SOME NONPARAMETRIC TESTS BASED ON THE ORDER STRATIFICATION METHOD FOR THE TWO-SAMPLE PROBLEM

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https://doi.org/10.5109/13102

出版情報：統計数理研究. 17 (1/2), pp.33-47, 1976-03. Research Association of Statistical Sciences
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SOME NONPARAMETRIC TESTS BASED ON THE ORDER STRATIFICATION METHOD FOR THE TWO-SAMPLE PROBLEM

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(Received February 10, 1975)

§ 0. Summary.

In this paper several rank order statistics for the univariate two-sample testing problem are proposed and their asymptotic relative efficiencies (ARE's) w. r. t. the Wilcoxon test in the location problem and also w. r. t. the Mood test in the scale problem for important symmetric distributions are investigated. The ARE's of our tests w. r. t. the Wilcoxon test for the exponential and the gamma alternatives are also investigated in the scale case. Our statistics are constructed with the aid of the idea of McIntyre [5] and of Takahashi and Wakimoto [9], which may be called the order stratification method. When the order of the magnitude of the observations between a small member is intuitively found without measurement by e. g. a visual inspection and the measurement is costly, our test statistics are available and we can conclude that the tests based on only a half or a third of the samples have high ARE's. Furthermore even if we use all the samples our method is powerful especially in the scale and the scale slippage model.

§ 1. Introduction and basic theory of multivariate rank order statistics.

Let $X_1, \cdots, X_{km}$ and $Y_1, \cdots, Y_{kn}$ be independent variables with cdf's $F(x)$ and $G(x)$ having density functions $f(x)$ and $g(x)$ respectively. Here k is a fixed positive integer. Let $N=m+n$, $\lambda=m/N$ and assume that there exists a positive number λ_0 such that $\lambda_0 \leqq \lambda \leqq 1-\lambda_0$. To test the hypothesis $H: F=G$, there are available many rank order statistics such as Wilcoxon, normal score etc. for location alternatives and Mood, Freund and Ansari, Siegel and Tukey, Klotz normal score etc. for scale alternatives. All these statistics are based on the rank scores of the X_i's among the pooled sample of size kN. Each of them is asymptotically optimal for some parametric family. However, they are not optimal for other families. Thus, we feel that there remains room to improve them by utilizing other informations. Furthermore there are many practical cases where the order of the magnitude between a small member of the samples can be found without measurement. In such a case it is desirable to utilize

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this information to save the measurement cost. This paper applies an artificial stratification of the samples by means of ordering to the nonparametric problem on the two-sample case.

Let $X_{i,1} \leqq \cdots \leqq X_{i,k}$ be the order statistic of $X_{(i-1)k+1}, \cdots, X_{ik}$ and put $\boldsymbol{X}_i=(X_{i,1}, \cdots, X_{i,k})$ for $i=1, \cdots, m$. Then $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_m$ is a random sample of size m from a k-variate population with a density function $\tilde{f}(\boldsymbol{x})=k!\,\mathrm{I}(x_1 \leqq \cdots \leqq x_k)\prod_{i=1}^{k} f(x_i)$, I denoting the indicator function. The random vectors $\boldsymbol{Y}_i=(Y_{i,1}, \cdots, Y_{i,k})$, $i=1, \cdots, n$ are similarly defined and $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_n$ is a random sample of size n from a population with a density function $\tilde{g}(\boldsymbol{y})=k!\,\mathrm{I}(y_1 \leqq \cdots \leqq y_k)\prod_{i=1}^{k} g(y_i)$. Then the null hypothesis H is equivalent to the hypothesis $\tilde{H}: F_j=G_j$ for all $j=1, \cdots, k$ where F_j and G_j stand for the j-th marginal cdf's of $\tilde{f}$ and $\tilde{g}$ respectively. Moreover the location alternative $K_1(\theta): G(x)=F(x-\theta)$ is equivalent to the alternative $\tilde{K}_1(\theta): G_j(x)=F_j(x-\theta)$, $j=1, \cdots, k$ while the scale alternative $K_2(\theta): G(x)=F(e^{-\theta}x)$ is equivalent to $\tilde{K}_2(\theta): G_j(x)=F_j(e^{-\theta}x)$, $j=1, \cdots, k$. Thus, to test H against $K_1(\theta)$ $(K_2(\theta))$ we may consider the tests based on the sets $W_j=\{X_{1,j}, \cdots, X_{m,j}, Y_{1,j}, \cdots, Y_{n,j}\}$, $j \in A$ where A is a subset of $(1, \cdots, k)$. In this paper we deal with only the one-sided alternative $\theta>0$ and a test is said to be based on S if it has a critical region of the form $S \leqq c$.

Let us propose multivariate rank order statistic $\boldsymbol{S}_N=(S_{N1}, \cdots, S_{Nk})$,

$$S_{Nj}=\sum_{i=1}^{m} a_{Nj}(R_{Nij}) \tag{1.1}$$

where R_{Nij} is the rank of $X_{i,j}$ among the set W_j and $a_{Nj}(1), \cdots, a_{Nj}(N)$ are given constants such that for some nonconstant square integrable function $L_j(u)$,

$$\lim_{N\to\infty}\int_0^1 [L_j(u)-a_{Nj}(1+[uN])]^2 du=0\,, \qquad j=1, \cdots, k\,. \tag{1.2}$$

The tests based on only S_{Nj} should be, if it is powerful, recommended in the cases mentioned before. The reader may wonder the legitimacy of the statement 'tests based on only S_{Ni}, because appearently the whole observations are needed in order to obtain S_{Nj}. However, if a grance of k members may reveal the order of them without measurement we can adopt S_{Nj} without measurement of the samples not used. In [9] Takahashi and Wakimoto considered several circumstances and McIntyre [5] discussed the application of the similar method to the pasture measurement.

In the following sections we shall propose specific statistics in the case $k=2$ or 3 and investigate their ARE's w. r. t. the Wilcoxon or the Mood test. Throughout this paper ARE is conceived in the Hájek and Šidák's sense [4, p. 267].

To calculate the ARE we need asymptotic theory. Patel [6] proved the asymptotic normality of the multivariate linear rank statistics under H and also under the contiguous regression alternatives. Here the asymptotic normality under the contiguous scale alternatives is also required. So we give Theorem 1.1 and 1.2 below to be used in the two-sample problem.

