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SUFFICIENT CONDITIONS FOR SWITCHING FUNCTIONS TO BE THRESHOLD ONES AND THEIR APPLICATIONS

By

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§ 1. Introduction.

In this paper we shall present two types of sufficient conditions under which a switching function can be represented by a threshold gate, i. e., a threshold function, with n variables for an arbitrary positive integer n. In deriving these sufficient conditions, a new concept called an orientating vector is introduced in this paper and it will play an important role in our discussion since it gives an insight into the structure of a threshold gate.

We shall begin with notations and preliminaries in Section 2. In Section 3, we shall introduce the notion of orientating vectors in terms of which we give the sufficient conditions for threshold functions. Indeed, an orientating vector can be used to classify the set of all the input vectors $\{x^{(1)}, x^{(2)}, \cdots, x^{(2^n)}\}$ into two subsets where one is a set of true (i. e., on) vectors and the other a set of false (i. e., off) vectors. If we arrange all input vectors in an inverse lexical order (see Kitagawa [3]), then, for any p, $1 \leqq p \leqq 2^n$, a classification of all the input vectors into the two sets $\{x^{(1)}, x^{(2)}, \cdots, x^{(p)}\}$ and $\{x^{(p+1)}, x^{(p+2)}, \cdots, x^{(2^n)}\}$ represents a threshold function as stated in Proposition 3.2. In Section 4, it is described that the combination of two sufficient conditions turn out to be necessary so far as p in the above classification is not greater than 4. This is the reason why the combination of these two sufficient conditions amounts to be necessary and sufficient so far as n is not greater than 3 as given in Section 5.

Our results can be compared with the notion of 2-asummability due to Elgot [2] and Chow [1] which gives the necessary and sufficient condition that a switching function is a threshold function when n in not greater than 8. It is noted that the results of this paper can be used to get all the possible digraphs associated with dynamical behaviors of the neuronic equation

$$x(t+1)=1[\sum_{k=0}^{n-1} a_k x(t-k)-\theta]$$

for n which is not greater than 3, appealing to our necessary and sufficient conditions obtained in Section 5, as we shall show in a later occasion.

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§ 2. Notations and preliminaries.

The real valued function $f(x)=f(x_1, \cdots, x_n)$ with real variables is called a *switching function* if every variable x_i and the values of the function assume 0 or 1.

DEFINITION 1 (Muroga [4]). A *threshold gate* is defined as a logic gate in which each variable input $x_i (i=1, 2, \cdots, n)$ is 1 or 0 and for which there is a set of real numbers $a_1, \cdots, a_n$ and θ such that the output of the gate is

$$f(x^{(j)})=1 \qquad \text{for } \sum_{i=1}^{n} a_i x_i^{(j)} > \theta,\ j=1, 2, \cdots, p,$$

$$f(x^{(j)})=0 \qquad \text{for } \sum_{i=1}^{n} a_i x_i^{(j)} \leqq \theta,\ j=p+1, \cdots, 2^n.$$

The set of $a_1, \cdots, a_n$ and θ is called the *structure* of a threshold gate and is denoted by $[a; \theta]$, where $a=(a_1, \cdots, a_n)$. A switching function $f(x_1, \cdots, x_n)$, represented by the output of a threshold gate, is called a *threshold function*.

There is a characterization of a class of threshold functions.

DEFINITION 2 (Muroga [4]). A switching function f of n variables is called to be *m-summable* for a given m, if for some k, such that $2 \leqq k \leqq m$, there exist two sets,

$$\{x^{(j)} | f(x^{(j)})=1,\ j=1, \cdots, k\} \quad \text{and} \quad \{y^{(j)} | f(y^{(j)})=0,\ j=1, \cdots, k\},$$

such that

$$\sum_{j=1}^{k} x^{(j)} = \sum_{j=1}^{k} y^{(j)}$$

holds. If f is not *m-summable*, f is called to be *m-asummable*. If f is *m-summable* for some m $(\geqq 2)$, f is called to be *summable*, otherwise f is said to be *asummable*.

THEOREM 1 (Asummability theorem) (Elgot [2], Chow [1]). *A necessary and sufficient condition for a switching function to be a threshold function is that f is asummable.*

THEOREM 2 (Muroga [4], pp. 191-201). *A switching function is a threshold function if and only if it is 2-asummable whenever the number of variables of a function is less than or equal to 8.*

Let X be a set of all possible n-dimensional vectors whose components assume 0 or 1. We shall denote the element x of X by

$$x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i=0 \text{ or } 1.$$

Since the switching function $f(x)$ assumes 0 or 1, it divides the set X into two subsets, i. e.,

$$\{x \in X | f(x)=1\} \quad \text{and} \quad \{y \in X | f(y)=0\},$$

and these two subsets are disjoint. Conversely, each classification of vectors in X corresponds to a switching function and this correspondence is 1 to 1. We would like to obtain threshold functions by means of these classifications.

