{G}^{-1}$ is uniformly continuous with bounded first derivative.

THEOREM 1. *Assume that the following conditions hold:*

(i) *f is uniformly continuous on S.*

(ii) *k is a density function of bounded variation and is uniformly continuous on the real line.*

(iii) *For any $\delta>0$, $\sum_{n=1}^{\infty}\exp(-\delta na_n^2)<\infty$.*

Then

$$\sup_{x\in S}|\hat{r}_{FG}(x)-r_{FG}(x)|\rightarrow 0 \text{ w. p. 1.}, \quad as \quad n\rightarrow\infty, \tag{9}$$

where the notation $X_n\rightarrow X$ w. p. 1 means that X_n converges to X with probability one. Conversely, if conditions (ii) *and* (iii) *are satisfied and if for some measurable function $q(x)$, we have*

$$\sup_{x\in S}|\hat{r}_{FG}(x)-q(x)|\rightarrow 0 \text{ w.p.1}, \quad as \quad n\rightarrow\infty. \tag{10}$$

Then $q(x)$ is uniformly continuous and $q(x)=r_{FG}(x)$, for all $x\in S$.

Note that if we take g to be uniform [0, 1], Theorem 1 reduces to Theorem 3.11 of Schuster (1969), and if we take it to be exponential, it reduces to a theorem for the usual failure rate function $r(x)$ given by (1).

Basu (1971), extended the notion of failure rate function to the bivariate case. Let (X, Y) be a bivariate random vector with (joint) p. d. f. h and (joint) d. f. H. Let $\bar{H}(x, y)=P[X>x, r>y]$, and define the bivariate failure rate function by

$$\rho(x, y)=h(x, y)/\bar{H}(x, y), \tag{11}$$

for all (x, y) in the support of $\bar{H}$. Definition (11) can be generalized as in the univariate case to give the generalized bivariate failure rate function (GBFRF) as follows:

$$\rho_{HG}(x, y)=h(x, y)/g\bar{G}^{-1}\bar{H}(x, y), \tag{12}$$

for all (x, y) in the support S^* of $g\bar{G}^{-1}\bar{H}$.

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from h. Let k_i, $i=1, 2$ be known p. d. f.'s satisfying Condition (3). The kernel estimate of $h(x, y)$ using $k_1(u)$ and $k_2(v)$ is given by:

$$\hat{h}(x, y)=\hat{h}(x, y; X_1, Y_1, \cdots, X_n, Y_n)=a_n^{-2}\iint k_1[(x-u)/a_n[k_2[(y-v)/a_n]dH_n(u, v)$$

$$=(na_n^2)^{-1}\sum_{i=1}^{n} k_1[(x-X_i)/a_n]k_2[(y-Y_i)/a_n], \quad (13)$$

where $H_n(u, v)$ is the bivariate empirical d. f. From (13), the kernel estimate of $\bar{H}(x, y)$ may be given as

$$\hat{\bar{H}}(x, y)=\hat{\bar{H}}(x, y; X_1, Y_1, \cdots, X_n, Y_n)=\int_x^\infty\int_y^\infty \hat{h}(u, v)dudv$$

$$=(na_n^2)^{-1}\sum_{i=1}^{n} \bar{K}_1[(x-X_i)/a_n]\bar{K}_2[(y-Y_i)/a_n], \quad (14)$$

where

$$\bar{K}_1[(x-X_i)/a_n]=\int_x^\infty k_1[(u-X_i)/a_n]du,$$

and

$$\bar{K}_2[(y-X_i)/a_n]=\int_y^\infty k_2[(v-Y_i)/a_n]dv.$$

From (13) and (14) we may estimate the BGFRF by

$$\hat{\rho}_{HG}(x, y)=\hat{\rho}_{HG}(x, y; X_1, Y_1, \cdots, X_n, Y_n)=\hat{h}(x, y)/g\bar{G}^{-1}\hat{\bar{H}}(x, y), \quad (15)$$

for all $(x, y)\in S^*$. Again assuming that $g\bar{G}^{-1}$ in uniformly continuous with bounded first derivative we state