Let $h(\boldsymbol{x}\,;\boldsymbol{\theta})=h(x_1, \cdots, x_k\,;\theta_1, \cdots, \theta_r)$ be a k-variate r-parameter density function and let H_j be the j-th marginal cdf of $h(\boldsymbol{x}\,;0)$. We need the following notations and Assumptions.

$$\sigma_j^2=\sigma_{jj}=\int_0^1[L_j(u)-\bar{L}_j]^2du\,, \qquad \bar{L}_j=\int_0^1 L_j(u)du\,,$$

(1.3) $$\sigma_{ij}=\int_0^1\int_0^1[L_i(u)-\bar{L}_i][L_j(v)-\bar{L}_j]dP(H_i(X_i)\leqq u,\ H_j(X_j)\leqq v)$$

where $(X_1, \cdots, X_k)$ is distributed according to $h(\boldsymbol{x}\,;\boldsymbol{0})$,

$$\Sigma=(\gamma_{ij}) \qquad \text{where} \quad \gamma_{ij}=\sigma_{ij}/\sigma_i\sigma_j\,, \quad i, j=1, \cdots, k\,.$$

Assumption 1. Σ is positive definite.
Assumption 2. There exist

(1.4) $$\psi_j(\boldsymbol{x}\,;\boldsymbol{\theta})=(\partial/\partial\theta_j)h(\boldsymbol{x}\,;\boldsymbol{\theta}) \qquad j=1, \cdots, r$$

satisfying the following two conditions:

(1.5) $$\lim_{\|\theta\|\to 0}\int|h_j(\boldsymbol{x}\,;\boldsymbol{\theta})|d\boldsymbol{x}=\int|h_j(\boldsymbol{x}\,;\boldsymbol{0})|d\boldsymbol{x}\,,$$

(1.6) $$\lim_{\|\theta\|,\,\|\theta'\|\to 0}\int|h_i(\boldsymbol{x}\,;\boldsymbol{\theta})h_j(\boldsymbol{x}\,;\boldsymbol{\theta}')|/\sqrt{h(\boldsymbol{x}\,;\boldsymbol{\theta})h(\boldsymbol{x}\,;\boldsymbol{\theta}')}d\boldsymbol{x}$$

$$=\int|h_i(\boldsymbol{x}\,;\boldsymbol{0})h_j(\boldsymbol{x}\,;\boldsymbol{0})|/h(\boldsymbol{x}\,;\boldsymbol{0})d\boldsymbol{x}\,.$$

Let $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_m$ be a sample from $h(\boldsymbol{x}\,;\boldsymbol{0})$ and let $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_n$ be a sample from $h(\boldsymbol{x}\,;N^{-\frac{1}{2}}\boldsymbol{\theta}_0)$ for a certain specified vector $\boldsymbol{\theta}_0=(\theta_{01}, \cdots, \theta_{0r})$ where $N=m+n$. Furthermore let us normalize (1.1) and denote it by $\boldsymbol{T}_N=(T_{N1}, \cdots, T_{Nk})$,

(1.7) $$T_{Nj}=(var_0 S_{Nj})^{-\frac{1}{2}}(S_{Nj}-E_0 S_{Nj}) \qquad j=1, \cdots, k\,,$$

where E_0 and var_0 are performed under $h(\boldsymbol{x}\,;\boldsymbol{0})$. We give the following theorems.

THEOREM 1.1. *Under the above condition, the asymptotic normality*

(1.8) $$\boldsymbol{T}_N\sim N(\boldsymbol{0}, \Sigma)$$

holds under H where the sign $\sim$ denotes the asymptotic equivalency in law.

THEOREM 1.2. *Under the above conditions, the asymptotic normality*

(1.9) $$\boldsymbol{T}_N\sim N(\boldsymbol{\mu}, \Sigma)$$

holds for $\boldsymbol{\theta}_0$. Here $\boldsymbol{\mu}=(\mu_1, \cdots, \mu_k)$ is defined as

(1.10) $$\mu_j=-\sqrt{\lambda(1-\lambda)}\sigma_j^{-1}\sum_{i=1}^{r}\theta_{0i}\int L_j(u_j)\varphi_i(u_1, \cdots, u_k)$$

$$dP(H_\alpha(X_\alpha)\leqq u_\alpha,\ \alpha=1, \cdots, k)$$

where $(X_1, \cdots, X_k)$ is distributed according to $h(\boldsymbol{x}\,;\boldsymbol{0})$ and where

(1.11) $$\varphi_i(u_1, \cdots, u_k)=-\psi_i(H_1^{-1}(u_1), \cdots, H_k^{-1}(u_k)\,;\boldsymbol{0})/h(H_1^{-1}(u_1), \cdots, H_k^{-1}(u_k)\,;\boldsymbol{0})\,,$$

$$i=1, \cdots, r\,.$$

A proof of Theorem 1.1 was given by Patel [6] and also by the auther [8]. A proof of Theorem 1.2 was given by the auther [8] and is similar to that of the location case given by Patel [6]. The proofs are based on the contiguity of $h(\boldsymbol{x}\,;N^{-\frac{1}{2}}\boldsymbol{\theta}_0)$ to

$h(\boldsymbol{x}\,;\boldsymbol{0})$ and the multivariate version of LeCam's lemma. We also needed the convergence theorems in Hájek and Šidák [4, p. 64 and 154] which was guaranteed by (1.5) and (1.6).

For our order stratification it holds that

$$\varphi_j(u_1, \cdots, u_k) = \varphi(u_j\,; f) \equiv -f'(F^{-1}(u_j))/f(F^{-1}(u_j)) \qquad j=1, \cdots, k, \tag{1.12}$$

for the location problem and

$$\varphi_j(u_1, \cdots, u_k) = \varphi_1(u_j\,; f) \equiv -1 - F^{-1}(u_j) f'(F^{-1}(u_j))/f(F^{-1}(u_j)) \qquad j=1, \cdots, k, \tag{1.13}$$

for the scale problem since in our situation we can interpret as $r=k$ and $\boldsymbol{\theta}=(\theta, \cdots, \theta)$. Thus, in our situation

$$\begin{aligned}\sigma_{ij} &= \int_0^1 \int_0^v L_i\Big(\sum_{d=i}^{k} \binom{k}{d} u^d (1-u)^{k-d}\Big) L_j\Big(\sum_{d=j}^{k} \binom{k}{d} v^d (1-v)^{k-d}\Big) \\ &\qquad d\Big(\sum_{d=i}^{k} \sum_{c=\max(j-d,0)}^{k-d} \frac{k!}{d!\,c!\,(k-c-d)!} u^d (v-u)^c (1-v)^{k-c-d}\Big) - \bar{L}_i \bar{L}_i \\ &\equiv \int_0^1 \int_0^v L_i(A_i(u)) L_j(A_j(v)) dA_{ij} - \bar{L}_i \bar{L}_j, \qquad \text{say},\end{aligned} \tag{1.14}$$