We call $\bar{x}_i$ the conjugate component which is defined by

$$\bar{x}_i=\begin{cases}0 & \text{if } x_i=1\\ 1 & \text{if } x_i=0\,.\end{cases}$$

For each vector x, every component of $\bar{x}$ is a conjugate component. It is evident that

$$x_i\bar{x}_i=0$$

(2.1) $$(x_i)^r+(\bar{x}_i)^r=1 \quad \text{for} \quad r=1,\,2,\,\cdots,\ i=1,\,\cdots,\,n\,.$$

We may use the following notation:

$$x_i^{\varepsilon_i}=\begin{cases}x_i & \text{if } \varepsilon_i=0\\ \bar{x}_i & \text{if } \varepsilon_i=1\,,\end{cases}$$

and

$$x^{\varepsilon}=\begin{pmatrix}x_1^{\varepsilon_1}\\ x_2^{\varepsilon_2}\\ \vdots\\ x_n^{\varepsilon_n}\end{pmatrix} \quad \text{for} \quad x=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} \quad \text{and} \quad \varepsilon=\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix}.$$

The set X is arrayed in an *inverse lexical order* if it is enumerated as follows:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	……	(2^n-3)	(2^n-2)	(2^n-1)	(2^n)
0	1	0	1	0	1	0	1	……	0	1	0	1
0	0	1	1	0	0	1	1		0	0	1	1
0	0	0	0	1	1	1	1		1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮
0	0	0	0	0	0	0	0	……	1	1	1	1.

If $\{\varepsilon^{(j)}\}_{j=1,\,\cdots,\,2^n}$ is arrayed in an inverse lexical order, then the set of ordered vectors

(2.2) $$x^{\varepsilon(1)},\ x^{\varepsilon(2)},\ \cdots,\ x^{\varepsilon(2^n)}$$

is also called to be arrayed in an *inverse lexical order*.

§ 3. Sufficient conditions for switching functions to be threshold functions.

First of all, we shall introduce the orientating vector.

DEFINITION 3.1. We define the *(p-th) orientating vector*[1] $[x^{(1)},\,\cdots,\,x^{(p)}]$ for vectors $x^{(1)},\,\cdots,\,x^{(p)}\in X$ and a positive number p, $1\leqq p\leqq 2^n$, by

$$[x^{(1)},\,\cdots,\,x^{(p)}]=\frac{1}{p}\sum_{i=1}^{p}(x^{(i)}-\bar{x}^{(i)})\,.$$

The i-th component of $[x^{(1)},\,\cdots,\,x^{(p)}]$ is denoted by $[x^{(1)},\,\cdots,\,x^{(p)}]_i$.

PROPOSITION 3.1. *If the set X is divided into two subsets $\{x^{(1)},\,\cdots,\,x^{(p)}\}$ and $\{x^{(p+1)},\,\cdots,\,x^{(2^n)}\}$, $2\leqq p\leqq n+1$, such that*

$$x^{(1)}=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix},\quad x^{(i)}=\begin{pmatrix}x_1\\ \vdots\\ x_{i-2}\\ \bar{x}_{i-1}\\ x_i\\ \vdots\\ x_n\end{pmatrix},\quad i=2,\,3,\,\cdots,\,p\,,$$

1) This vector is closely related to Chow parameters.

then this classification represents a threshold function. Note that we take it for granted that all threshold functions obtained by permuting subscripts belong to the same type.

PROOF. It is clear, by the definition of the orientating vector, that

$$[x^{(1)}, \cdots, x^{(p)}]_i = \begin{cases} \frac{p-2}{p}(x_i - \bar{x}_i) & \text{if} \quad i=1, 2, \cdots, p-1, \\ x_i - \bar{x}_i & \text{if} \quad i=p, \cdots, 2^n. \end{cases}$$

