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ON A CONVERGENCE OF MINIMIZING SEQUENCES FOR UNIFORMLY CONVEX FUNCTIONS

By

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§ 1. Introduction.

In this paper, we shall introduce the notion of a uniformly convex function defined on a normed linear space, which is different from the one introduced by Levitin and Poljak [2]. We shall show that the uniform convexity is a sufficient condition for a convergence of minimizing sequences and for the existence of a unique minimum on a closed convex set in a real Banach space under some additional assumptions.

In Section 2, we shall introduce a uniformly quasi-convex function which is different from that presented by Poljak [3] and investigate the difference between the uniform convexity in our sense and strict quasi-convexity. In Section 3, we shall give some properties of convex and uniformly convex functions. We shall show that every minimizing sequence converges to a unique minimum for a uniformly convex function on a closed convex set in a Banach space under some additional assumptions in Section 4. It is also described that the projection theorem in a Hilbert space falls under the category of nonlinear programming introduced in [5] with the aid of the above results in Section 5.

We can immediately apply this argument to linear optimal control problems with uniformly convex cost criteria (for examples, L_p norm ($p\geqq 2$), integral quadratic form and so on) and to some problems of best approximation theory. This argument makes us possible to treat many optimization problems by a unified approach.

§ 2. Uniformly quasi-convex functions.

In this section, we shall introduce a new notion of uniformly quasi-convex functions and compare them with the other classes of functions.

Let X be a normed linear space over a real number field R and let f be a real-valued function defined on X.

DEFINITION 2.1. A real-valued function f on X is called to be uniformly quasi-convex if for every real number d and every positive number ε, there exists a positive number $\delta=\delta(d, \varepsilon)$ such that

$$f\left(\frac{x+y}{2}\right)\leqq d-\delta \qquad \text{whenever} \quad f(x)\leqq d,\ f(y)\leqq d,\ \|x-y\|\geqq\varepsilon,\ x, y\in X\,.$$

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We shall present some classes of functions in order to clear the relations among them and our uniformly quasi-convex functions.

A function $f(x)$ is called to be uniformly quasi-convex in the sense of Poljak if $f\left(\frac{x+y}{2}\right) \leqq \max\{f(x), f(y)\} - \mu(\|x-y\|)$, where $\mu(\tau)$ is a real function, $\mu(0)=0$ and $\mu(\tau)>0$ for $\tau>0$. Then, it is evident that every uniformly quasi-convex function in the sense of Poljak is a uniformly quasi-convex function in our sense. But the converse is not true: For consider a real function $f(x)=e^x$. This function is uniformly quasi-convex in our sense, but not in the sense of Poljak.

A functional $f(x)$ is called to be strictly quasi-convex if

$$f\left(\frac{x+y}{2}\right) < \max\{f(x), f(y)\} \qquad \text{for all} \quad x, y\ (\neq x) \in X.$$

It is then clear that every uniformly quasi-convex function is strictly quasi-convex. But the converse is not true: For consider the function $f\colon R^2 \to R^1$ defined by

$$f(z)=f(x, y)=e^{x^2+y} \qquad \text{for all} \quad z=(x, y) \in R^2.$$

It is clear that this function is strictly quasi-convex. We shall show that this function is not uniformly quasi-convex in our sense. Let d and ε be arbitrary positive numbers and $w=(u, v)$ and $z=(x, y)$ two points in R^2 satisfying

$$e^{x^2+y}=d, \qquad e^{u^2+v}=d$$

or

$$y=-x^2+\log d, \qquad v=-u^2+\log d,$$

moreover

$$\|w-z\| = \sqrt{(x-u)^2+(y-v)^2} = \varepsilon.$$

Then we have

$$(x-u)^2 = \frac{\varepsilon^2}{1+(x+u)^2}$$

and hence

$$(x-u)^2 \longrightarrow 0 \qquad \text{as} \quad x, u \to \infty.$$

It is valid that

$$f\left(\frac{w+z}{2}\right) = d \cdot e^{-\left(\frac{x-u}{2}\right)^2},$$

so that

$$f\left(\frac{w+z}{2}\right) \longrightarrow d \qquad \text{as} \quad x, u \to \infty,$$

where $f(w)=f(z)=d$, and $\|w-z\|=\varepsilon$. This fact implies that this function $f(z)$ is not uniformly quasi-convex in our sense.

§ 3. Uniformly convex functions.

In this section we shall introduce the notion of uniformly convex functions and investigate the properties of these functions.

DEFINITION 3.1. Let X be a normed linear space. The real-valued function f defined over X is called to be *uniformly convex* if f is uniformly quasi-convex in the sense of Definition 2.1 and if f is convex.

First of all, we shall present some properties of continuous convex functions defined over a normed linear space X. Throughout this section, we shall denote a real-valued convex function defined over a normed linear space X by f unless it is specifically mentioned. We shall presume the following assumption.

ASSUMPTION 1. There exists a real number r_0 such that the set $G_{r_0} \equiv \{x \in X \mid f(x) < r_0\}$ is a non-empty, bounded open set in X.

We now state the properties of a convex function under Assumption 1.

LEMMA 3.1. *Under Assumption* 1, *f is continuous on X.*

PROOF. This is an immediate consequence of Proposition 21 in Bourbaki [1; Ch. 2, § 2, n°10]. Q. E. D.

LEMMA 3.2. *Under Assumption* 1, *choose a vector $x_0 \in X$ such that $f(x_0) < r_0$. Then, for every vector $x \in X$ which is not the origin, there exists a positive number λ_x satisfying $f(x_0+\lambda_x x)=r_0$. Moreover, λ_x is unique to each $x \in X$.*

PROOF. Since the set G_{r_0} is a bounded set in X, there is a positive number b_0 such that $\|x\| \leqq b_0$ for all $x \in G_{r_0}$. Hence, we have

$$\|x\| > b_0 \qquad \text{implies} \quad x \notin G_{r_0} \ (\text{or } f(x) \geqq r_0)\,.$$

Now, it is valid that to each vector $x\ (\neq 0) \in X$,

$$\|x_0+\lambda x\| > b_0 \qquad \text{for sufficiently large } \lambda > 0\,,$$

and, therefore,

$$f(x_0+\lambda x) \geqq r_0\,.$$

Since f is continuous on X and since $f(x_0) < r_0$, by Lemma 3.1, it is true that there is a small number $\lambda > 0$ satisfying $f(x_0+\lambda x) < r_0$. Since $f(x_0+\lambda x)$ is continuous in λ, it follows from the Mean-value Theorem that there is a positive number λ_x such that $f(x_0+\lambda_x x)=r_0$.