for $i<j$ and

$$\begin{aligned}\mu_j &= -\theta \sqrt{\lambda(1-\lambda)}\,\sigma_j^{-1} \Big[\sum_{i=1}^{j-1} \int_0^1 \int_0^u L_j(A_j(u)) \varphi(v\,; f) dA_{ij}(u, v) \\ &\quad + \int_0^1 L_j(A_j(u)) \varphi(u\,; f) dA_j(u) + \sum_{i=j+1}^{k} \int_0^1 \int_0^v L_j(A_j(u)) \varphi(v\,; f) dA_{ji}(u, v)\end{aligned} \tag{1.15}$$

for the alternative $K_1(N^{-\frac{1}{2}}\theta)$. For $K_2(N^{-\frac{1}{2}}\theta)$, we may replace $\varphi(u\,; f)$ in (1.15) with $\varphi_1(u\,; f)$ defined in (1.13).

It should be noted that $\boldsymbol{T}_N$ is distribution-free under H, since

$$P(\boldsymbol{T}=\boldsymbol{t}) = 2^k \int_{(*)} \prod_{j=1}^{k} \prod_{i=1}^{N} du_{ij} \quad \text{where} \quad (*) = \{\boldsymbol{T}=\boldsymbol{t},\ u_{i1} \leqq \cdots \leqq u_{ik},\ i=1, \cdots, N\}.$$

If F is symmetric, it seems convenient to choose score functions satisfying symmetric property in some sense. Thus, we give the following theorems.

THEOREM 1.3. *If* $\tilde{L}_i(u)\tilde{L}_j(v) = \tilde{L}_{k+1-i}(1-u)\tilde{L}_{k+1-j}(1-v)$ *where* $\tilde{L}_i(u) = L_i(u) - \bar{L}_i$, *then*

$$\sigma_{ij} = \sigma_{k+1-i,k+1-j} \qquad i, j=1, \cdots, k. \tag{1.16}$$

PROOF. Since $A_j(u) = 1 - A_{k+1-j}(1-u)$,

$$\begin{aligned}\sigma_{ij} &= \int_0^1 \int_0^v \tilde{L}_{k+1-i}(A_{k+1-i}(1-u)) \tilde{L}_{k+1-j}(A_{k+1-j}(1-v)) dA_{ij}(u, v) \\ &= \int_0^1 \int_0^u \tilde{L}_{k+1-i}(A_{k+1-i}(u)) \tilde{L}_{k+1-j}(A_{k+1-j}(v)) dA_{ij}(1-u, 1-v).\end{aligned}$$

Let (Z_1, Z_2, Z_3) be a trinomial variable with parameter $(k\,; 1-u, u-v, v)$, then

$$\begin{aligned}&A_{ij}(1-u, 1-v) \\ &= P(i \leqq Z_1 \leqq k,\ j \leqq Z_1 + Z_2 \leqq k) \\ &= P(k+1-j \leqq Z_3,\ Z_2 + Z_3 \geqq k-i+1) + P(Z_2 + Z_3 \leqq k-i) - P(k-j+1 \leqq Z_3).\end{aligned} \tag{1.17}$$

The second and the third terms in the right hand side of (1.17) are independent of u, while the first terms is $A_{k+1-i,k+1-j}(v, u)$. Thus, we have $dA_{ij}(1-u, 1-v)= dA_{k+1-i,k+1-j}(v, u)$ which yields the theorem.

THEOREM 1.4. *If* $\varphi(u)$ *which is defined by either* (1.12) *or* (1.13) *satisfies*

$$\int_0^1 \varphi(u)du=0 \tag{1.18}$$

and

$$\tilde{L}_j(u)\varphi(v)=\tilde{L}_{k+1-j}(1-u)\varphi(1-v)\,, \tag{1.19}$$

then it holds that $\mu_j=\mu_{k+1-j}$, $j=1, \cdots, k$.

PROOF. Write (1.15) as

$$\mu_j=-\theta\sqrt{\lambda(1-\lambda)}\sigma_j^{-1}\sum_{i=1}^{k}\mu_{ji}\,.$$

Then similarly as in the proof of Theorem 1.3 we have

$$\mu_{ji}=\mu_{k+1-j,k+1-i} \qquad i, j=1, \cdots, k\,.$$

This implies the conclusion.

From the form of the asymptotic distributions, we should adopt a linear combination of $T_{N1}, \cdots, T_{Nk}$ as a test statistic when we measure all the samples and use $\boldsymbol{T}_N$.

§ 2. A statistic for the location alternatives when $k=2$.

In this section we consider only symmetric distributions with $\varphi(u\,;f)$ satisfying (1.18). The proposing statistic is $\boldsymbol{S}_1=\left(\sum_{i=1}^{m}R_{i1}, \sum_{i=1}^{m}R_{i2}\right)$. Normalizing $\boldsymbol{S}_1$, we have $\boldsymbol{T}_1=(T_1, T_2)$ where

$$T_j=\left[\frac{1}{12}mn(N+1)\right]^{-\frac{1}{2}}\left[\sum_{i=1}^{m}R_{ij}-\frac{1}{2}m(N+1)\right] \qquad j=1, 2\,.$$

In this case $L_1(u)=L_2(u)=u$. Computing (1.14) and (1.15) we have

$$\begin{aligned}
&\boldsymbol{T}_1\sim N\left[\boldsymbol{0},\ \Sigma=\begin{bmatrix}1 & \frac{7}{15}\\ \frac{7}{15} & 1\end{bmatrix}\right] && \text{under } H\,,\\
&\boldsymbol{T}_1\sim N(\boldsymbol{\mu}, \Sigma) && \text{under } K_1(N^{-\frac{1}{2}}\theta)
\end{aligned} \tag{2.1}$$

where $\boldsymbol{\mu}=(\mu_1, \mu_2)$,

$$\mu_1=\mu_2=-\frac{8}{3}\theta\sqrt{3\lambda(1-\lambda)}\int_0^1 u^3\varphi(u\,;f)du\,. \tag{2.2}$$