If we set

$$\begin{aligned} \alpha &\equiv x^{(1)} \cdot [x^{(1)}, \cdots, x^{(p)}] \\ &= \frac{p-2}{p}(x_1^2 + \cdots + x_{p-1}^2) + x_p^2 + \cdots + x_n^2, \end{aligned}$$

then it is easy to show that

$$\begin{aligned} x^{(i)} \cdot [x^{(1)}, \cdots, x^{(p)}] &= \frac{p-2}{p}(x_1^2 + \cdots + (-\bar{x}_{i-1}^2) + \cdots + x_{p-1}^2) + x_p^2 + \cdots + x_n^2 \\ &= \frac{p-2}{p}(x_1^2 + \cdots + (x_{i-1}^2 - 1) + \cdots + x_{p-1}^2) + x_p^2 + \cdots + x_n^2 \\ & \qquad \text{(by (2.1))} \\ &= \alpha - \frac{p-2}{p}, \qquad \text{for} \quad i=2, 3, \cdots, p. \end{aligned}$$

When we consider the set $\{x^{(p+1)}, \cdots, x^{(2^n)}\}$, there are two cases:

Case 1. If the vector $x^{(q)}$, $p+1 \leqq q \leqq 2^n$, has only one conjugate component $\bar{x}_i$ as i-th component, $p \leqq i \leqq n$, then it follows that

$$x^{(q)} \cdot [x^{(1)}, \cdots, x^{(p)}] \leqq \alpha - 1 < \alpha - \frac{p-2}{p}.$$

Case 2. Suppose that the vector $x^{(q)}$, $p+1 \leqq q \leqq 2^n$, has $\bar{x}_i$ and $\bar{x}_j$ as its i-th and j-th components, respectively, $1 \leqq i, j \leqq p-1$. It is then true that

$$x^{(q)} \cdot [x^{(1)}, \cdots, x^{(p)}] \leqq \alpha - \frac{2(p-2)}{p} < \alpha - \frac{p-2}{p}.$$

In both cases, the hyperplane

$$\left\{ y \in R^n \,|\, y \cdot [x^{(1)}, \cdots, x^{(p)}] = \alpha - \frac{p-2}{p} \right\}$$

divides the set X into two subsets. Note that the orientating vector $[x^{(1)}, \cdots, x^{(p)}]$ gives the structure of the threshold gate. Q. E. D.

PROPOSITION 3.2. *Let the set $X = \{x^{(1)}, \cdots, x^{(2^n)}\}$ be arrayed in an inverse lexical order as stated in (2.2). Then, for every p, $1 \leqq p \leqq 2^n$, the classification $\{x^{(1)}, \cdots, x^{(p)}\}$ $\cdot \{x^{(p+1)}, \cdots, x^{(2^n)}\}$ always represents a threshold function.*

Before we prove the proposition, we shall prepare some lemmas.

Let the set $X = \{x^{(1)}, x^{(2)}, \cdots, x^{(2^n)}\}$ be arrayed as in Proposition 3.2. Then, we have the following Lemmas 3.1 and 3.2.

LEMMA 3.1. *For every p, $1 \leqq p \leqq 2^n$, the orientating vector $[x^{(1)}, \cdots, x^{(p)}]$ is represented as follows:*

$$[x^{(1)}, \cdots, x^{(p)}]_s = \frac{1}{p} f(p)(x_s - \bar{x}_s) \qquad \textit{for} \quad s=1, 2, \cdots, n, \tag{3.1}$$

where

(3.2) $$f_s(p)=2^{s-1}-|\lambda_{p,s}-2^{s-1}|\ ,$$

and

(3.3) $$p=2^s\delta_{p,s}+\lambda_{p,s}\ ,$$

where $0\leqq\lambda_{p,s}<2^s$ *and* $\delta_{p,s}$ *is a nonnegative integer. The above* $f_s(p)$ *has the following properties:*

(a) $$0\leqq f_s(p)\leqq 2^{s-1}\ ,$$

(b) *If* $1\leqq p\leqq 2^{s-1}$, *then* $f_s(p)=p$,

(c) $$f_s(p)\leqq f_{s+1}(p)\qquad \text{for}\quad s=1,\ 2,\cdots,n-1\ .$$

PROOF. It is immediate that the orientating vector $[x^{(1)},\cdots,x^{(p)}]$ is represented by (3.1). It is also clear that (a) and (b) hold. We shall prove (c). It is true, by (3.2), that

$$f_{s+1}(p)-f_s(p)=2^{s-1}+|\lambda_{p,s}-2^{s-1}|-|\lambda_{p,s+1}-2^s|\ .$$

Now, there are two cases:

Case 1. Suppose that $0\leqq\lambda_{p,s+1}<2^s$. Then, we obtain, by (3.3),

$$2\delta_{p,s+1}=\delta_{p,s}\quad\text{and}\quad\lambda_{p,s+1}=\lambda_{p,s}\ .$$