We shall show the uniqueness of λ_x. To show a contradiction, suppose that it is not unique, i. e., there are positive numbers λ'_x, λ''_x, such that

$$0 < \lambda'_x < \lambda''_x \quad \text{and} \quad f(x_0+\lambda'_x x) = f(x_0+\lambda''_x x) = r_0\,.$$

It is then true, by the convexity of f, that

$$\begin{aligned} r_0 &= f(x_0+\lambda'_x x) \\ &= f\Big(\Big(1-\frac{\lambda'_x}{\lambda''_x}\Big)x_0 + \frac{\lambda'_x}{\lambda''_x}(x_0+\lambda''_x x)\Big) \\ &\leqq \Big(1-\frac{\lambda'_x}{\lambda''_x}\Big)f(x_0) + \frac{\lambda'_x}{\lambda''_x} f(x_0+\lambda''_x x) \\ &< \Big(1-\frac{\lambda'_x}{\lambda''_x}\Big)r_0 + \frac{\lambda'_x}{\lambda''_x} r_0 = r_0\,, \end{aligned}$$

since $0 < 1-\dfrac{\lambda'_x}{\lambda''_x} < 1$ and $f(x_0) < r_0$. This is a contradiction. Q. E. D.

LEMMA 3.3. *Under Assumption* 1, *we have the following relations. For each vector* $x\ (\neq 0) \in X$,

i) $f(x_0+x) \leqq r_0$ *if and only if* $\lambda_x \geqq 1$,

ii) $f(x_0+x) \geqq r_0$ *if and only if* $0<\lambda_x \leqq 1$.

PROOF. i) "only if" part. For every $x\ (\neq 0)$ satisfying $f(x_0+x) \leqq r_0$, if we suppose that $0<\lambda_x<1$, then we obtain

$$\begin{aligned} r_0 &= f(x_0+\lambda_x x) \\ &= f((1-\lambda_x)x_0+\lambda_x(x_0+x)) \\ &\leqq (1-\lambda_x)f(x_0)+\lambda_x f(x_0+x)<r_0 , \end{aligned}$$

since $r_0>f(x_0)$ and $0<\lambda_x<1$. This is a contradiction.

"if" part. For each $x\ (\neq 0)$ satisfying $\lambda_x \geqq 1$, if we assume that $f(x_0+x)>r_0$, then we have

$$\begin{aligned} r_0<f(x_0+x) &= f\Big(\Big(1-\frac{1}{\lambda_x}\Big)x_0+\frac{1}{\lambda_x}(x_0+\lambda_x x)\Big) \\ &\leqq \Big(1-\frac{1}{\lambda_x}\Big)f(x_0)+\frac{1}{\lambda_x}f(x_0+\lambda_x x) \leqq r_0 . \end{aligned}$$

This is a contradiction.

ii) "only if" part. For every x satisfying $f(x_0+x) \geqq r_0$, if we assume that $\lambda_x>1$, then we obtain

$$\begin{aligned} r_0 \leqq f(x_0+x) &= f\Big(\Big(1-\frac{1}{\lambda_x}\Big)x_0+\frac{1}{\lambda_x}(x_0+\lambda_x x)\Big) \\ &\leqq \Big(1-\frac{1}{\lambda_x}\Big)f(x_0)+\frac{1}{\lambda_x}f(x_0+\lambda_x x) \\ &< \Big(1-\frac{1}{\lambda_x}\Big)r_0+\frac{1}{\lambda_x}r_0=r_0 , \end{aligned}$$

since $0<1-\dfrac{1}{\lambda_x}<1$ and $f(x_0)<r_0$. This is a contradiction.

"if" part. Suppose that $f(x_0+x)<r_0$ for every x satisfying $0<\lambda_x \leqq 1$. It is then valid that

$$\begin{aligned} r_0 &= f(x_0+\lambda_x x) \\ &= f((1-\lambda_x)x_0+\lambda_x(x_0+x)) \\ &\leqq (1-\lambda_x)f(x_0)+\lambda_x f(x_0+x) \\ &< (1-\lambda_x)r_0+\lambda_x r_0=r_0 . \end{aligned}$$

This is a contradiction. Q. E. D.

LEMMA 3.4. *Under Assumption* 1, f *is bounded below on the entire space* X.

PROOF. To show the contradiction, assume that f is not bounded below on X, i. e., there exists a sequence $\{x_n\} \subset G_{r_0}$ such that $f(x_n) \to -\infty$ as $n \to \infty$. Without loss of generality, suppose that $f(x_n)<f(x_0)\ (<r_0)$ for all n. For each x_n defined above, it follows from Lemmas 3.2 and 3.3 that there is a unique positive number $\lambda_{x_0-x_n}$ such that $f(x_n+\lambda_{x_0-x_n}(x_0-x_n))=r_0$ and $\lambda_{x_0-x_n}>1$. If we suppose that $\lambda_{x_0-x_n} \to 1$ as $n \to \infty$, then we have

$$\begin{aligned}\|x_n+\lambda_{x_0-x_n}(x_0-x_n)-x_0\| &= (\lambda_{x_0-x_n}-1)\|x_0-x_n\| \\ &\leqq (\lambda_{x_0-x_n}-1)(\|x_0\|+\|x_n\|) \\ &\leqq (\lambda_{x_0-x_n}-1)2b_0 \longrightarrow 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

and hence

$$x_n+\lambda_{x_0-x_n}(x_0-x_n) \longrightarrow x_0 \quad \text{as} \quad n\to\infty .$$

Since f is continuous by Lemma 3.1, it is clear that

$$r_0=f(x_n+\lambda_{x_0-x_n}(x_0-x_n)) \underset{n\to\infty}{\longrightarrow} f(x_0)<r_0 .$$

This contradiction shows that there exist a positive number ε_1 and a subsequence $\{\lambda_n\}$ of $\{\lambda_{x_0-x_n}\}$ such that $\lambda_n>1+\varepsilon_1$ for all n. It is then true that

$$0<\frac{1}{\lambda_n}<\frac{1}{1+\varepsilon_1}<1, \qquad 1-\frac{1}{\lambda_n}>\frac{\varepsilon_1}{1+\varepsilon_1}>0,$$

$$\begin{aligned}f(x_0) &= f\Big(\Big(1-\frac{1}{\lambda_n}\Big)x_n+\frac{1}{\lambda_n}(x_n+\lambda_n(x_0-x_n))\Big) \\ &\leqq \Big(1-\frac{1}{\lambda_n}\Big)f(x_n)+\frac{1}{\lambda_n}f(x_n+\lambda_n(x_0-x_n)) \\ &= \Big(1-\frac{1}{\lambda_n}\Big)f(x_n)+\frac{1}{\lambda_n}r_0 .\end{aligned}$$

Since $f(x_n)<0$ for all sufficiently large n, it is easily verified that

$$\begin{aligned}f(x_0) &\leqq \Big(1-\frac{1}{\lambda_n}\Big)f(x_n)+\frac{1}{\lambda_n}r_0 \\ &\leqq \frac{\varepsilon_1}{1+\varepsilon_1}f(x_n)+|r_0| \longrightarrow -\infty \quad \text{as} \quad n\to\infty .\end{aligned}$$

This is a contradiction. Q. E. D.