THEOREM 2. *Assume that the following condition hold:*

(iv) *h is uniformly continuous on S^*.*

(v) *k_i, $i=1, 2$, are density functions of bounded variation and are uniformly contінous on the real line, and*

(vi) *For any $\gamma>0$, $\sum_{n=1}^{\infty}\exp(-\gamma na_n^4)<\infty$.*

Then

$$\sup_{(x,y)\in S^*}|\hat{\rho}_{HG}(x, y)-\rho_{HG}(x, y)|\to 0 \text{ w. p. 1,} \quad as \quad n\to\infty. \quad (16)$$

Conversely, if conditions (v) *and* (vi) *are satisfied and if for some measurable function $p(x, y)$, we have*

$$\sup_{(x,y)\in S^*}|\hat{\rho}_{HG}(x, y)-p(x, y)|\to 0 \text{ w. p. 1,} \quad as \quad n\to\infty. \quad (17)$$

Then $p(x, y)$ is uniformly continuous and $p(x, y)=\rho_{HG}(x, y)$ for all $(x, y)\in S^*$.

Note that taking g to be uniform [0, 1], Theorem 2 reduces to a Theorem of Samanta (1973), and if g is taken to be exponential, it reduces to a theorem concerning Basu's bivariate failure rate function $\rho(x, y)$ as defined in (11).

2. Proofs of theorems

PROOF OF THEOREM 1: Sufficiency. Let sup (inf) denote the supremum (infemum) over S, any $\sup_x$ denote the supremum over the entire real line. It is easy to see that

$$\sup|\hat{r}_{FG}(x)-r_{FG}(x)|\leqq I_{1n}+I_{3n}+I_{3n}\,, \quad \text{say}$$

where

$$I_{1n}=\sup|\hat{f}(x)-E\hat{f}(x)|/g\bar{G}^{-1}\hat{\bar{F}}(x)\,,$$

$$I_{2n}=\sup E\hat{f}(x)|[g\bar{G}^{-1}\hat{\bar{F}}(x)]^{-1}-[g\bar{G}^{-1}\bar{F}(x)]^{-1}|,$$

and

$$I_{3n}=\sup|E\hat{f}(x)-f(x)|g\bar{G}^{-1}\bar{F}(x)\,.$$

It suffices to show that $I_{jn}\to 0$ w. p. 1, as $n\to\infty$, $j=1, 2, 3$. Let α, α_1, α_2, and β_1 be defined by:

$$g\bar{G}^{-1}\bar{F}(\alpha)=\inf g\bar{G}^{-1}\bar{F}(x)\,, \qquad g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)=\inf g\bar{G}^{-1}\hat{\bar{F}}(x)\,,$$

$$g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2))=\inf g\bar{G}^{-1}(E\hat{\bar{F}}(x))\,, \quad \text{and} \quad g\bar{G}^{-1}F(\beta_1)=\sup g\bar{G}^{-1}\hat{\bar{F}}(x)\,.$$

Then $g\bar{G}^{-1}\bar{F}(\alpha)>0$, and since $a_n^{-1}\bar{K}(x/a_n)\to I_{(0,\infty)}(x)$, it can be shown that $\sup_x|E\hat{\bar{F}}(x)-\bar{F}(x)|\to 0$, thus $g\bar{G}^{-1}E\hat{\bar{F}}(\alpha_2)\to g\bar{G}^{-1}\bar{F}(\alpha)>0$, as $n\to\infty$, since $g\bar{G}^{-1}$ is uniformly continuous. Furthermore, we claim that $g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)\to g\bar{G}^{-1}\bar{F}(\alpha)$ w. p. 1 as $n\to\infty$. First we show that $\sup_x|\hat{\bar{F}}(x)-E\hat{\bar{F}}(x)|\to 0$ w.p.1 as $n\to\infty$,