(i) If $\lambda_{p,s}=\lambda_{p,s+1}<2^{s-1}$, then we have

$$f_{s+1}(p)-f_s(p)=2^{s-1}-(\lambda_{p,s}-2^{s-1})+(\lambda_{p,s+1}-2^s)=0\ .$$

(ii) If $\lambda_{p,s}=\lambda_{p,s+1}\geqq 2^{s-1}$, then we have

$$f_{s+1}(p)-f_s(p)=2\lambda_{p,s}-2^s\geqq 2.2^{s-1}-2^s=0\ .$$

Case 2. Assume that $2^s\leqq\lambda_{p,s+1}<2^{s+1}$. It is then valid, by (3.3), that

$$\delta_{p,s}=2\delta_{p,s+1}+1\quad\text{and}\quad\lambda_{p,s}=\lambda_{p,s+1}-2^s\ .$$

(i) If $\lambda_{p,s}>2^{s-1}$, then it is true that

$$f_{s+1}(p)-f_s(p)=2^{s-1}+(\lambda_{p,s}-2^{s-1})-(\lambda_{p,s+1}-2^s)=0\ ,$$

(ii) If $0\leqq\lambda_{p,s}\leqq 2^{s-1}$, then we have

$$f_{s+1}(p)-f_s(p)=2^s-2\lambda_{p,s}\geqq 2^s-2.2^{s-1}=0\ .$$

In any case, we can conclude that

$$f_{s+1}(p)-f_s(p)\geqq 0\qquad\text{for}\quad s=1,\ 2,\cdots,n-1,\quad 1\leqq p\leqq 2^n.$$

Q. E. D.

LEMMA 3.2. *For each* p, $1\leqq p\leqq 2^n$, *it is true that*

$$x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}]=\min_{1\leqq r\leqq p}x^{(r)}\cdot[x^{(1)},\cdots,x^{(p)}]\ .$$

PROOF. The positive integer p is written as follows:

$$p=2^{q_1}+2^{q_2}+\cdots+2^{q_m},\quad 0\leqq q_1<q_2<\cdots<q_m\leqq n\ .$$

First of all, we shall derive $[x^{(1)},\cdots,x^{(p)}]$. Suppose that $q_1\geqq 1$. (It is similar for $q_1=0$). For every s, $1\leqq s\leqq q_1$, we have

$$p=2^s\delta_{p,s}\quad\text{and}\quad f_s(p)=0\ .$$

For every s, $q_1+1 \leqq s \leqq q_2$, we obtain

$$p=2^s(2^{q_2-s}+\cdots+2^{q_m-s})+2^{q_1} \quad \text{and} \quad f_s(p)=2^{q_1}.$$

For $s=q_2+1$, we have

$$p=2^s(2^{q_3-s}+\cdots+2^{q_m-s})+2^{q_1}+2^{q_2} \quad \text{and} \quad f_{q_2+1}(p)=2^{q_2}-2^{q_1}.$$

For every s, $q_2+2 \leqq s \leqq q_3$, we obtain

$$p=2^s(2^{q_3-s}+\cdots+2^{q_m-s})+2^{q_1}+2^{q_2} \quad \text{and} \quad f_s(p)=2^{q_1}+2^{q_2}.$$

In general, for every s, $q_i+2 \leqq s \leqq q_{i+1}$, $1 \leqq i \leqq m-1$, we have

$$p=2^s(2^{q_{i+1}-s}+\cdots+2^{q_m-s})+2^{q_1}+\cdots+2^{q_i} \quad \text{and} \quad f_s(p)=2^{q_1}+\cdots+2^{q_i}.$$

It is valid that, for $s=q_i+1$,

$$p=2^{q_i+1}(2^{q_{i+1}-q_i-1}+\cdots+2^{q_m-q_i-1})+2^{q_1}+\cdots+2^{q_i},$$

and

$$f_{q_i+1}(p)=2^{q_i}-(2^{q_1}+\cdots+2^{q_{i-1}}) \qquad \text{for} \quad i=2,\cdots,m.$$

It follows from Lemma 3.1 (b) that

$$f_s(p)=p \qquad \text{for } s=q_m+2,\cdots,n,$$

since $0 \leqq p \leqq 2^{s-1}$.