LEMMA 3.5. *Suppose that Assumption* 1 *holds. Let us define the real-valued function* $p(x)$ *on* X *by*

$$p(x)=\begin{cases}\dfrac{1}{\lambda_x} & \text{for} \quad x\neq 0 \\ 0 & \text{for} \quad x=0,\end{cases}$$

where λ_x *is stated in Lemma* 3.2. *Then, the function* $p(x)$ *is sublinear, i.e.,*

$$p(\mu x)=\mu p(x) \qquad \text{for all} \quad \mu\geqq 0,$$

$$p(x+y)\leqq p(x)+p(y) \qquad \text{for all} \quad x, y\in X .$$

PROOF. For every $\mu>0$ and every x $(\neq 0)\in X$, it follows from Lemma 3.2 that there are positive numbers λ_x and $\lambda_{\mu x}$ such that

$$f(x_0+\lambda_x x)=r_0=f(x_0+\lambda_{\mu x}\mu x) .$$

By virtue of uniqueness of λ_x for each x, we have $\lambda_x=\mu\lambda_{\mu x}$, and, therefore, it is valid that

$$p(\mu x)=\frac{1}{\lambda_{\mu x}}=\frac{\mu}{\lambda_x}=\mu p(x) .$$

Consequently, we can conclude that $p(x)$ is positively homogeneous, i. e.,

$$p(\mu x)=\mu p(x) \qquad \text{for all} \quad \mu\geqq 0 \text{ and all } x\in X .$$

We shall show that

$$p(x+y)\leqq p(x)+p(y) \qquad \text{for every} \quad x, y\in X . \tag{3.1}$$

If at least one of three vectors x, y, and $x+y$ is the origin, the inequality (3.1) holds since $p(x)\geqq 0$ for all $x\in X$, and $p(0)=0$. Now, assume that none of x, y and $x+y$ is the origin. It is easily verified that

$$x_0+\frac{\lambda_x\lambda_y}{\lambda_x+\lambda_y}(x+y)=\frac{\lambda_y}{\lambda_x+\lambda_y}(x_0+\lambda_x x)+\frac{\lambda_x}{\lambda_x+\lambda_y}(x_0+\lambda_y y) ,$$

$$\begin{aligned} f\Big(x_0+\frac{\lambda_x\lambda_y}{\lambda_x+\lambda_y}(x+y)\Big)&\leqq\frac{\lambda_y}{\lambda_x+\lambda_y}f(x_0+\lambda_x x)+\frac{\lambda_x}{\lambda_x+\lambda_y}f(x_0+\lambda_y y)\\ &=\frac{\lambda_y}{\lambda_x+\lambda_y}r_0+\frac{\lambda_x}{\lambda_x+\lambda_y}r_0=r_0 . \end{aligned}$$

It follows from Lemma 3.3 that

$$\lambda_{\frac{\lambda_x\lambda_y}{\lambda_x+\lambda_y}(x+y)}\geqq 1 .$$

The positive homogeneity of $p(x)$ implies

$$1\leqq\lambda_{\frac{\lambda_x\lambda_y}{\lambda_x+\lambda_y}(x+y)}=\frac{\lambda_x+\lambda_y}{\lambda_x\lambda_y}\lambda_{x+y} ,$$

or

$$\frac{1}{\lambda_{x+y}}\leqq\frac{1}{\lambda_x}+\frac{1}{\lambda_y} ,$$

and hence $p(x+y)\leqq p(x)+p(y)$. Q. E. D.

LEMMA 3.6. *Suppose that Assumption* 1 *holds. It is then valid that*

i) $f(x_0+x)\leqq r_0$ *if and only if* $0\leqq p(x)\leqq 1$,

ii) $f(x_0+x)\geqq r_0$ *if and only if* $p(x)\geqq 1$.

Note that x may be the origin.

PROOF. This is an immediate consequence of Lemma 3.3 and the definition of $p(x)$. Q. E. D.

REMARK. The above $p(x)$ is compared with the Minkovski functional of the set $\{x\in X\,|\,f(x)\leqq r_0\}-x_0$.

LEMMA 3.7. *Suppose that Assumption* 1 *holds. Then $p(x)$ is continuous on X.*

PROOF. It follows from Lemma 3.6 that $\{x\in X\,|\,p(x)<1\}=G_{r_0}-x_0$. Since G_{r_0} is a non-empty open set, $p(x)$ is continuous on X. Q. E. D.

LEMMA 3.8. *Suppose that Assumption* 1 *holds. If we define the functional $\bar{p}(x)$ by*

$$\bar{p}(x)=p(x)(r_0-f(x_0))+f(x_0) \qquad \textit{for all} \quad x\in X ,$$

then we have

i) $f(x_0+x)\leqq r_0$ *implies* $\bar{p}(x)\geqq f(x_0+x)$,

ii) $f(x_0+x)\geqq r_0$ *implies* $\bar{p}(x)\leqq f(x_0+x)$ *for all* $x\in X$.

PROOF. i) If $x=0$, then it holds by the definition of $\bar{p}(x)$. Suppose that $x\neq 0$. It is easily verified, by Lemma 3.3, that

$$\begin{aligned} f(x_0+x)&=f\Big(\Big(1-\frac{1}{\lambda_x}\Big)x_0+\frac{1}{\lambda_x}(x_0+\lambda_x x)\Big)\\ &\leqq\Big(1-\frac{1}{\lambda_x}\Big)f(x_0)+\frac{1}{\lambda_x}f(x_0+\lambda_x x)\\ &=(1-p(x))f(x_0)+p(x)r_0=\bar{p}(x)\,, \end{aligned}$$

since $\lambda_x\geqq 1$ whenever $f(x_0+x)\leqq r_0$.

ii) It follows from Lemma 3.3 that $0<\lambda_x\leqq 1$ whenever $f(x_0+x)\geqq r_0$. It is then true that

$$\begin{aligned} f(x_0+\lambda_x x)&=f((1-\lambda_x)x_0+\lambda_x(x_0+x))\\ &\leqq(1-\lambda_x)f(x_0)+\lambda_x f(x_0+x) \end{aligned}$$

and hence

$$f(x_0+x)\geqq r_0 p(x)-(p(x)-1)f(x_0)=\bar{p}(x)\,.$$

Q. E. D.

LEMMA 3.9. *Under Assumption* 1, *for every real number* r *such that the set* $G_r\equiv\{x\in X\mid f(x)<r\}$ *is non-empty, the set* G_r *is bounded and open in* X.

PROOF. The openness immediately follows from Lemma 3.1. It is true, by Assumption 1, that there is a positive number b_0 such that $\|x\|\leqq b_0$ for all $x\in G_{r_0}$. For each $r\leqq r_0$, G_r is bounded since G_{r_0} is bounded.