$$|\hat{\bar{F}}(x)-E\hat{\bar{F}}(x)|\leqq a_n^{-1}\left|\int\bar{K}[(x-u)/a_n]dF_n(u)-\int\bar{K}[(x-u)/a_n]dF(u)\right|$$

$$\leqq a_n^{-2}\sup_x|F_n(x)-F(x)|\int k[(x-u)/a_n]du$$

$$\leqq a_n^{-1}\sup_x|F_n(x)-F(x)|\,.$$

Thus it follows from Lemma 2.1 of Schuster (1969), that for any $\delta>0$

$$P[\sup_x|\hat{\bar{F}}(x)-E\hat{\bar{F}}(x)|>\delta]\leqq P[\sup_x|F_n(x)-F(x)|>\delta a_n]$$

$$\leqq C_1\exp(-\delta_1 n a_n^2)\,.$$

Hence from (iii) we have $\sup_x|\hat{\bar{F}}(x)-E\hat{\bar{F}}(x)|\to 0$, w. p. 1 as $n\to\infty$, thus since $g\bar{G}^{-1}$ is uniformly continuous our claim is proved. Now I_{1n} follows from the above discussion, the fact that $g\bar{G}^{-1}\bar{F}(\alpha)>0$, and the inequality

$$I_{1n}\leqq[g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)]^{-1}\sup_x|\hat{f}(x)-E\hat{f}(x)|,$$

by using Theorem 1 of Nadaraya (1965). Next, we show that $I_{2n}\to 0$ w. p. 1, as $n\to\infty$.

$$I_{2n}\leqq\sup E\hat{f}(x)\sup|[g\bar{G}^{-1}\hat{\bar{F}}(x)]^{-1}-[g\bar{G}^{-1}\bar{F}(x)]^{-1}|$$

$$\leqq[g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)g\bar{G}^{-1}\bar{F}(\alpha)]^{-1}\sup_x E\hat{f}(x)\sup|g\bar{G}^{-1}\hat{\bar{F}}(x)-g\bar{G}^{-1}\bar{F}(x)|\,.$$

Since from (3), $\sup_x E\hat{f}(x)<\infty$, and $g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)\to g\bar{G}^{-1}F(\alpha)$ w. p. 1, as $n\to\infty$, if we show that $\Delta_n=\sup|g\bar{G}^{-1}\hat{\bar{F}}(x)-g\bar{G}^{-1}\bar{F}(x)|\to 0$ w. p. 1, as $n\to\infty$, then $I_{2n}\to 0$ w. p. 1, as $n\to\infty$.

But using Taylor expansion, uniform boundedness of $(g\bar{G}^{-1})'$, we have

$$\Delta_n \leqq C_2 \sup_x |\hat{\bar{F}}(x)-\bar{F}(x)| \to 0 \text{ w. p. 1} \quad \text{as} \quad n\to\infty .$$

Finally,

$$I_3 \leqq [g\bar{G}^{-1}\bar{F}(\alpha)]^{-1} \sup_x |E\hat{f}(x)-f(x)| \to 0 \quad \text{as} \quad n\to\infty ,$$

by equation (3.11) of Parzen (1962) and the fact that $g\bar{G}^{-1}\bar{F}(\alpha)>0$. This terminates the proof of sufficiency.

Necessity. The proof will follow from the following four lemmas.

LEMMA 1. *Let* $\zeta_n(x)=E\hat{f}(x)/g\bar{G}^{-1}(E\hat{\bar{F}}(x))$. *Then*

$$\text{suy} |\hat{r}_{FG}(x)-\zeta_n(x)| \to 0 \text{ w. p. 1}, \quad \text{as} \quad n\to\infty . \tag{18}$$