Consequently, we can write $[x^{(1)},\cdots,x^{(p)}]$ as follows:

$$[x^{(1)},\cdots,x^{(p)}]_s=\begin{cases} 0 & \text{if } 1 \leqq s \leqq q_1 \\ \dfrac{2^{q_1}}{p}(x_s-\bar{x}_s) & \text{if } q_1+1 \leqq s \leqq q_2 \\ \dfrac{2^{q_i}-(2^{q_1}+\cdots+2^{q_{i-1}})}{p}(x_s-\bar{x}_s) & \text{if } s=q_i+1,\ i=2,\cdots,m, \\ \dfrac{2^{q_1}+\cdots+2^{q_i}}{p}(x_s-\bar{x}_s) & \text{if } q_i+2 \leqq s \leqq q_{i+1},\ i=1,\cdots,m-1 \\ x_s-\bar{x}_s & \text{if } q_m+2 \leqq s \leqq n. \end{cases}$$

It is evident that

$$x_s^{(p)}=\begin{cases} \bar{x}_s & \text{if } 1 \leqq s \leqq q_1, \\ \bar{x}_s & \text{if } s=q_i+1,\ i=2,\cdots,m \\ x_s & \text{otherwise}. \end{cases}$$

Therefore, we have

$$x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}]=\alpha-\frac{2^{q_m}-(m-1)2^{q_1}-(m-3)2^{q_2}-(m-4)2^{q_3}-\cdots-2^{q_{m-2}}}{p},$$

where $\alpha \equiv x^{(1)}\cdot[x^{(1)},\cdots,x^{(p)}]$.

It is also verified that for every i, $1 \leqq i \leqq m$,

$$x_s^{(2^{q_m}+2^{q_{m-1}}+\cdots+2^{q_i})}=\begin{cases} \bar{x}_s & \text{if } 1 \leqq s \leqq q_i \\ \bar{x}_s & \text{if } s=q_j+1,\ j=i+1,\cdots,m \\ x_s & \text{otherwise}. \end{cases}$$

and that

$$x^{(2^{q_m}+\cdots+2^{q_i})}\cdot[x^{(1)},\cdots,x^{(p)}]=\alpha-(q_2-q_1)\frac{2^{q_1}}{p}-\frac{2^{q_2}-2^{q_1}}{p}-(q_3-q_2-1)\frac{2^{q_1}+2^{q_2}}{p}-\cdots$$

$$-(q_i-q_{i-1}-1)\frac{2^{q_1}+\cdots+2^{q_{i-1}}}{p}+\frac{2^{q_i}-(2^{q_1}+\cdots+2^{q_{i-1}})}{p}-\cdots$$

$$-\frac{2^{q_m}-(2^{q_1}+\cdots+2^{q_{m-1}})}{p}.$$

Therefore we have

$$x^{(2^{q_m}+\cdots+2^{q_1})}\cdot[x^{(1)},\cdots,x^{(p)}]-x^{(p)}\cdot[x^{(1)}\cdots,x^{(p)}]$$

$$=\frac{1}{p}\{2^{q_i}-(q_i-q_1-i+3)2^{q_1}-(q_i-q_2-i+3)2^{q_2}-(q_i-q_3-i+4)2^{q_3}-\cdots$$

$$-(q_i-q_{i-1})2^{q_{i-1}}\}.$$

We can show that

$$(q_i-q_{i-1})2^{p_{i-1}}\leqq 2^{q_{i-1}}$$

$$\vdots$$

$$(q_i-q_3-i+4)2^{q_3}\leqq 2^{q_i-i+3}$$

$$(q_i-q_2-i+3)2^{q_2}\leqq 2^{q_i-i+2}$$

$$(q_i-q_1-i+3)2^{q_1}\leqq 2^{q_i-i+2},$$

since

$$2^y-y-1\geqq 0 \qquad \text{whenever} \quad y\geqq 0.$$

Consequently, we can conclude that

$$x^{(2^{q_m}+\cdots+2^{q_i})}\cdot[x^{(1)},\cdots,x^{(p)}]-x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}]$$

$$\geqq\frac{1}{p}\{2^{q_i}-2^{q_i-i+2}-2^{q_i-i+3}-\cdots-2^{q_{i-1}}\}=0, \quad i=2,\cdots,m.$$

It is easily seen that for every s satisfying

$$2^{q_m}+\cdots+2^{q_{i+1}}+1\leqq s\leqq 2^{q_m}+\cdots+2^{q_i},$$

$$x^{(s)}\cdot[x^{(1)},\cdots,x^{(p)}]\geqq x^{(2^{q_m}+\cdots+2^{q_i})}\cdot[x^{(1)},\cdots,x^{(p)}],$$

and hence we can obtain

$$x^{(s)}\cdot[x^{(1)},\cdots,x^{(p)}]\geqq x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}] \qquad \text{for} \quad s=1,\cdots,p.$$

Q. E. D.

Now, we shall give the proof of Proposition 3.2.