For every $r>r_0$, we shall show that

$$P_r\equiv\{x\in X\mid \bar{p}(x-x_0)<r\}\supset G_r\,, \tag{3.2}$$

where the functional $\bar{p}(x)$ is defined in Lemma 3.8. For every $x\in G_r$ satisfying $f(x)\leqq r_0$, it follows from Lemma 3.6 that $p(x-x_0)\leqq 1$, so that

$$\begin{aligned} \bar{p}(x-x_0)&=p(x-x_0)(r_0-f(x_0))+f(x_0)\\ &\leqq(r_0-f(x_0))+f(x_0)=r_0<r\,, \end{aligned}$$

and hence $x\in P_r$. For every $x\in G_r$ such that $r_0<f(x)<r$, we have, by Lemma 3.8, $\bar{p}(x-x_0)\leqq f(x)<r$, which implies that $x\in P_r$. Therefore, it suffices to show that the set P_r is bounded. For each $x\in P_r$, we have

$$p(x-x_0)<\frac{r-f(x_0)}{r_0-f(x_0)}$$

or

$$p\Big(\frac{r_0-f(x_0)}{r-f(x_0)}(x-x_0)\Big)<1\,.$$

It follows from Lemma 3.6 that

$$f\Big(x_0+\frac{r_0-f(x_0)}{r-f(x_0)}(x-x_0)\Big)<r_0\,,$$

so that

$$\Big\|x_0+\frac{r_0-f(x_0)}{r-f(x_0)}(x-x_0)\Big\|\leqq b_0\,,$$

and hence

$$\frac{r_0-f(x_0)}{r-f(x_0)}\|x\|\leqq b_0+\left(1-\frac{r_0-f(x_0)}{r-f(x_0)}\right)\|x_0\|$$

which implies the boundedness of P_r. Q. E. D.

Under Assumption 1, there is a real number d_0 such that $d_0=\inf_{x\in X} f(x)$ since the convex function $f(x)$ is bounded below by Lemma 3.4. For every $d>d_0$, the set G_d is a non-empty, bounded open set in X by Lemma 3.9. Moreover, it is valid, by Lemma 3.2, that to every $x\in X$ $(x\neq 0)$, there corresponds a unique positive number λ_x such that $f(x_0+\lambda_x x)=d$, where the vector x_0 satisfies $f(x_0)<d$. Similarly, for each $\tilde{d}>d$, there is a unique positive number $\tilde{\lambda}_x$ such that $f(x_0+\tilde{\lambda}_x x)=\tilde{d}$. Then, we can state the following lemmas:

LEMMA 3.10. *Under Assumption* 1, $\tilde{\lambda}_x>\lambda_x$ *for all* $x\in X$ $(x\neq 0)$.

PROOF. To show the contradiction, suppose that $0<\tilde{\lambda}_x\leqq\lambda_x$. It is then true that

$$\begin{aligned}\tilde{d}&=f(x_0+\tilde{\lambda}_x x)\\&=f\left(\left(1-\frac{\tilde{\lambda}_x}{\lambda_x}\right)x_0+\frac{\tilde{\lambda}_x}{\lambda_x}(x_0+\lambda_x x)\right)\\&\leqq\left(1-\frac{\tilde{\lambda}_x}{\lambda_x}\right)f(x_0)+\frac{\tilde{\lambda}_x}{\lambda_x}f(x_0+\lambda_x x)\leqq d\,.\end{aligned}$$

This contradicts to the fact that $\tilde{d}>d$. Q. E. D.

LEMMA 3.11. *Suppose that Assumption* 1 *holds. Then, there is a positive number* $\tilde{c}>0$ *such that* $\tilde{c}>\sup_{\|x\|=1}\tilde{\lambda}_x$.

PROOF. It follows from Lemma 3.9 that $G_{\tilde{d}}$ is bounded in X and hence so is the set $F_{\tilde{d}}\equiv\{x\in X\,|\,f(x)\leqq\tilde{d}\}$, i.e., there is a positive number $\tilde{b}$ such that

$$\|x\|\leqq\tilde{b}\qquad\text{for all}\quad x\in F_{\tilde{d}}\,. \tag{3.3}$$

To show the contradiction, suppose that there is a sequence of vectors $\{x_n\}$ such that $\|x_n\|=1$ for all $n=1,2,\cdots$, and $\tilde{\lambda}_{x_n}\to\infty$ as $n\to\infty$. Since

$$\|x_0+\tilde{\lambda}_{x_n}x_n\|\geqq\|\tilde{\lambda}_{x_n}x_n\|-\|x_0\|=\tilde{\lambda}_{x_n}-\|x_0\|\,,$$

there is a sufficiently large positive integer N such that $\|x_0+\tilde{\lambda}_{x_N}x_N\|>\tilde{b}$. Then, (3.3) implies that $f(x_0+\tilde{\lambda}_{x_N}x_N)>\tilde{d}$, which contradicts to the fact that $\tilde{d}=f(x_0+\tilde{\lambda}_{x_N}x_N)$.

Q. E. D.

LEMMA 3.12. *Let a real-valued function f defined over a linear space X be convex. Then, it is valid that*

$$f(y+\varepsilon(y-x))\geqq f(y)+\varepsilon(f(y)-f(x))\qquad\textit{for all}\quad x,y\in X\ \textit{and all}\ \varepsilon>0\,.$$

PROOF. It is clear that

$$0<\frac{\varepsilon}{1+\varepsilon}\,,\qquad\frac{1}{1+\varepsilon}<1\,,$$

$$y=\frac{\varepsilon}{1+\varepsilon}x+\frac{1}{1+\varepsilon}(y+\varepsilon(y-x))\qquad\text{for all}\quad\varepsilon>0\ \text{and all}\ x,y\in X\,.$$

We obtain

$$f(y)\leqq\frac{\varepsilon}{1+\varepsilon}f(x)+\frac{1}{1+\varepsilon}f(y+\varepsilon(y-x)),$$

which implies

$$f(y+\varepsilon(y-x))\geqq f(y)+\varepsilon(f(y)-f(x)).$$

Q. E. D.

We now introduce another assumption for f.

ASSUMPTION 2. Let us define the set S_t in a normed linear space X by $S_t\equiv\{x\in X\mid \|x\|\leqq t\}$ for each $t>0$. The function $f: X\to R^1$ is bounded above on S_t for all $t>0$.