Let W denote the usual normalized Wilcoxon statistic which is obtained by replacing m, N and R_{ij} with $2m, 2N$ and R_i: the rank of X_i among the pooled sample, respectively. It holds that

$$W \sim N(0, 1) \qquad \text{under } H,$$

(2.3)

$$W \sim N\left(-\theta\sqrt{24\lambda(1-\lambda)}\int_0^1 u\varphi(u\,;f)du,\, 1\right) \qquad \text{under } K_1(N^{-\frac{1}{2}}\theta).$$

We compare T_1 with W. It seems unfair to compare T_1 with W, since the number of the samples to calculate them are different. However they are calculated based on the same samples. From (2.2) we should take T_1+T_2 as a test statistic if we use all the samples for symmetric distributions. From (2.1), (2.2) and (2.3) we have

$$e(T_1, W) = e(T_2, W) = \frac{8}{9}\left[\int u^3\varphi(u\,;f)du \Big/ \int u\varphi(u\,;f)du\right]^2$$

$$= \frac{8}{9}J_1, \qquad \text{say},$$

and

$$e(T_1+T_2, W) = \frac{15}{11}e(T_1, W).$$

The bound of the ARE above is given by the following theorem.

THEOREM 2.1. *If* $\varphi(u\,;f)<0$ *for* $u<\frac{1}{2}$, *then* $\frac{9}{16} \leqq J_1 \leqq 1$.

PROOF. Let us consider $\int (u^3-au)\varphi(u\,;f)du$ and put $g(u)=u^3-au$. Then $\bar{g}(u) \equiv g(u)-g(1-u) = 2u^3-3u^2-(2a-3)u+a-1$ is always not smaller than zero if $a \geqq 1$ and is larger than zero if $a \leqq \frac{3}{4}$. This means that $\frac{3}{4} \leqq \sqrt{J_1} \leqq 1$, which yields the conclusion.

This theorem implies that, for symmetric unimodal distributions, tests based on only T_1 or T_2 have more than a half of the information as compared with the Wilcoxon test though they use only a half of the samples. Some numerical value's of the ARE are given in Table 1.

Table 1. ARE of the tests based on $\boldsymbol{T_1}$ w.r.t. the Wilcoxon test.

	normal	Cauchy	Logistic	Double exponential
$e(T_1, W)$	.740	.639	.72	.681
$e(T_1+T_2, W)$	1.009	.872	.982	.928

When we use T_2+T_2, it may be efficient to use the exact covariance $cov\,(T_1, T_2)$ which is given by the following theorem.

THEOREM 2.2. *Under H, it holds that*

$$cov\,(T_1, T_2) = (N+1)^{-1}\left(\frac{7}{15}N + \frac{1}{15}\right).$$

PROOF. Let $(X_1, Y_1), \cdots, (X_N, Y_N)$ be an iid sequence. Let R_{i1} be the rank of X_i among $(X_1, \cdots, X_N)$ and let R_{i2} be the rank of Y_i among $(Y_1, \cdots, Y_N)$. Define $u(x)=1$ for $x \geqq 0$ and 0 for $x<0$. Then

$$E(\sum_{i=1}^{m} R_{i1} \sum_{i=1}^{m} R_{i2}) = m(m-1)\Big[-Eu(X_1-X_2)u(Y_1-Y_2)$$
$$-(N-2)Eu(X_1-X_2)u(Y_1-Y_3)+\frac{1}{4}N^2+\frac{3}{4}N\Big]$$
$$+m[(N-1)(N-2)Eu(X_1-X_2)u(Y_1-Y_3)$$
$$+(N-1)Eu(X_1-X_2)u(Y_1-Y_2)+N] .$$

Put

(2.4) $$A=Eu(X_1-X_2)u(Y_1-Y_2), \qquad B=Eu(X_1-X_2)u(Y_1-Y_3).$$

Then the correlation coefficient of $\sum_{i=1}^{m} R_{i1}$ and $\sum_{i=1}^{m} R_{i2}$ is given by

(2.5) $$cor\,(\sum_{i=1}^{m} R_{i1}, \sum_{i=1}^{m} R_{i2}) = 3(N+1)^{-1}[(4B-1)N-(8B-4A-1)] .$$

In our case, by a short calculation we have $A=\frac{1}{3}$ and $B=\frac{13}{45}$. Substitutions of the above values to (2.5) prove the theorem.

§ 3. A statistic for the location alternatives when $k=3$.

The proposing statistic in this section is $\boldsymbol{S}_2=(\sum_{i=1}^{m} R_{i1}, \sum_{i=1}^{m} R_{i2}, \sum_{i=1}^{m} R_{i3})$. Like Section 2 we deal with only symmetric distributions with $\varphi(u\,;f)$ defined by (1.12) satisfying (1.18). Denote the normalized statistic of $\boldsymbol{S}_2$ by $\boldsymbol{T}_2=(T_1, T_2, T_3)$,

$$T_j=\Big[\frac{1}{12}mn(N+1)\Big]^{-\frac{1}{2}}\Big[\sum_{i=1}^{m} R_{ij}-\frac{1}{2}m(N+1)\Big] \qquad j=1, 2, 3 .$$

As in Section 2 we have

(3.1)
$$\boldsymbol{T}_2 \sim N\left(\boldsymbol{0},\ \Sigma=\begin{bmatrix} 1 & \frac{11}{20} & \frac{41}{140} \\ \frac{11}{20} & 1 & \frac{11}{20} \\ \frac{41}{140} & \frac{11}{20} & 1 \end{bmatrix}\right) \qquad \text{under } H,$$
$$\boldsymbol{T}_2 \sim N(\boldsymbol{\mu}, \Sigma) \qquad \text{under } K_1(N^{-\frac{1}{2}}\theta),$$

where $\boldsymbol{\mu}=(\mu_1, \mu_2, \mu_3)$,

(3.2) $$\mu_1=\mu_3=-\frac{18}{5}\theta\sqrt{3\lambda(1-\lambda)}\int_0^1 u^5\varphi(u\,;f)du$$

and

(3.3) $$\mu_2=-\theta\sqrt{3\lambda(1-\lambda)}\int_0^1\Big(\frac{72}{5}u^5-48u^3+36u\Big)\varphi(u\,;f)du .$$

The normalized Wilcoxon statistic W satisfies

(3.4)
$$W \sim N(0, 1) \qquad \text{under } H,$$
$$W \sim N\Big(-6\theta\sqrt{\lambda(1-\lambda)}\int_0^1 u\varphi(u\,;f)du,\ 1\Big) \qquad \text{under } K_1(N^{-\frac{1}{2}}\theta) .$$

It follows that

$$e(T_1, W)=e(T_3, W)=\frac{27}{25}\left[\int_0^1 u^5\varphi(u\,;f)du\Big/\int_0^1 u\varphi(u\,;f)du\right]^2$$

$$=\frac{27}{25}J_2, \qquad \text{say},$$

and

$$e(T_2, W)=\frac{12}{25}\left[\int_0^1 (6u^5-20u^3+15u)\varphi(u\,;f)du\Big/\int_0^1 u\varphi(u\,;f)du\right]^2$$

$$=\frac{12}{25}J_3, \qquad \text{say}.$$

THEOREM 3.1. *If* $\varphi(u\,;f)<0$ *for* $u<\frac{1}{2}$, *then* $\frac{25}{256}\leqq J_2\leqq 1$, $1\leqq J_3\leqq\frac{225}{64}$ *and* $1\leqq J_3/J_2\leqq 36$.