PROOF. It is easy to see that

$$\begin{aligned}\sup |\hat{r}_{FG}(x)-\zeta_n(x)| &\leqq \sup [g\bar{G}^{-1}\hat{\bar{F}}(x)]^{-1} |\hat{f}(x)-E\hat{f}(x)| \\ &\quad + \sup E\hat{f}(x)[g\bar{G}^{-1}\hat{\bar{F}}(x)\cdot g\bar{G}^{-1}E\hat{\bar{F}}(x)]^{-1} |g\bar{G}^{-1}\hat{\bar{F}}(x)-g\bar{G}^{-1}(E\hat{\bar{F}}(x))| \\ &\leqq [g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)]^{-1} \sup_x |\hat{f}(x)-E\hat{f}(x)| + [g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1) g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2))]^{-1} \\ &\quad \sup E\hat{f}(x) \sup_x |g\bar{G}^{-1}\hat{\bar{F}}(x)-g\bar{G}^{-1}(E\hat{\bar{F}}(x))| ,\end{aligned}$$

where

$$g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2)) \to g\bar{G}^{-1}\bar{F}(\alpha)>0 \quad \text{as} \quad n\to\infty .$$

Now, the first term in the above upper bound converges to 0 w. p. 1, as $n\to\infty$, from Theorem 1 of Nadaraya (1965), while the second term, upon using Taylor expansion and the boundedness of $(g\bar{G}^{-1})'$, is bounded above by

$$C_3[g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1) g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2))]^{-1} \sup_x E\hat{f}(x) \cdot \sup_x |\hat{\bar{F}}(x)-E\hat{\bar{F}}(x)| ,$$

and thus converges to 0 w. p. 1 as $n\to\infty$.

LEMMA 2. *The function* $\zeta_n(x)$ *defined in Lemma* 1 *is uniformly continuous on, for all* n *sufficiently large.*

PROOF. Given $\varepsilon>0$, we wish to show that there exists a $\delta>0$, such that for all x, y, $|x-y|<\delta$ implies that $|\zeta_n(x)-\zeta_n(y)|<\varepsilon$ for n sufficiently large.

$$\begin{aligned}|\zeta_n(x)-\zeta_n(y)| &\leqq [g\bar{G}^{-1}[E\hat{\bar{F}}(x))]^{-1} |E\hat{f}(x)-E\hat{f}(y)| \\ &\quad + Ef(y) |[g\bar{G}^{-1}(E\hat{\bar{F}}(x))]^{-1}-[g\bar{G}^{-1}\bar{F}(y)]^{-1}| \\ &\leqq [g\bar{G}^{-1}E\hat{\bar{F}}(\alpha_2)]^{-1} |E\hat{f}(x)-E\hat{f}(y)| \\ &\quad + E\hat{f}(y)[g\bar{G}^{-1}[E(\hat{\bar{F}}(\alpha_2))]^{-2} |g\bar{G}^{-1}(E\hat{\bar{F}}(x))-g\bar{G}^{-1}(E\hat{\bar{F}}(y))| \\ &\leqq [g\bar{G}^{-1}E\hat{\bar{F}}(\alpha_2)]^{-1} |E\hat{f}(x)-E\hat{f}(y)| \\ &\quad + C_4 E\hat{f}(y)[g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2))]^{-2} |E\hat{\bar{F}}(x)-E\hat{\bar{F}}(y)| .\end{aligned}$$

Since $g\bar{G}^{-1}E\hat{\bar{F}}(\alpha_2)>0$ for n sufficiently large, it suffices to show that for all sufficiently large n, and

$$\varepsilon_1=\varepsilon g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2))/2, \quad \text{and} \quad \varepsilon_2=\varepsilon[g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2))]^2/C_4 \sup_x E\hat{f}(x),$$

$$|E\hat{f}(x)-E\hat{f}(y)|<\varepsilon_1 \quad \text{and} \quad |E\hat{\bar{F}}(x)-E\hat{\bar{F}}(y)|<\varepsilon_2,$$

whenever $|x-y|<\delta$. Using Condition (ii) on $k(u)$, implies that $|k(x)-k(y)|<\varepsilon$, whenever $|x-y|<\delta_1$. Define $\delta=\delta_1 a_n$, so that where $|x-y|<\delta_1$ we have

$$|E\hat{f}(x)-E\hat{f}(y)|\leqq a_n^{-1}\int|k[(x-u)/a_n]-k[(y-u)/a_n]|\,dF(u)<\varepsilon_1.$$

Since $E\hat{\bar{F}}(x)=\int_{-\infty}^{x}E\hat{f}(u)du$, the second part follows from the continuity and boundedness of $E\hat{\bar{F}}(x)$.