LEMMA 3.13. *Suppose that Assumptions* 1 *and* 2 *hold. There is a positive number* ν_0 *such that*

$$\inf_{\|x\|=1}(\tilde{\lambda}_x-\lambda_x)\geqq\nu_0 \qquad \textit{whenever} \quad \tilde{d}>d>d_0.$$

PROOF. First of all, note that $\tilde{\lambda}_x>\lambda_x$ for all $x\ (\neq 0)\in X$. To show the contradiction, suppose that $\inf_{\|x\|=1}(\tilde{\lambda}_x-\lambda_x)=0$. Then, there is a sequence $\{x_n\}\subset X$ such that $\|x_n\|=1$ for all $n=1, 2, \cdots$, and $\tilde{\lambda}_{x_n}-\lambda_{x_n}\to 0$ as $n\to\infty$. It follows from Lemma 3.12 that for every positive number ε, we have

$$\begin{aligned}
&f(x_0+\lambda_x x+(1+\varepsilon)(\tilde{\lambda}_x-\lambda_x)x)\\
&\quad\geqq(1+\varepsilon)[f(x_0+\tilde{\lambda}_x x)-f(x_0+\lambda_x x)]+f(x_0+\lambda_x x)\\
&\quad=(1+\varepsilon)(\tilde{d}-d)+d=\tilde{d}+\varepsilon(\tilde{d}-d) \qquad \text{for each} \quad x\ (\neq 0)\in X.
\end{aligned}$$

If we define the positive numbers ε_n, $n=1, 2, \cdots$, by

$$\varepsilon_n\equiv\frac{\tilde{c}-\tilde{\lambda}_{x_n}}{\tilde{\lambda}_{x_n}-\lambda_{x_n}},$$

then $\varepsilon_n\to\infty$ as $n\to\infty$. It is evident that

$$\begin{aligned}
f(x_0+\tilde{c}x_n)&=f\Big(x_0+\lambda_{x_n}x_n+\Big(1+\frac{\tilde{c}-\tilde{\lambda}_{x_n}}{\tilde{\lambda}_{x_n}-\lambda_{x_n}}\Big)(\tilde{\lambda}_{x_n}-\lambda_{x_n})x_n\Big)\\
&\geqq\tilde{d}+\varepsilon_n(\tilde{d}-d)\longrightarrow\infty \qquad \text{as} \quad n\to\infty.
\end{aligned}$$

Since $\|x_0+\tilde{c}x_n\|\leqq\|x_0\|+\tilde{c}$, it is valid, by Assumption 2, that there is a positive number M such that $f(x_0+\tilde{c}x_n)\leqq M$ for all $n=1, 2, \cdots$, which is a contradiction. Q. E. D.

LEMMA 3.14. *Let the real-valued function* $f(x)$ *on* X *be a sublinear function which satisfies*

$$f(x)\geqq 0 \qquad \textit{for all} \quad x\in X,$$

$$f(x)=0 \textit{ if and only if } x=0.$$

Then, the function $f(x)$ *is uniformly convex if and only if for an arbitrary positive number* ε, *there exists a positive number* $\delta=\delta(\varepsilon)$ *such that*

$$f(x)\leqq 1,\quad f(y)\leqq 1,\quad \|x-y\|\geqq\varepsilon \quad \textit{implies} \quad f\Big(\frac{x+y}{2}\Big)\leqq 1-\delta. \tag{3.4}$$

PROOF. It is clear that every uniformly convex function $f(x)$ satisfies the condition (3.4). We shall show the contrary. There is no vector x satisfying $f(x)<d$ for every $d<0$. If $d=0$, then there is no pair of vectors x and y such that $f(x)\leqq 0$, $f(y)\leqq 0$, and $\|x-y\|\geqq\varepsilon$ for any $\varepsilon>0$. For every $d>0$ and every $\varepsilon>0$, let the vectors x and y satisfy $f(x)\leqq d$, $f(y)\leqq d$, and $\|x-y\|\geqq\varepsilon$, and hence we have

$$f\left(\frac{x}{d}\right)\leqq 1\,,\quad f\left(\frac{y}{d}\right)\leqq 1\quad\text{and}\quad\left\|\frac{x}{d}-\frac{y}{d}\right\|\geqq\frac{\varepsilon}{d}\,.$$

It is valid, by (3.4), that there is a positive number $\delta\left(\frac{\varepsilon}{d}\right)$ such that

$$f\left(\frac{x+y}{2}\right)\leqq d-d\delta\left(\frac{\varepsilon}{d}\right).$$

Consequently, the function f is uniformly convex with $\delta(d,\varepsilon)\equiv d\delta\left(\frac{\varepsilon}{d}\right)>0$. Q. E. D.

LEMMA 3.15. *Suppose that Assumptions* 1 *and* 2 *hold. For arbitrary positive number* η_1 *and* η_2 *satisfying* $0<\eta_1\leqq\eta_2$, *there is a positive number* ν_1 *such that*

$$\inf_{\eta_1\leqq\|x\|\leqq\eta_2}(\tilde{\lambda}_x-\lambda_x)\geqq\nu_1\,.$$

PROOF. It follows from Lemma 3.13 that there is a positive number ν_0 such that $\inf_{\|x\|=1}(\tilde{\lambda}_x-\lambda_x)\geqq\nu_0$ we shall show that

$$\inf_{\eta_1\leqq\|x\|\leqq\eta_2}(\tilde{\lambda}_x-\lambda_x)=\inf_{\eta_1\leqq\eta\leqq\eta_2}\inf_{\|x\|=\eta}(\tilde{\lambda}_x-\lambda_x)\,. \tag{3.5}$$

If we define the sets E and E_η by

$$E\equiv\{x\in X\mid\eta_1\leqq\|x\|\leqq\eta_2\}\,,$$

$$E_\eta\equiv\{x\in X\mid\|x\|=\eta\}\qquad\text{for every}\quad\eta>0\,,$$

then it is immediately seen that

$$E=\bigcup_{\eta_1\leqq\eta\leqq\eta_2}E_\eta\,,$$

so that

$$\inf_{x\in E}(\tilde{\lambda}_x-\lambda_x)\leqq\inf_{\eta_1\leqq\eta\leqq\eta_2}\inf_{x\in E\eta}(\tilde{\lambda}_x-\lambda_x)\,.$$

On the other hand, for every vector $x\in E$, there is a positive number η, $\eta_1\leqq\eta\leqq\eta_2$, such that $\|x\|=\eta$, and hence we have

$$\inf_{\eta_1\leqq\eta\leqq\eta_2}\inf_{x\in E\eta}(\tilde{\lambda}_x-\lambda_x)\leqq\tilde{\lambda}_x-\lambda_x\,.$$

This establishes the equality (3.5).

Note that, by Lemma 3.5,

$$\tilde{\lambda}_{\eta x}=\frac{\tilde{\lambda}_x}{\eta}\,,\quad\lambda_{\eta x}=\frac{\lambda_x}{\eta}\qquad\text{for all}\quad\eta>0\,.$$

We then obtain

$$\begin{aligned}\inf_{\eta_1\leqq\|x\|\leqq\eta_2}(\tilde{\lambda}_x-\lambda_x)&=\inf_{\eta_1\leqq\eta\leqq\eta_2}\inf_{\|x\|=\eta}(\tilde{\lambda}_x-\lambda_x)\\&=\inf_{\eta_1\leqq\eta\leqq\eta_2}\inf_{\|x\|=1}(\tilde{\lambda}_{\eta x}-\lambda_{\eta x})\end{aligned}$$

$$= \inf_{\eta_1 \leqq \eta \leqq \eta_2} \frac{1}{\eta} \inf_{\|x\|=1} (\tilde{\lambda}_x - \lambda_x) \geqq \inf_{\eta_1 \leqq \eta \leqq \eta_2} \frac{\nu_0}{\eta} \geqq \frac{\nu_0}{\eta_2}.$$

Therefore, the condition of Lemma 3.15 holds with $\nu_1 \equiv \nu_0/\eta_2$. Q. E. D.