PROOF OF PROPOSITION 3.2. It follows from Lemma 3.2 that

$$x^{(r)}\cdot[x^{(1)},\cdots,x^{(p)}]\geqq x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}] \qquad \text{for} \quad r=1,\cdots,p.$$

Therefore, if we can show that

$$x^{(r)}\cdot[x^{(1)},\cdots,x^{(p)}]< x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}] \qquad \text{for} \quad r=p+1,\cdots,n,$$

then this classification represents a threshold function.

Note that

$$x_s^{(p+1)}=x_s^{(2^{q_m}+\cdots+2^{q_1}+1)}=\begin{cases}\bar{x}_s & \text{if} \quad s=q_i+1,\ i=1,\cdots,m\,,\\ x_s & \text{otherwise}\,.\end{cases}$$

It is easy to verify that

$$x^{(p+1)}\cdot[x^{(1)},\cdots,x^{(p)}]-x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}]=-\frac{2^{q_1}}{p}<0\,.$$

Since for every r, $p+1\leqq r\leqq 2^{q_m}+\cdots+2^{q_1+1}$,

$$x^{(r)}\cdot[x^{(1)},\cdots,x^{(p)}]\leqq x^{(p+1)}\cdot[x^{(1)},\cdots,x^{(p)}]\,,$$

we have

$$x^{(r)}\cdot[x^{(1)},\cdots,x^{(p)}]<x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}]$$

$$\text{for} \quad r=p+1,\cdots,2^{q_m}+\cdots+2^{q_1+1}\,.$$

Consider the vector $x^{(2^{q_m}+\cdots+2^{q_{i+1}}+1)}$, $1\leqq i\leqq m$, whose components are

$$x_s^{(2^{q_m}+\cdots+2^{q_{i+1}}+1)}=\begin{cases}\bar{x}_s & \text{if} \quad s=q_i+2\,,\\ \bar{x}_s & \text{if} \quad s=q_j+1,\ j=i+1,\cdots,m\,,\\ x_s & \text{otherwise}\,.\end{cases}$$

Then we have

$$x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}]-x^{(2^{q_m}+\cdots+2^{q_{i+1}}+1)}\cdot[x^{(1)},\cdots,x^{(p)}]$$

$$=\frac{2^{q_m}+\cdots+2^{q_{i-1}}+(i-1)2^{q_1}+(i-1)2^{q_2}+(i-2)2^{q_3}+\cdots+2^{q_{i-2}}}{p}>0\,.$$

Since, for every r satisfying

$$2^{q_m}+\cdots+2^{q_{i+1}}+1\leqq r\leqq 2^{q_m}+\cdots+2^{q_{i+1}+1}\,,$$

$$x^{(r)}\cdot[x^{(1)},\cdots,x^{(p)}]\leqq x^{(2^{q_m}+\cdots+2^{q_{i+1}}+1)}\cdot[x^{(1)},\cdots,x^{(p)}]\,,$$

we can conclude that

$$x^{(r)}\cdot[x^{(1)},\cdots,x^{(p)}]<x^{(p)}\cdot[x^{(1)},\cdots,x^{(p)}] \qquad \text{for} \quad r=p+1,\cdots,n\,.$$

Q. E. D.

REMARK. In Propositions 3.1 and 3.2, the vector $[x^{(1)},\cdots,x^{(p)}]$ essentially gives the structure of the threshold function.

PROPOSITION 3.3. *Let X be divided into two subsets $\{x^{(1)},\cdots,x^{(p)}\}$, $\{x^{(p+1)},\cdots,x^{(2^n)}\}$. Let d be a positive integer such that $2^{d-1}<p\leqq 2^d$. If there are two vectors in the set $\{x^{(1)},\cdots,x^{(p)}\}$ such that one vector differs by at least $(d+1)$ components from the other, then this classification does not represent any threshold function.*

PROOF. Suppose that $x^{(1)}$ and $x^{(2)}$ satisfy the condition of Proposition 3.3 i.e., say

$$x^{(1)}=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} \quad \text{and} \quad x^{(2)}=\begin{pmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_{d+1}\\ x_{d+2}\\ \vdots\\ x_n\end{pmatrix}.$$