LEMMA 3. *The function $q(x)$ satisfying Condition* (10) *is uniformly Continuous on S.*

PROOF. Let $x, y\in S$ with $|x-y|<\delta$ and let $\varepsilon>0$ be given. Then,

$$\begin{aligned}|q(x)-q(y)|&\leqq|q(x)-\zeta_n(x)|+|\zeta_n(x)-\zeta_n(y)|+|q(y)-\zeta_n(y)|\\ &\leqq|\hat{r}_{FG}(x)-q(x)|+|\hat{r}_{FG}(x)-\zeta_n(x)+|\zeta_n(x)-\zeta_n(y)|\\ &\quad+|\hat{r}_{FG}(y)-q(y)|+|\hat{r}_{FG}(y)-\zeta_n(y)|,\\ &\leqq 2\sup|\hat{r}_{FG}(x)-q(x)|+2\sup|\hat{r}_{FG}(x)-\zeta_n(x)|\\ &\quad+|\zeta_n(x)-\zeta_n(y)|.\end{aligned}$$

It follows from Lemma 1 with Condition (10), and Lemma 2, that the last expression is less ε for all n sufficiently large.

LEMMA 4. *If Condition* (10) *is satisfied, then $q(x)=r_{FG}(x)$ for all $x\in S$.*

PROOF. Since for any distribution function $F(x)$, $F'(x)$ exists almost everywhere, and since $g\bar{G}^{-1}$ is uniformly continuous then $r_{FG}(x)=(\bar{G}^{-1}\bar{F}(x))'$ exists almost everywhere. Let $x\in S$ be such that $r_{FG}(x)$ exists.

Then

$$\begin{aligned}|\hat{f}(x)-q(x)g\bar{G}^{-1}\bar{F}(x)|&\leqq g\bar{G}^{-1}\hat{\bar{F}}(\beta_1)|\hat{r}_{FG}(x)-q(x)g\bar{G}^{-1}\bar{F}(x)/g\bar{G}^{-1}\hat{\bar{F}}(x)|\\ &\leqq g\bar{G}^{-1}\hat{\bar{F}}(\beta_1)\{|\hat{r}_{FG}(x)-q(x)|\\ &\quad+g(x)|1-g\bar{G}^{-1}\bar{F}(x)/g\bar{G}^{-1}\hat{\bar{F}}(x)|\}\\ &\leqq g\bar{G}^{-1}\hat{\bar{F}}(\beta_1)\{|\hat{r}_{FG}(x)-q(x)|\\ &\quad+[g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)]^{-1}|g\bar{G}^{-1}\hat{\bar{F}}(x)-g\bar{G}^{-1}\bar{F}(x)|\}.\end{aligned}$$

Thus

$$\begin{aligned}\sup|\hat{f}(x)-q(x)g\bar{G}^{-1}\bar{F}(x)|\leqq g\bar{G}^{-1}\hat{\bar{F}}(\beta_1)\{&\sup|\hat{r}_{FG}(x)-q(x)|+[g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)]^{-1}\\ &\sup q(x)\sup|g\bar{G}^{-1}\hat{\bar{F}}(x)-g\bar{G}^{-1}\bar{F}(x)|\}.\end{aligned}$$

The first term of the above upper bound converges to 0 w. p. 1, as $n\to\infty$ by Condition (10), and since $(g\bar{G}^{-1})'$ is bounded, the second term is bounded above by $C_5[g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1)]^{-1}\sup q(x)\sup_x|\hat{\bar{F}}(x)-\bar{F}(x)|\to 0$ w. p. 1, as $n\to\infty$. Thus Lemma 3.4 of