The proof of this theorem goes along the same line as Theorem 2.1 and is omitted. By this theorem $\frac{57}{256}\leqq e(T_1, W)=e(T_3, W)\leqq\frac{27}{25}$, $\frac{12}{25}\leqq e(T_2, W)\leqq\frac{27}{16}$ and $\frac{12}{27}\leqq e(T_2, T_1)\leqq 16$. Thus, when the underlying distribution is symmetric unimodal, T_2 which uses only a third of the sample, contains asymptotically more than 48 percent of the information as compared with the Wilcoxon test. Though T_2 contains tolerable information, we can not declare that T_2 is always better than T_1 and T_3. However, Table 2 will tell us that T_2 is more preferable than T_1 and T_3 for well known distributions.

Table 2. ARE of the tests based on $\boldsymbol{T_2}$ w.r.t. the Wilcoxon test.

	normal	Cauchy	Logistic	Double exponential	$\frac{1}{2}(1+\|x\|)^{-2}$
$e(T_1, W)$	.605	.351	.551	.450	.280
$e(T_2, W)$	.746	1.026	.793	.908	1.159
$e(T_1+T_2+T_3, W)$	1.012	.835	.975	.910	.787
$e(B, W)$	1.013	1.030	.979	.971	1.170

When we are to use all the samples by adopting a linear combination of T_1, T_2 and T_3, occurs a question of finding the best weighted statistic B with the ARE $e(B, W)$. Though the best weight depend on F, some experiences on examples indicates that $e(B, W)$ is close to $e(T_2, W)$ when the latter is large enough and to $e(T_1+T_2+T_3, W)$ otherwise, as is shown in Table 2. Another interesting weight is the weight which gives the most stringent test considered by Schaafsma [7]. When the ratio of the weight of T_1 or T_3 to T_2 is $\frac{1}{2}(6\sqrt{121.5}-\sqrt{349})/(\sqrt{349}-\sqrt{121.5})\fallingdotseq 3.098$, the asymptotically most stringent test for symmetric unimodal distributions is given.

Exact values of the covariances of T_1, T_2 and T_3 are given in the following theorem.

THEOREM 3.2. *It holds that*

$$cov\,(T_1, T_2)=cov\,(T_2, T_3)=(N+1)^{-1}\left(\frac{11}{20}N+\frac{1}{10}\right), \tag{3.5}$$

and

$$cov\,(T_1, T_3)=(N+1)^{-1}\left(\frac{41}{140}N+\frac{1}{70}\right). \tag{3.6}$$

PROOF. We may calculate A and B in (2.4). They become $A=\frac{7}{20}$ and $B=\frac{71}{240}$ for $cov(T_1, T_2)$ which give (3.5). They become $A=\frac{3}{10}$ and $B=\frac{481}{1680}$ for $cov(T_1, T_3)$ which give (3.6).

§ 4. Statistics for the scale alternatives.

In this section we consider the problem of testing H against $K_2(N^{-\frac{1}{2}}\theta)$, $\theta>0$ when $k=3$. It seems that $k=2$ is too small to extract the merit of the order stratification method for the scale problem. Usual Mood test is based on the statistic $\sum_{i=1}^{3m}\left[R_i-\frac{3}{2}(N+1)\right]^2$ where R_i is the rank of X_i among the pooled sample. The Mood statistic has the mean $\frac{1}{4}m(9N^2-1)$ and the variance $\frac{1}{20}mn(3N+1)(9N^2-4)$ in our situation. We denote by M the normalized Mood statistic. As in the previous sections we deal with only symmetric distributions with $\varphi_1(u;f)$ defined by (1.13) satisfying (1.18).

Let us propose four statistics $S_\alpha=(S_{1\alpha}, S_2, S_{3\alpha})$, $\alpha=1, 2, 3, 4$ where

$$S_{1\alpha}=\sum_{i=1}^{m}(N+1-R_{i1})^\alpha$$

$$S_2=\sum_{i=1}^{m}\left[R_{i2}-\frac{1}{2}(N+1)\right]^2$$

and

$$S_{3\alpha}=\sum_{i=1}^{m}R_{i3}^\alpha .$$

Then $L_1(u)=(1-u)^\alpha$, $L_2(u)=\left(u-\frac{1}{2}\right)^2$ and $L_3(u)=u^\alpha$. For the case $\alpha\geqq 5$, we have failed to find better properties than for those $\alpha\leqq 4$, so we deal with only $\alpha=1, 2, 3$ and 4. The exact means and variances under H are given as follows,

$$ES_2=\frac{1}{12}m(N^2-1), \qquad var\, S_2=\frac{1}{180}mn(N+1)(N^2-4),$$

$$ES_{1\alpha}=ES_{3\alpha}=\frac{1}{2}m(N+1) \qquad \alpha=1,$$

$$=\frac{1}{6}m(N+1)(2N+1) \qquad \alpha=2,$$

$$=\frac{1}{4}mN(N+1)^2 \qquad \alpha=3,$$

$$=\frac{1}{30}m(N+1)(2N+1)(3N^2+3N-1) \qquad \alpha=4,$$

$$var\, S_{1\alpha}=var\, S_{3\alpha}=\frac{1}{12}mn(N+1) \qquad \alpha=1,$$

$$=\frac{1}{180}mn(N+1)(2N+1)(8N+11) \qquad \alpha=2,$$

$$=\frac{1}{336}mn(N+1)(27N^4+84N^3+69N^2-8) \qquad \alpha=3,$$

$$=\frac{1}{900}mn(N+1)(2N+1)(32N^5+119N^4+100N^3-65N^2-62N+31) \qquad \alpha=4.$$