Now we shall review the preceding arguments. Let $f(x)$ be a convex function defined on a normed linear space X. Suppose that Assumptions 1 and 2 hold. Since f is bounded below by Lemma 3.4, there is a real number d_0 such that $d_0 = \inf_{x \in X} f(x)$. It is valid, by Lemma 3.9, that the set G_d is a non-empty, bounded open set in X for every $d > d_0$. It then follows from Lemma 3.2 that for each $d > d_0$, and every $x\ (\neq 0) \in X$, there exists a unique positive number λ_x such that $f(x_0 + \lambda_x x) = d$, where $f(x_0) < d$. Let $p(x)$ be the functional defined in Lemma 3.5. Then, we can describe the following proposition:

PROPOSITION 3.1. *Under the conditions stated above, if $f(x)$ is uniformly convex, then $p(x)$ is also uniformly convex.*

PROOF. It follows from Lemma 3.5 that $p(x)$ is sublinear on X and

$$p(x) \geqq 0 \quad \text{for all} \quad x \in X,$$

$$p(x) = 0 \quad \text{if and only if} \quad x = 0.$$

By virtue of Lemma 3.14, it is sufficient to show that for every positive number ε there is a positive number $\delta = \delta(\varepsilon)$ such that

$$p(x) \leqq 1, \quad p(y) \leqq 1 \quad \text{and} \quad \|x-y\| \geqq \varepsilon \quad \text{implies} \quad p\left(\frac{x+y}{2}\right) \leqq 1-\delta, \tag{3.6}$$

in order to point out the uniform convexity of $p(x)$. To show the contradiction, suppose that there is a positive number ε_0 such that for every positive number δ there exist vectors x_δ and y_δ satisfying

$$p(x_\delta) \leqq 1, \quad p(y_\delta) \leqq 1, \quad \|x_\delta - y_\delta\| \geqq \varepsilon_0, \quad \text{and} \quad 1 \geqq p\left(\frac{x_\delta + y_\delta}{2}\right) > 1-\delta.$$

Since $p(x)$ is continuous and since the origin is an interior point of the set $\left\{x \in X \mid p(x) \leqq \frac{1}{2}\right\}$, there is a positive number η_1 such that

$$\left\|\frac{x_\delta + y_\delta}{2}\right\| \geqq \eta_1 \quad \text{whenever} \quad 0 < \delta < \frac{1}{2}.$$

It follows from Lemma 3.6 that

$$f\left(x_0 + \frac{x_\delta + y_\delta}{2}\right) \leqq d \quad \text{whenever} \quad 0 < \delta < \frac{1}{2}.$$

It is true, by Lemma 3.9, that the set $F_d \equiv \{x \in X \mid f(x) \leqq d\}$ is bounded in X and hence there is a positive number η_2 satisfying

$$\eta_2 \geqq \left\|\frac{x_\delta + y_\delta}{2}\right\| \geqq \eta_1 \quad \text{whenever} \quad 0 < \delta < \frac{1}{2}.$$

Let us fix an arbitrary real number $\tilde{d}$ satisfying $\tilde{d} > d$. Then, for each vector $x\ (\neq 0) \in X$, there exists a unique positive number $\tilde{\lambda}_x$ such that $f(x_0 + \tilde{\lambda}_x x) = \tilde{d}$. It is then valid, by virtue of Lemma 3.15, that there is a positive number ν_1 such that

$$\inf_{\eta_1 \leqq \|x\| \leqq \eta_2} (\tilde{\lambda}_x - \lambda_x) \geqq \nu_1 ,$$

which implies

$$\tilde{\lambda}_{\frac{x_\delta + y_\delta}{2}} - \lambda_{\frac{x_\delta + y_\delta}{2}} \geqq \nu_1 \quad \text{whenever} \quad 0 < \delta < \frac{1}{2} . \tag{3.7}$$

Since f is uniformly convex, for the real numbers d and $\varepsilon_0 > 0$, there is a positive number $\delta(d, \varepsilon_0)$ such that

$$f(x) \leqq d , \quad f(y) \leqq d , \quad \|x-y\| \geqq \varepsilon_0 \quad \text{implies} \quad f\Big(\frac{x+y}{2}\Big) \leqq d - \delta(d, \varepsilon_0) . \tag{3.8}$$

It is then true, by Lemma 3.6, that $f(x_0 + x_\delta) \leqq d$ and $f(x_0 + y_\delta) \leqq d$, so that

$$f\Big(x_0 + \frac{x_\delta + y_\delta}{2}\Big) = f\Big(\frac{(x_0 + x_\delta) + (x_0 + y_\delta)}{2}\Big) \leqq d - \delta(d, \varepsilon_0) < d ,$$

which implies

$$p\Big(\frac{x_\delta + y_\delta}{2}\Big) < 1 \quad \text{or} \quad \frac{1}{1-\delta} > \lambda_{\frac{x_\delta + y_\delta}{2}} > 1 \quad \text{whenever} \quad 0 < \delta < \frac{1}{2} .$$

It is easily seen, by Lemma 3.12, that

$$\begin{aligned} & f\Big(x_0 + \lambda_{\frac{x_\delta + y_\delta}{2}} \frac{x_\delta + y_\delta}{2} + \varepsilon(\lambda_{\frac{x_\delta + y_\delta}{2}} - \tilde{\lambda}_{\frac{x_\delta + y_\delta}{2}}) \frac{x_\delta + y_\delta}{2}\Big) \\ & \geqq f\Big(x_0 + \lambda_{\frac{x_\delta + y_\delta}{2}} \frac{x_\delta + y_\delta}{2}\Big) + \varepsilon\Big(f\Big(x_0 + \lambda_{\frac{x_\delta + y_\delta}{2}} \frac{x_\delta + y_\delta}{2}\Big) - f\Big(x_0 + \tilde{\lambda}_{\frac{x_\delta + y_\delta}{2}} \frac{x_\delta + y_\delta}{2}\Big)\Big) \\ & = d + \varepsilon(d - \tilde{d}) \qquad \text{for every} \quad \varepsilon > 0 . \end{aligned}$$