Schuster (1969) applies and $F'(x)=g(x)g\bar{G}^{-1}\bar{F}(x)$ almost everywhere (with respect to the Lebesgue measure μ, i. e., $r_{FG}(x)=q(x)$ a. e. (μ). This and the fact that $|\xi_n(x)-q(x)|\to 0$ a. e. (μ) implies that $\zeta_n(x)\to r_{FG}(x)$ a. e. (μ). Since $\zeta_n(x)\leqq[g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2)]^{-1}E\hat{f}(x)$ which is integrable with respect to μ and convergence to $[g\bar{G}^{-1}\bar{F}(\alpha)]^{-1}f(x)$ which is also integrable. Thus by the Lebesgue Dominated Convergence theorem, we have

$$\int\xi_n(x)dx\to\int r_{FG}(x)dx\,. \tag{19}$$

On the other hand from Lemma 1 and Condition (10) we have

$$\sup|\zeta_n(x)-q(x)|\to 0 \qquad \text{as} \quad n\to\infty\,.$$

Thus

$$\int_S\zeta_n(x)dx\to\int_S q(x)dx\,, \qquad \text{as} \quad n\to\infty\,. \tag{20}$$

Therefore from (19) and (20), we have

$$\int_S q(x)dx=\int_S r_{FG}(x)dx\,, \tag{21}$$

in the Lebesgue sense. Since $q(x)$ is uniformly continuous on S, equation (21) also hold in Rieman sense. Hence by the fundamental theorem of Calculus, we conclude that $r_{FG}(x)=q(x)$ everywhere for all $x\in S$. This finishes the proof of necessity and the Theorem.

For the proof of Theorem 2, we shall only show how the proof of Theorem 1 can be modified, and thus will be brief whenever possible.

PROOF OF THEOREM 2. Sufficiency. The crucial steps in its proof are the following

$$\sup|\hat{f}(x, y)-f(x, y)|\to 0 \text{ w. p. 1}, \qquad \text{as} \quad n\to\infty\,. \tag{22}$$

and

$$\sup|\hat{\bar{F}}(x, y)-\bar{F}(x, y)|\to 0 \text{ w. p. 1}, \qquad \text{as} \quad n\to\infty\,, \tag{23}$$

after defining

$$g\bar{G}^{-1}\bar{F}(\alpha, \tilde{\alpha})=\inf_{(x,y)\in S^*} g\bar{G}^{-1}\bar{F}(x, y)\,,$$

$$g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1, \tilde{\alpha}_1)=\inf_{(x,y)\in S^*} g\bar{G}^{-1}\hat{\bar{F}}(x, y)\,,$$

$$g\bar{G}^{-1}(E\hat{\bar{F}}(\alpha_2, \alpha_2))=\inf_{(x,y)\in S^*} g\bar{G}^{-1}(E\hat{\bar{F}}(x, y))\,,$$

and

$$g\bar{G}^{-1}\hat{\bar{F}}(\beta_1, \tilde{\beta}_1)=\sup_{(x,y)\in S^*} g\bar{G}^{-1}F(x, y)\,,$$

and noting that $g\bar{G}^{-1}\bar{F}(\alpha'\,\tilde{\alpha})>0$, $g\bar{G}^{-1}\hat{\bar{F}}(\alpha_1, \alpha_1)\to g\bar{G}^{-1}\bar{F}(\alpha, \tilde{\alpha})$ w. p. 1 as $n\to\infty$, and $g\bar{G}^{-1}E\hat{\bar{F}}(\alpha_2, \alpha_2)\to g\bar{G}^{-1}\bar{F}(\alpha, \tilde{\alpha})$, as $n\to\infty$. Using (22) and (23) with the above quantities the proof of sufficiency proceed as that of Theorem 1. Since (22) is given by Nadaraya (1970), we shall only show (23). Since it can be easily seen that $\sup|E\hat{\bar{F}}(x, y)-\bar{F}(x, y)|\to 0$, as $n\to\infty$, and