Consider the pairs $\{y^{(i)}, z^{(i)}\}$ satisfying

$$y^{(i)}+z^{(i)}=\begin{pmatrix} x_1+\bar{x}_1 \\ \vdots \\ x_{d+1}+\bar{x}_{d+1} \\ x_{d+2}+x_{d+2} \\ \vdots \\ x_n+x_n \end{pmatrix}.$$

It is clear that there are 2^d pairs $\{y^{(i)}, z^{(i)}\}$. If one pair $\{y^{(k)}, z^{(k)}\}$ is in $\{x^{(1)}, \cdots, x^{(p)}\}$, then there is at least one pair $\{y^{(j)}, z^{(j)}\}$ in $\{x^{(p+1)}, \cdots, x^{(2^n)}\}$ since there are $(p-2)$ vectors in $\{x^{(1)}, \cdots, x^{(p)}\}$ without $\{y^{(k)}, z^{(k)}\}$ whereas there are (2^d-1) pairs $\{y^{(i)}, z^{(i)}\}$ and since

$$p-2\leqq 2^d-2<2^d-1.$$

Therefore, it follows that

$$y^{(k)}+z^{(k)}=y^{(j)}+z^{(j)}$$

which immediately implies that the corresponding swiching function is 2-summable and hence this classification does not represent any threshold function. Q. E. D.

§ 4. Necessary and sufficient conditions for switching functions to be threshold functions for $1\leqq p\leqq 4$ and any n.

In this section, we shall show that the sufficient conditions given in Section 3 are necessary whenever $1\leqq p\leqq 4$.

PROPOSITION 4.1. *The classification*

$$\{x^{(1)}\}\{x^{(2)}, \cdots, x^{(2^n)}\}$$

always represents a threshold function.

PROOF. This is an immediate consequence of Proposition 3.2. Q. E. D.

PROPOSITION 4.2. *The classification*

$$\{x^{(1)}, x^{(2)}\}\{x^{(3)}, \cdots, x^{(2^n)}\}$$

can represents a threshold function if and only if it is of the type

$$x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

PROOF. "if" part. This holds by Proposition 3.2.

"only if" part. In any other type, vectors $x^{(1)}$ and $x^{(2)}$ have at least two different components. It follows from Proposition 3.3 that this classification cannot represent any threshold function. Q. E. D.

PROPOSITION 4.3. *The classification*

$$\{x^{(1)}, x^{(2)}, x^{(3)}\} \quad \{x^{(4)}, \cdots, x^{(2^n)}\}$$

represents a threshold function if and only if it is of the following type:

$$x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad x^{(3)}=\begin{pmatrix} x_1 \\ \bar{x}_2 \\ \vdots \\ x_n \end{pmatrix}.$$

PROOF. "if" part. This is an immediate consequence of Proposition 3.1 (or 3.2).

"only if" part. Suppose that the vector $x^{(1)}$ differs from the vectors $x^{(2)}$ and $x^{(3)}$ in three components. Without loss of generality, we can consider the case where first three components are different. By virtue of Proposition 3.3, all remaining combinations of $x^{(2)}$ and $x^{(3)}$ are

$$\begin{pmatrix} \bar{x}_{i_1} \\ \bar{x}_{i_2} \\ x_{i_3} \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_{i_1} \\ \bar{x}_{i_2} \\ \bar{x}_{i_3} \\ \vdots \\ x_n \end{pmatrix},$$

where (i_1, i_2, i_3) runs over all permutations of $\{1, 2, 3\}$. It is valid, on the basis of asummability theorem, that these classifications do not represent any threshold function. It is clear, by Proposition 3.3, that if the vector $x^{(1)}$ differs from the vectors $x^{(2)}$ and $x^{(3)}$ in more than three components, then any threshold function cannot be represented by these classifications. The remaining cases are the cases where the vectors $x^{(2)}$ and $x^{(3)}$ are two vectors taken among following three vectors

$$\begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}.$$

Each combination of all the possible three combinations define the same type of classification in the sense that the rearrangement of $x^{(r)}$, $r=1, 2, 3$, in any one combination implies the other two ones. Q. E. D.

PROPOSITION 4.4. *The classification*

$$\{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\} \quad \{x^{(5)}, \cdots, x^{(2^n)}\}$$

represents a threshold function if and only if it is of the types

$$(4.1) \qquad x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x^{(3)}=\begin{pmatrix} x_1 \\ \bar{x}_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x^{(4)}=\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$(4.2) \qquad x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad x^{(3)}=\begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \quad x^{(4)}=\begin{pmatrix} x_1 \\ x_2 \\ \bar{x}_3 \\ \vdots \\ x_n \end{pmatrix}.$$

PROOF. "if" part. This holds by Propositions 3.1 and 3.2.