Let $\boldsymbol{T}_\alpha=(T_{1\alpha}, T_2, T_{3\alpha})$ denote the normalized statistic of $\boldsymbol{S}_\alpha$, then calculating (1.14) we have the asymptotic covariance matrix Σ_α of $\boldsymbol{T}_\alpha$:

(4.1)
$$\Sigma_1=\begin{bmatrix} 1 & .1132 & -.2929 \\ .1132 & 1 & .1132 \\ -.2929 & .1132 & 1 \end{bmatrix}, \quad \Sigma_2=\begin{bmatrix} 1 & .1835 & -.2168 \\ .1835 & 1 & .1835 \\ -.2168 & .1835 & 1 \end{bmatrix},$$
$$\Sigma_3=\begin{bmatrix} 1 & .2065 & -.1669 \\ .2065 & 1 & .2065 \\ -.1669 & .2065 & 1 \end{bmatrix}, \quad \Sigma_4=\begin{bmatrix} 1 & .2121 & -.1346 \\ .2121 & 1 & .2121 \\ -.1346 & .2121 & 1 \end{bmatrix}$$

up to the fourth decimal place. Hence from Theorem 1.1 and 1.2,

(4.2)
$$\boldsymbol{T}_\alpha \sim N(\boldsymbol{0}, \Sigma_\alpha) \qquad \text{under } H,$$
$$\boldsymbol{T}_\alpha \sim N(\boldsymbol{\mu}_\alpha, \Sigma_\alpha) \qquad \text{under } K_2(N^{-\frac{1}{2}}\theta),$$

where $\boldsymbol{\mu}_\alpha=(\mu_{1\alpha}, \mu_2, \mu_{3\alpha})$,

(4.3)
$$\mu_{1\alpha}=\mu_{3\alpha}=-\theta\sqrt{(2\alpha+1)\lambda(1-\lambda)}\int_0^1[9(\alpha+1)^{3\alpha+2}/(3\alpha+2)]\varphi_1(u;f)du,$$

and

(4.4)
$$\mu_2=-\theta\sqrt{180\lambda(1-\lambda)}\int_0^1(-18u^8+72u^7-96u^6+36u^5+18u^4-12u^3)\varphi_1(u;f)du.$$

On the other hand we have

(4.5)
$$M \sim N(0,1) \qquad \text{under } H,$$
$$M \sim N\left(-\theta\sqrt{540\lambda(1-\lambda)}\int_0^1\left(u-\frac{1}{2}\right)^2\varphi(u;f)du, 1\right) \qquad \text{under } K_2(N^{-\frac{1}{2}}\theta).$$

We are again faced with a question of how to give weights to each element of $\boldsymbol{T}_\alpha$. The examples in Table 3 indicate that $T_{11}+T_2+T_{31}$ for $\alpha=1$ and $T_{1\alpha}+\frac{1}{2}T_2+T_{3\alpha}$ for $\alpha=2$, 3 and 4 have high ARE's (especially for $\alpha=3$ and 4), though simple enough. Furthermore, since the (1.3) element of each Σ_α is negative, $T_{1\alpha}+T_{3\alpha}$ may be attractive. In fact Table 3 and the covariance matrices (4.1) show that $e(T_{1\alpha}+T_{3\alpha}, M)$ is close to $e(B_\alpha, M)$ where B_α denotes the best weighted $\boldsymbol{T}_\alpha$. There are cases in which $T_{1\alpha}+T_{3\alpha}$ is more preferable than M though only two thirds of the samples are used.

Some results concerned to the bound of the ARE are given in the following theorem.

THEOREM 4.1. *Let us consider the absolutely continuous symmetric distributions with* $\varphi_1(u;f)$ *satisfying* (1.18) *such that* $\int xf^2(x)dx<\infty$, *then* $0\leqq e(T_2, M)\leqq\frac{243}{64}$ *and* $\frac{3}{20}\times 2^{-2(3\alpha+1)}(\alpha+1)^2(2\alpha+1)(3\alpha+1)\leqq e(T_{1\alpha}, M)\leqq\frac{3}{80}(\alpha+1)^2(2\alpha+1)$.

Table 3. ARE of the tests based on T_α w.r.t. the Mood test.

α	$e(T_2, M)$	$e(T_{1\alpha}+T_2+T_{3\alpha}, M)$	$e\left(T_{1\alpha}+\frac{1}{2}T_2+T_{3\alpha}, M\right)$	$e(T_{1\alpha}+T_{3\alpha}, M)$	$e(B, M)$	
1	.331	.859	.869	.699	.879	Normal case
2	.331	.984	1.051	.962	1.052	
3	.331	1.047	1.133	1.073	1.139	
4	.331	1.083	1.173	1.122	1.205	
1	.565	.991		.617	.993	Cauchy case
2	.565	.953		.666	.953	
1	.365	.879	.874	.684	.891	Logistic case
2	.365	.975	1.020	.904	1.020	
3	.365	1.005	1.066	.974	1.066	
4	.365	1.021	1.078	.988	1.078	
1	.379	.889	.878	.680	.899	Double exponential case
2	.379	.978	1.015	.890	1.016	
3	.379	1.009	1.057	.954	1.057	
4	.379	1.020	1.068	.969	1.068	

PROOF. We may consider the bound of

$$\int_0^1 u^\alpha \varphi_1(u\,;f)\,du \Big/ \int_0^1 \left(u-\frac{1}{2}\right)^2 \varphi_1(u\,;f)\,du \qquad \text{for} \quad e(T_{1\alpha}, M)\,. \tag{4.6}$$

To derive the bound of (4.6) we may obtain the range of c for which

$$\int_0^1 \left[u^j - c\left(u-\frac{1}{2}\right)^2\right] \varphi_1(u\,;f)\,du \tag{4.7}$$

is zero for some $\varphi_1(u\,;f)$. Since (4.7) is

$$\int \left[jF^{j-1} - 2c\left(F-\frac{1}{2}\right)\right] xf^2 dx\,,$$

we can obtain $j(j-1)2^{-j+1} \leqq c \leqq \frac{1}{2}k$ which yields the bound of $e(T_{1\alpha}, M)$. For $e(T_2, M)$ we can show similarly.

§ 5. Statistics for the exponential and the gamma alternatives.