$$\sup|\hat{\bar{F}}(x, y)-E\hat{\bar{F}}(x, y)|=a_n^{-1}\sup\left|\iint\bar{K}_1[(x-y)/a_n]\bar{K}_2[(g-v)/a_n]dH_n(u, v)\right.$$
$$\left.-\iint\bar{K}_1[(x-y)/a_n]\bar{K}_2[(y-v)/a_n]dH(u, v)\right|$$
$$=a_n^{-4}\left|\iint[H_n(u, v)-H(u, v)]k_1[(x-y)/a_n]k_2[(y-v)/a_n]dudv\right|$$
$$\leqq a_n^{-2}\sup|H_n(x, y)-H(x, y)|.$$

Hence using a result of Kiefer and Wolfowitz (1958), we have

$$\rho[\sup|\hat{\bar{F}}(x, y)-E\bar{F}(x, y)|\geqq\delta]\leqq P[\sup|H_n(x, y)-H(x, y)|\geqq\varepsilon a_n^2]$$
$$\leqq\exp(-C_1na_n^4).$$

The result follows in view of Boel-Cantelli lemma and Condition (vi).

Necessity. Will be given in four lemmas analogous to those of Theorem 1.

LEMMA 5. *Let*

$$\eta_n(x, y)=E\hat{f}(x, y)/g\bar{G}^{-1}(E\hat{\bar{F}}(x, y)),$$

then

$$\sup|\hat{\rho}_{HG}(x, y)-\eta_n(x, y)|\to 0 \text{ w. p. 1} \quad as \quad n\to\infty. \tag{24}$$

PROOF. Proceed analogous to this of Lemma 1 except we use Theorem 2 of Nadaraya (1970) for this case.

LEMMA 6. *The function $\eta_n(x, y)$ defined in Lemma* 5 *is uniformly continuous on S^*, for all n sufficiently large.*

PROOF. Proceeds exactly as the analogous Lemma 2.

LEMMA 7. *The function $p(x, y)$ satisfying Condition* (17) *is uniformly continuous on S^*.*

PROOF. Again analogous to this of Lemma 3.

LEMMA 8. *If Condition* (17) *is satisfied, then $p(x, y)=\rho_{HG}(x, y)$ for all $x, y\in S^*$.*

PROOF. Proceeds as Lemma 4 except we use the result of Samanta (1973) to conclude that $(\partial^2F(x, y)/\partial x\partial y)=p(x, y)g\bar{G}^{-1}\bar{F}(x, y)$ almost everywhere.

References

[1] BARLOW, R. E. and Van ZWET, W.: *Asymptotic properties of isotonic regression estimates for the generalized failure function. Part I. Strong Consistency*, Proc. of the First Int. Symp. on Nonparametric Statistical Inference. M. L. Puri (ed), Cambridge University Press (1970).

[2] BASU, A. P.: *Bivariate failure Rate.* J. Amer. Statist. Assoc. **61**, 103-105 (1971).

[3] KIEFER, J. and WOLFOWITZ, J.: *On the deviation of the empiric distribution of vector of chance variables.* Trans. Amer. Math. Soc. **87**, 173-186 (1958).

[4] NADARAYA, E. A.: *On nonparametric estimates of density function and regression curve.* Theory Prob. Appl. **10**, 186-190 (1965).

[5] NADARAYA, E. A.: *Remarks on nonparametric estimates for donsity functions and regression curves.* Theory Prob. Appl. **15**, 134-136 (1970).

[6] PARZEN, E.: *On estimation of a probability density and mode.* Ann. Math. Statist. **33**, 1065-1076 (1962).

[7] SAMANTA, M.: *A note on uniform Strong Convergence of bivariate density estimates.* Z. Wahrscheinlichkeitstheorie verw. Geb. **28**, 85-88 (1973).