"only if" part. It follows from Proposition 3.3 that any pair in $x^{(r)}$, $r=1, 2, 3, 4$, has at most two different components, and hence each of $x^{(r)}$, $r=2, 3, 4$, has at most two conjugate components. Let h denotes the number of the set of components of $x^{(1)}$ each of which differs from at least one of the corresponding components of $x^{(r)}$, $r=2, 3, 4$.

(1) The case where $h=2$ gives us the type (4.1).

(2) Suppose that $h=3$. The following cases (i), (ii), (iii) and (iv) exhaust all the possibilities. (i) If each of $x^{(r)}$, $r=2, 3, 4$, has just one conjugate component, then this is of the type (4.2). (ii) The case where only one of the three vectors $x^{(r)}$, $r=2, 3, 4$, has two conjugate components and each of other two vectors has just one conjugate component leads us to a contradiction in view of Proposition 3.3. (iii) The case where two of the three vectors $x^{(r)}$, $r=2, 3, 4$, have two conjugate components, respectively, and the other one has one conjugate component can be reduced by the permutation of component numbers to the case

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \quad \begin{pmatrix} x_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \vdots \\ x_n \end{pmatrix} \quad \begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix},$$

which is of the type (4.2). (iv) The case where all of $x^{(r)}$, $r=2, 3, 4$, have two conjugate components, respectively, can not be concerned with any threshold function by the asummability theorem.

(3) Suppose that $h=4$. If at least one of $x^{(r)}$, $r=2, 3, 4$, has only one conjugate component, then all these cases do not represent any threshold function by Proposition 3.3. The case where each of $x^{(r)}$, $r=2, 3, 4$, has just two conjugate components does not represent any threshold function by the asummability theorem.

(4) It is clear that any case where $h \geqq 5$ does not represent any threshold function. Q. E. D.

§ 5. A characterization of threshold functions with three variables.

We shall now present all types of threshold functions with three variables by appealing to the preceding arguement. In this case the set X contains eight vectors $x^{(1)}, x^{(2)}, \cdots, x^{(8)}$.

1. $\{\phi\}\{x^{(1)}, x^{(2)}, \cdots, x^{(8)}\}$ $\qquad 1$

2. $\{x^{(1)}\}\{x^{(2)}, x^{(3)}, \cdots, x^{(8)}\}$ $\qquad 2^3=8$

3. $\{x^{(1)}, x^{(2)}\}\{x^{(3)}, x^{(4)}, \cdots, x^{(8)}\}$

$$x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \frac{3\,!}{2\,!}2^2=12$$

4. $\{x^{(1)}, x^{(2)}, x^{(3)}\}\{x^{(4)}, \cdots, x^{(8)}\}$

$$x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_2 \end{pmatrix}, \quad x^{(3)}=\begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix} \qquad \frac{3!}{2!}2^3=24$$

5. $\{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\}\{x^{(5)}, \cdots, x^{(8)}\}$

$$x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x^{(3)}=\begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix}, \quad x^{(4)}=\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix} \qquad \frac{3!}{2!}2^1=6$$

and

$$x^{(1)}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x^{(2)}=\begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x^{(3)}=\begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix}, \quad x^{(4)}=\begin{pmatrix} x_1 \\ x_2 \\ \bar{x}_3 \end{pmatrix} \qquad \frac{3!}{3!}2^3=8$$

6. $\{x^{(1)}, x^{(2)}, \cdots, x^{(5)}\}\{x^{(6)}, x^{(7)}, x^{(8)}\}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \bar{x}_3 \end{pmatrix} \qquad \frac{3!}{2!}2^3=24$$

7. $\{x^{(1)}, x^{(2)}, \cdots, x^{(6)}\}\{x^{(7)}, x^{(8)}\}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \bar{x}_3 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ x_2 \\ \bar{x}_3 \end{pmatrix} \qquad \frac{3!}{2!}2^2=12$$

8. $\{x^{(1)}, x^{(2)}, \cdots, x^{(7)}\}\{x^{(8)}\}$ $\qquad \frac{3!}{3!}2^3=8$

9. $\{x^{(1)}, x^{(2)}, \cdots, x^{(8)}\}\{\phi\}$ $\qquad 1$

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