If we set

$$\varepsilon \equiv \frac{1 - \lambda_{\frac{x_\delta + y_\delta}{2}}}{\lambda_{\frac{x_\delta + y_\delta}{2}} - \tilde{\lambda}_{\frac{x_\delta + y_\delta}{2}}} ,$$

then we have

$$f\Big(x_0 + \frac{x_\delta + y_\delta}{2}\Big) \geqq d + \frac{\lambda_{\frac{x_\delta + y_\delta}{2}} - 1}{\tilde{\lambda}_{\frac{x_\delta + y_\delta}{2}} - \lambda_{\frac{x_\delta + y_\delta}{2}}} (d - \tilde{d}) .$$

It is then valid, by (3.7), that

$$f\Big(x_0 + \frac{x_\delta + y_\delta}{2}\Big) \geqq d + \frac{\lambda_{\frac{x_\delta + y_\delta}{2}} - 1}{\nu_1} (d - \tilde{d}) \qquad \text{whenever} \quad 0 < \delta < \frac{1}{2} .$$

For the positive number $\delta(d, \varepsilon_0)$, there is a positive number δ_0, $0 < \delta_0 < \frac{1}{2}$ such that

$$0 < \frac{\lambda_{\frac{x_{\delta_0} + y_{\delta_0}}{2}} - 1}{\nu_1} (\tilde{d} - d) < \delta(d, \varepsilon_0) ,$$

which implies

$$f\Big(x_0 + \frac{x_{\delta_0} + y_{\delta_0}}{2}\Big) > d - \delta(d, \varepsilon_0) .$$

This contradicts to (3.6). Q. E. D.

PROPOSITION 3.2. *Let f be uniformly convex and suppose that Assumptions* 1 *and* 2 *hold. If we set $F_d \equiv \{x \in X \mid f(x) \leqq d\}$ for every $d > d_0$, then for each positive number ε, there is a positive number $\delta = \delta(\varepsilon)$ such that*

$$\textit{conuex set } A \subset F_{d+\delta} \cap F_d^{\,c}, \quad x, y \in A \quad \textit{implies} \quad \|x-y\| < \varepsilon,$$

where $F_d^{\,c}$ denotes the complement of F_d.

PROOF. Note that the functional $p(x)$ is uniformly convex by Proposition 3.1. To show the contradiction, suppose that there is a positive number ε_0 such that for every $\delta > 0$, there exist the convex set A_δ and the vectors x_δ, $y_\delta \in A_\delta$ satisfying

$$A_\delta \subset F_{d+\delta} \cap F_d^{\,c}, \quad x_\delta, y_\delta \in A \quad \text{and} \quad \|x_\delta - y_\delta\| \geqq \varepsilon_0 .$$

It is then evident, by Lemma 3.6, that $p(x_\delta - x_0) > 1$ and $p(y_\delta - x_0) > 1$. It follows from Lemma 3.8 that

$$p(x_\delta - x_0)(d - f(x_0)) + f(x_0) \leqq f(x_\delta) \leqq d + \delta ,$$

$$p(y_\delta - x_0)(d - f(x_0)) + f(x_0) \leqq f(y_\delta) \leqq d + \delta$$

and hence

$$p(x_\delta - x_0) \leqq d(\delta) \quad \text{and} \quad p(y_\delta - x_0) \leqq d(\delta) ,$$

where $d(\delta) \equiv \dfrac{d + \delta - f(x_0)}{d - f(x_0)} > 1$. It is true that

$$\left\| \frac{x_\delta - x_0}{d(\delta)} - \frac{y_\delta - x_0}{d(\delta)} \right\| = \frac{\|x_\delta - y_\delta\|}{d(\delta)} \geqq \frac{\varepsilon_0}{d(\delta)} \geqq \frac{\varepsilon_0}{d(1)}, \qquad \text{for all} \quad \delta,\ 0 < \delta < 1 ,$$

and

$$p\Big(\frac{x_\delta - x_0}{d(\delta)}\Big) \leqq 1 \quad \text{and} \quad p\Big(\frac{y_\delta - x_0}{d(\delta)}\Big) \leqq 1 \qquad \text{for every} \quad \delta,\ 0 < \delta < 1 ,$$

which implies the existence of a positive number μ_0 such that

$$p\left(\frac{\dfrac{x_\delta + x_0}{d(\delta)} + \dfrac{y_\delta - x_0}{d(\delta)}}{2}\right) \leqq 1 - \mu_0 ,$$

or

$$p\Big(\frac{x_\delta + y_\delta}{2} - x_0\Big) \leqq d(\delta)(1 - \mu_0) \qquad \text{whenever} \quad 0 < \delta < 1 ,$$

since $p(x)$ is uniformly convex. It is valid that there is a sufficiently small number δ_0, $0 < \delta_0 < 1$, satisfying

$$p\Big(\frac{x_{\delta_0} + y_{\delta_0}}{2} - x_0\Big) \leqq d(\delta_0)(1 - \mu_0) < 1 ,$$

which implies

$$f\Big(\frac{x_{\delta_0} + y_{\delta_0}}{2}\Big) > d .$$

This contradicts to the fact that

$$\frac{x_{\delta_0} + y_{\delta_0}}{2} \in A_{\delta_0} \subset F_{d+\delta_0} \cap F_d^{\,c} .$$

Q. E. D.

§ 4. Convergent minimizing sequences.

In this section, we shall observe that the uniform convexity leads to the convergence of minimizing sequence and that it assures the existence and uniqueness of minimum.

Let X be a normed linear space and let f be a real-valued function defined over X. We shall set up two assumptions.

ASSUMPTION 3. For every positive number ε, there is a vector $x_\varepsilon \in X$ such that $f(x_\varepsilon) < \inf_{\|x\|=1} f(x_\varepsilon + \varepsilon x)$.

ASSUMPTION 4. The function f is bounded above on the set $\{x \in X \mid \|x\| = t\}$ for every $t > 0$.

PROPOSITION 4.1. *Let f be a real-valued convex function defined on a normed linear space X. If we suppose that Assumption 4 holds, then Assumption 3 is equivalent to the following condition:*

(*) *For every positive number ε, there exists a real number d ($> \inf_{x \in X} f(x)$) such that $f(x) \leqq d$ and $f(y) \leqq d$ imply $\|x-y\| < \varepsilon$.*

PROOF. First of all, we shall show that Assumption 3 implies the condition (*). Since f is bounded below on X, by Lemma 3.4, then there is a real number d_0 such that $d_0 = \inf_{x \in X} f(x)$. Assumption 3 assures that for every $\varepsilon > 0$, there exists a vector $x_{\varepsilon/2} \in X$ such that $f(x_{\varepsilon/2}) < \inf_{\|x\|=1} f\left(x_{\varepsilon/2} + \frac{\varepsilon}{2} x\right)$. Let d be the number $\inf_{\|x\|=1} f\left(x_{\varepsilon/2} + \frac{\varepsilon}{2} x\right)$. For each vector $x \in X$ satisfying $\|x\| > 1$, we have

$$d \leqq f\left(x_{\varepsilon/2} + \frac{\varepsilon}{2} \frac{x}{\|x\|}\right)$$
$$\leqq \left(1 - \frac{1}{\|x\|}\right) f(x_{\varepsilon/2}) + \frac{1}{\|x\|} f\left(x_{\varepsilon/2} + \frac{\varepsilon}{2} x\right),$$

and hence $f\left(x_{\varepsilon/2} + \frac{\varepsilon}{2} x\right) > d$. Consequently, it is valid that

$$f\left(x_{\varepsilon/2} + \frac{\varepsilon}{2} x\right) \leqq d \qquad \text{implies} \quad \|x\| \leqq 1 .$$