Let us consider the problem of testing H against $K_2(N^{-\frac{1}{2}}\theta)$ for the exponential, Weibull and the gamma distribution. Though the parameter here is a scale parameter, it has a similar property to a location parameter. The Mood test is inefficient in this case, see Basu and Woodworth [1]. This is because, if the parameter is large, the observations from the second population have tendency to be large. Thus it will be appropriate to adopt the Wilcoxon test as the standard test. Woinsky [10] showed that the Wilcoxon test has high efficiency for a scale slippage problem. In this section we propose competitors to the Wilcoxon test and investigate their ARE's.

The proposing statistics are $S_{j_1j_2}=(\sum_{i=1}^m R_{i1}^{j_1} \sum_{i=1}^m R_{i2}^{j_2})$ for $k=2$ and $S_{j_1j_2j_3}=(\sum_{i=1}^m R_{i1}^{j_3}, \sum_{i=1}^m R_{i2}^{j_4},$

$\sum_{i=1}^{m} R_{i3}^{j_5})$ for $k=3$. Denote the normalized statistics of $\boldsymbol{S}_{j_1j_2}$ and $\boldsymbol{S}_{j_3j_4j_5}$ by $\boldsymbol{T}_{j_1j_2}=(T_{1i_1}, T_{2j_2})$ and $\boldsymbol{T}_{j_3j_4j_5}=(T_{3j_3}, T_{4j_4}, T_{5j_5})$ respectively. We deal with distributions having $\varphi_1(u;f)$ satisfying (1.18). Exponential, Weibull and gamma distributions are enjoying this property. The score functions of $\boldsymbol{T}_{j_1j_2}$ and $\boldsymbol{T}_{j_3j_4j_5}$ are (u^{j1}, u^{j2}) and (u^{j3}, u^{j4}, u^{j5}) respectively. The limit of the covariance matrix of each $\boldsymbol{T}$ can be obtained by (1.14) and are shown in Table 4-7.

From the results in Section 1, it follows that

$$\begin{aligned} &\boldsymbol{T}_{j_1j_2}\sim N(\boldsymbol{0}, \Sigma_{j_1j_2}) &&\text{under } H, \\ &\boldsymbol{T}_{j_1j_2}\sim N(\boldsymbol{\mu}_{j_1j_2}, \Sigma_{j_1j_2}) &&\text{under } K_2(N^{-\frac{1}{2}}\theta), \end{aligned} \tag{5.1}$$

where $\Sigma_{j_1j_2}$ is the asymptotic covariance matrix of $\boldsymbol{T}_{j_1j_2}$ obtainable from Table 4 and $\boldsymbol{\mu}_{j_1j_2}=(\mu_{1j_1}, \mu_{2j_2})$,

$$\begin{aligned} \mu_{1j_1} &= -4\theta j_1^{-1}(j_1+1)^{-1}\sqrt{(2j_1+1)\lambda(1-\lambda)}\int_0^1 \sum_{i=0}^{j_1}\binom{j_1}{i}(-1)^i i(2i+1)^{-1}(1-u)^{2i+1}\varphi_1(u;f)du\,, \end{aligned} \tag{5.2}$$

$$\mu_{2j_2}=-4\theta(j_2+1)(2j_2+1)^{-1}\sqrt{(2j_2+1)\lambda(1-\lambda)}\int_0^1 u^{2j_2+1}\varphi_1(u;f)du\,. \tag{5.3}$$

It also follows that

$$\begin{aligned} &\boldsymbol{T}_{j_3j_4j_5}\sim N(\boldsymbol{0}, \Sigma_{j_3j_4j_5}) &&\text{under } H, \\ &\boldsymbol{T}_{j_3j_4j_5}\sim N(\boldsymbol{\mu}_{j_3j_4j_5}, \Sigma_{j_3j_4j_5}) &&\text{under } K_2(N^{-\frac{1}{2}}\theta), \end{aligned} \tag{5.4}$$

where $\Sigma_{j_3j_4j_5}$ is the asymptotic covariance matrix of $\boldsymbol{T}_{j_3j_4j_5}$ obtainable from Table 5-7 and $\boldsymbol{\mu}_{j_3j_4j_5}=(\mu_{3j_3}, \mu_{4j_4}, \mu_{5j_5})$,

$$\mu_{3j_3}=-9\theta j_3^{-1}(j_3+1)\sqrt{(2j_3+1)\lambda(1-\lambda)}\int_0^1 \sum_{i=0}^{j_3}\binom{j_3}{i}(-1)^i i(3i+2)^{-1}(1-u)^{3i+2}\varphi_1(u;f)du, \tag{5.5}$$

$$\begin{aligned} \mu_{4j_4} &= -6\theta j_4^{-1}(j_4+1)\sqrt{(2j_4+1)\lambda(1-\lambda)}\int_0^1 \sum_{i=0}^{j_4}\binom{j_4}{i}3^{j_4-i}(-2)^i \\ &\quad\times\Big[\frac{2j_4+i}{2j_4+i+1}u^{2j_4+i+1}-\frac{2j_4+i}{2j_4+i+1}u^{2j_4+i+2}\Big]\varphi_1(u;f)du\,, \end{aligned} \tag{5.6}$$

$$\mu_{5j_5}=-9\theta(j_5+1)(3j_5+2)^{-1}\sqrt{(2j_5+1)\lambda(1-\lambda)}\int_0^1 u^{3j_5+2}\varphi_1(u;f)du\,. \tag{5.7}$$

From (5.3) and (5.7) we can find the bound of $e(T_{2j}, W)$ and $e(T_{5j}, W)$.

THEOREM 5.1. *If $f(x)>0$ for $x>0$ and $f(x)=0$ for $x\leqq 0$ such that φ_1 satisfies* (1.18), *then we have*

$$e(T_{2j}, W)\leqq \frac{2}{3}(j+1)^2(2j+1)\,, \qquad e(T_{5j}, W)\leqq \frac{9}{4}(j+1)^2(2j+1)\,.$$

The proof of this theorem is similar to that of Theorem 4.1 and is omitted.