For every $x, y \in X$ satisfying $f(x) \leqq d$ and $f(y) \leqq d$, we obtain

$$\|x - x_{\varepsilon/2}\| \leqq \frac{\varepsilon}{2} \quad \text{and} \quad \|y - x_{\varepsilon/2}\| \leqq \frac{\varepsilon}{2} ,$$

so that $\|x-y\| \leqq \varepsilon$, which implies that the condition (*) holds.

Conversely, we shall show that the condition (*) implies Assumption 3. We shall see that this case holds without convexity of f although it is assumed that f is convex. It is then noted that Assumption 3 is weaker than the condition (*) in this sense. There are following two cases:

Case 1. Suppose that f is not bounded below. For any $\varepsilon > 0$, there is a real number d which satisfies the condition (*). There is a vector $x_\varepsilon \in X$ such that $f(x_\varepsilon) < d$. It is valid that $f(x_\varepsilon + \varepsilon x) > d$ whenever $\|x\| = 1$, $x \in X$. For, if we assume that there is a vector $x \in X$, $\|x\| = 1$ satisfying $f(x_\varepsilon + \varepsilon x) \leqq d$, then we have, by the condition (*),

$$\varepsilon > \|(x_\varepsilon + \varepsilon x) - x_\varepsilon\| = \varepsilon \|x\| = \varepsilon ,$$

which is a contradiction. Therefore, we obtain $\inf_{\|x\|=1} f(x_\varepsilon + \varepsilon x) \geqq d > f(x_\varepsilon)$.

Case 2. Consider the case where f is bounded below. Then, there is a real number d_0 such that $d_0 = \inf_{x \in X} f(x)$. For every $\varepsilon > 0$, there is a real number $d\ (> d_0)$ which satisfies the condition (*). Then, there exists a vector $x_\varepsilon \in X$ such that $f(x_\varepsilon) < d$. By virtue of the argument parallel to that in Case 1, we have $\inf_{\|x\|=1} f(x_\varepsilon + \varepsilon x) \geqq d > f(x_\varepsilon)$. Q. E. D.

THEOREM 4.1. *Let X be a real Banach space, H a closed convex set in X and $f: X \to R^1$ a uniformly convex function. Under Assumptions* 3 *and* 4, *there exists a unique minimal point for f over H and every minimizing sequence converges* (*strongly*) *to the minimal point.*

PROOF. It turns out that Assumptions 1 and 2 are satisfied, so that all lemmas and propositions in Section 3 still hold. There are two cases.

Case 1. Assume that

$$d \equiv \inf_{x \in H} f(x) > d_0 \equiv \inf_{x \in X} f(x) . \tag{4.1}$$

If we define the sets A_n by

$$A_n \equiv \left\{ x \in H \,|\, f(x) \leqq d + \frac{1}{n} \right\} \qquad \text{for} \quad n = 1, 2, \cdots , \tag{4.2}$$

then all A_n's are non-empty, convex sets. It is valid that the decreasing sequence $\{A_n\}$ is a base of Cauchy filter on H since the diameter of the set A_n tends to 0 as $n \to \infty$, by Proposition 3.2. Since the set H is a closed subset of a Banach space X, H is separable and complete, and therefore, the base $\{A_n\}$ of Cauchy filter converges to a unique vector $\bar{x} \in H$. Clearly, $f(\bar{x}) = d$. We shall show that the set $A_\infty \equiv \{x \in H \,|\, f(x) \leqq d\}$ consists of a single vector $\bar{x}$. If we suppose that there is a vector $y \in A_\infty$ different from $\bar{x}$, then there is a positive number ε such that $\|y - \bar{x}\| \geqq \varepsilon$. The uniform convexity of f implies that there exists a positive number $\delta(\varepsilon)$ such that $f\left(\frac{x+y}{2}\right) \leqq d - \delta(\varepsilon) < d$. This contradicts to the definition of d since $\frac{x+y}{2} \in H$. Consequently, every minimizing sequence converges to a unique minimum $\bar{x}$.

Case 2. Suppose that $d = d_0$ where d and d_0 are defined in (4.1). Let A_n be the set in X defined by (4.2) for $n = 1, 2, \cdots$. It is valid, on the basis of Proposition 4.1, that the decreasing sequence $\{A_n\}$ is a base of Cauchy filter on H and hence there is a vector $\bar{x} \in X$ to which $\{A_n\}$ converges, such that $f(\bar{x}) = d$. It is clear that the uniform convexity of f implies the uniqueness of $\bar{x}$. Q. E. D.

§ 5. An application to the projection theorem in a Hilbert space.

In this section, we shall show that the preceding arguments combined with nonlinear programming stated in [5] result in the projection theorem in a Hilbert space.

THEOREM 5.1. *Let X be a Hilbert space and A a closed convex subset of X. If y is a vector in X, then there is a unique vector $\bar{x} \in A$ such that*

$$\|y - \bar{x}\| = \min_{x \in A} \|y - x\| . \tag{5.1}$$

Moreover, a necessary and sufficient condition for $\bar{x}\in A$ to be a unique minimal vector is that

$$(y-\bar{x}|x-\bar{x})\leqq 0 \quad \textit{for all} \quad x\in A. \tag{5.2}$$

PROOF. Let f be a real-valued function on X defined by $f(x)=\|y-x\|$. It is easily verified that $f(x)$ is a uniformly convex function in the sense of Definition 3.1 and hence it follows from Theorem 4.1 that there is a unique vector $\bar{x}\in A$ which satisfies (5.1). It is valid that the differential $f_{\bar{x}}(x)$ of f at $\bar{x}$ in the sense of Neustadt is as follows:

$$f_{\bar{x}}(x)=\begin{cases} \dfrac{-(y-\bar{x}|x)}{\|y-\bar{x}\|} & \text{if } y\neq\bar{x}, \\ \|x\| & \text{if } y=x. \end{cases}$$

It is then clear, on the basis of Theorem 3.2 in [5], that (5.2) holds if and only if $\bar{x}\in A$ satisfies (5.1). Q. E. D.

REMARK. If A is a closed subspace of X, then (5.2) implies that the vector $y-\bar{x}$ is orthogonal to A. Moreover, it may turn out that (5.2) implies the normal equations for the minimization problem.

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