EXAMPLE 5.1. Exponential distribution: $f(x)=e^{-x}(0)$ for $x>0$ $(\leqq 0)$. Although this is a special case of gamma distributions, it is quite important in many applications. We have $\varphi_1=-1-\log(1-u)$. The locally most powerful rank test is the Savage test

Table 4. $\lim_{N\to\infty} \text{cov}\,(T_{1j_1}, T_{2j_2})$.

j_2 \ j_1	1	2	3	4	5
1	.4667	.4150	.3734	.3412	.3156
2	.4887	.45	.4123	.3809	.3549
3	.4898	.4629	.4303	.4010	.3759
4	.4836	.4665	.4388	.4121	.3883
5	.4748	.4658	.4392	.4182	.3959

Table 5. $\lim_{N\to\infty} \text{cov}\,(T_{3j_3}, T_{4j_4})$.

j_4 \ j_3	1	2	3	4	5
1	.55	.5721	.5701	.5603	.5477
2	.5042	.5451	.5588	.5613	.5586
3	.4633	.5115	.5330	.5427	.5461
4	.4299	.4810	.5066	.5204	.5277

Table 6. $\lim_{N\to\infty} \text{cov}\,(T_{4j_4}, T_{5j_5})$.

j_5 \ j_4	1	2	3	4
1	.55	.5608	.5526	.5391
2	.4929	.5232	.5304	.5290
3	.4452	.4826	.4974	.5026

Table 7. $\lim_{N\to\infty} \text{cov}\,(T_{3j_3}, T_{5j_5})$.

j_5 \ j_3	1	2	3	4	5
1	.2929	.3119	.3169	.3209	.3141
2	.2553	.2775	.2866	.2901	.2909
3	.2276	.2500	.2602	.2653	.2676

$U=(km)^{-1}\sum_{i=1}^{m}\sum_{j=kN-R_i+1}^{kN} j^{-1}$ which has the score function φ_1 above. Recalling that the Wilcoxon test has the score function u, $e(W, U)=0.75$. Thus, the Wilcoxon test is not so inferior to the Savage test. Furthermore, by calculating $\boldsymbol{\mu}_{j_1j_2}$ and $\boldsymbol{\mu}_{j_3j_4j_5}$ we can conclude that our tests constructed from $\boldsymbol{T}_{j_1j_2}$ or $\boldsymbol{T}_{j_3j_4j_5}$ are not so inferior to the Savage test. However it is also the question as in Section 3 and 4 how to give weights to the components of $\boldsymbol{T}_{j_1j_2}$ or $\boldsymbol{T}_{j_3j_4j_5}$. We can find best weight for each statistic separately but it is tedious and yet not applicable, so we give only Table 8 showing $e(T_{ij_i}, W)$'s.

It is noted that T_{2j_2} and T_{5j_5} are quite good. The ARE of any linear combination

Table 8. ARE of the tests based on $\boldsymbol{T}_{j_1j_2}$ or $\boldsymbol{T}_{j_3j_4j_5}$ w.r.t. the Wilcoxon test for the exponential case.

i	1	2	3	4	5
$e(T_{1i}, W)$	.5	.579	.609	.618	.617
$e(T_{2i}, W)$	1.030	1.121	1.124	1.102	1.072
$e(T_{3i}, W)$	.333	.386	.406	.412	.411
$e(T_{4i}, W)$	.730	.792	.798	.786	
$e(T_{5i}, W)$	1.009	1.045	1.024		

of the components of each $\boldsymbol{T}$ can be found from Table 4-8. For example $e(T_{13}+2T_{24}, W)=1.256$. Although the best weight varies with the values of the j_i's, the statistics $T_{1j}+2T_{2j_2}$, $2T_{1j_1}+5T_{2j_2}$ and $T_{3j_3}+2T_{4j_4}+4T_{5j_5}$ seem to be good when we use all the samples. If the underlying distribution is Weibull with the known shape parameter, then the problem is reduced to the exponential case by means of the well known transformation.

EXAMPLE 5.2. Gamma distribution. Let us consider the density function $f_\alpha(x)=\Gamma^{-1}(\alpha)e^{-x}x^{\alpha-1}$ as $x>0$, and $=0$ as $x\leqq 0$ with $\alpha>0$. Then it follows that $\varphi_1(u;f)=F_\alpha^{-1}(u)-\alpha$ where F_α is the cdf of f_α. The locally most powerful rank test L_α has the score function φ_1 above and the ARE of the Wilcoxon test w.r.t. L_α is quite high. The numerical values of

$$e(W, L_\alpha)=12\alpha^{-1}\left[\int u[F_\alpha^{-1}(u)-\alpha]du\right]^2$$

are given by Table 9 where Breiter and Krishnaiah [2] is used. So we can adopt

Table 9. ARE of the Wilcoxon test w.r.t. the locally most powerfull rank test for the gamma case.

α	.5	1.5	2	2.5	3	3.5	4	4.5
$e(W, L_\alpha)$	.608	.75	.811	.844	.865	.879	.889	.897
α	5	5.5	6	6.5	7.5	8.5	9.5	10.5
$e(W, L_\alpha)$	.903	.908	.913	.919	.924	.927	.930	.933

the Wilcoxon test as a standard test. The ARE's of our tests $\boldsymbol{T}_{j_1j_2}$ and $\boldsymbol{T}_{j_3j_4j_5}$ w.r.t. the Wilcoxon test are given by Table 10 for $\alpha=2, 3, 4, 5$ where Gupta [3] is used. The ARE of any linear combination of the components of each $\boldsymbol{T}$ can be found similarly as in Example 5.1. If we use Breiter and Krishnaiah [2], we can add $e(T_{1j_1}, W)$, $e(T_{2j_2}, W)$, $j_1, j_2\leqq 4$, $e(T_{3j_3}, W)$, $e(T_{4j_4}, W)$, $e(T_{5j_5}, W)$, $j_3, j_4, j_5\leqq 2$ for $\alpha=0.5(1)10.5$.

Table 10. ARE of the tests based on $T_{j_1j_2}$ or $T_{j_3j_4j_5}$ w.r.t. the Wilcoxon test for the gamma case.

α	$e(T_{11}, W)$	$e(T_{12}, W)$	$e(T_{13}, W)$	$e(T_{14}, W)$	$e(T_{21}, W)$	$e(T_{22}, W)$
2	.564	.620	.630	.624	.946	.969
3	.595	.638	.639	.626	.905	.908
4	.614	.649	.644	.626	.881	.873
5	.626	.656	.647	.627	.865	.849

α	$e(T_{23}, W)$	$e(T_{24}, W)$	$e(T_{31}, W)$	$e(T_{32}, W)$	$e(T_{33}, W)$	$e(T_{34}, W)$
2	.943	.906	.401	.443	.451	.448
3	.872	.829	.438	.469	.472	.464
4	.831	.786	.456	.486	.484	.473
5	.805	.758	.470	.498	.493	.479

α	$e(T_{41}, W)$	$e(T_{42}, W)$	$e(T_{51}, W)$	$e(T_{52}, W)$
2	.739	.768	.872	.867
3	.742	.756	.817	.797
4	.743	.749	.785	.758
5	.744	.744	.765	.732

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