RENEWAL THEORETICAL APPROACH TO THE MISSION RELIABILITY OF A REDUNDANT REPAIRABLE SYSTEM WITH TWO DISSIMILAR UNITS

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バージョン：

RENEWAL THEORETICAL APPROACH TO THE MISSION RELIABILITY OF A REDUNDANT REPAIRABLE SYSTEM WITH TWO DISSIMILAR UNITS*

By

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Abstract

A system consisting of two dissimilar, redundant, repairable units is considered. We shall say that a major breakdown occurs in the system when both units fails. The system fails if one unit under repair is not repaired within a fixed time measured from the instant at which major breakdown occurs, or if the number of major breakdowns during the mission period exceeds a fixed number. As a special case, this number is allowed to be " infinite ".

The Laplace transforms of the reliability and the mean time to system failure are derived, and the explicit formulas in the special cases are exhibited.

§ 1. Model definition.

1. The system consists of two dissimilar redundant units A_1 and A_2.
2. There is only one repair station. When one unit fails, its repair begins at once, and when two units fail simultaneously, unit A_i is sent for repair with a specified constant probability α_i, $i=1, 2$, where $\alpha_1+\alpha_2=1$.

Concerning failure and repair we assume the following:

3. When the two units are good, unit-failures occur as three independent Poission processes with failure rates λ_1, λ_2 and λ_{12}. Events in the process with rate λ_i cause failure of unit A_i only, and events in the process with rate λ_{12} cause simultaneous failure of unit A_1 and A_2.
4. When only one unit, A_i is good, failure of the unit is Poisson with rate λ'_i, $\lambda'_i \neq \lambda_i$.
5. The repair time for each unit, A_i is independently distributed with general probability density function $f_i(t)$, but must be well behaved enough for the appropriate analytic operations to be performed.
6. The failure and repair processes for the two units are entirely independent.
7. The repaired unit is considered to be new again.

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8. A major breakdown occurs when both units fail. And the system fails if one unit under repair is not repaired within a fixed time, τ, measured from the instant at which major breakdown occurs. The number of major breakdowns during the mission period is limited as follows: We assume that for fixed n, any unit under repair after the $(n+1)$-st major breakdown is not repaired within allowable down time with probability 1. Therefore the system also fails if the number of major breakdown in $T-\tau$ exceeds n.

Recently J. Fukuta and M. Kodama [1] gave a mission reliability for a redundant repairable system with two dissimilar units using the method of supplementary variables. In [1], we assumed that the number of failures during the mission period is "infinite". In this paper, we assume that the number of major breakdowns during the mission period is a fixed number and is allowed to be "infinite". We study the above mentioned model by renewal theoretical approach introducing the probability functions depending on the number of major breakdowns. Our results include the results in [1] as special cases.

§ 2. Equations of the system.

Notation List:

i	subscript describing the 2 which $i=1,2$
α_i	probability that unit A_i is sent for repair, when both units fail simultaneously (see 2, § 1)
λ_i, λ'_i, λ_{12}	failure rates (see 3, § 1) ($\lambda_i>0$, $\lambda'_i>0$, $\lambda_{12}\geqq 0$)
τ	allowed down time (see § 1)
n	maximum number of major breakdown permitted during the mission period
$f_i(t)$	pdf of repair time of unit A_i
$F_i(t)$	cdf of repair time of unit A_i
$f^*(s)$	Laplace transform of $f(t)$
$f^\dagger(s)$	Integral $\int_0^\tau e^{-st}f(t)dt$
δ_{ij}	Kronecker symbol
$\alpha_i^*(s)$	$=s+\lambda_1+\lambda_2+\lambda_{12}-\lambda_i f_i^*(s+\lambda'_{3-i})$
$\beta_i^*(s;\tau)$	$=[e^{\lambda'_{3-1}\tau}-1]f_i^*(s+\lambda'_{3-i})-e^{\lambda'_{3-i}\tau}f_i^\dagger(s+\lambda'_{3-i})+f_i^\dagger(s)$
$\delta_i^*(s;\tau)$	$=\lambda'_{3-i}e^{-s\tau}\{e^{(s+\lambda'_{3-i})\tau}[f_i^*(s+\lambda'_{3-i})-f_i^\dagger(s+\lambda'_{3-i})]-f_i^*(s+\lambda'_{3-i})+F_i(\tau)\}/[s(s+\lambda'_{3-i})]-\beta_i^*(s;\tau)/s$
$\gamma_i^*(s;\tau)$	$=[s+\lambda'_{3-i}-\lambda_{3-i}e^{-s\tau}][1-f_i^*(s+\lambda'_{3-i})]/[s(s+\lambda'_{3-i})]+f_i^*(s+\lambda'_{3-i})[s+\lambda_{12}(1-e^{-s\tau})]/[s(s+\lambda_1+\lambda_2+\lambda_{12}]$
$\varepsilon_i^*(s;\tau)$	$=[1-\beta_i^*(s;\tau)-f_i^*(s+\lambda'_{3-i})]/(s+\lambda'_{3-i})+\lambda'_{3-i}[1-f_i^\dagger(s)-e^{-s\tau}(1-F_i(\tau))]/[s(s+\lambda'_{3-i})]$
$k_i^*(s;\tau)$	$=\lambda_{3-i}+\lambda_{12}\alpha_i f_i^\dagger(s)$
$\eta^*(s;\tau)$	$=\left\{s+\lambda_{12}\sum_{j=1}^{2}\alpha_j[1-f_j^\dagger(s)-e^{-s\tau}(1-F_j(\tau))]\right\}/s$
E_2	denotes unit A_i and unit A_{3-i} in operation
$E_1(i)$	denotes unit A_{3-i} in operation and unit A_i under repair
$E_0(i)$	denotes unit A_i under repair and unit A_{3-i} queneing for repair

$q_2^j(u)$ probability that system is state E_2 at $u=0$, has exactly j major breakdowns during the interval 0 to u, and no major breakdowns has lasted longer than τ. The major breakdown if it occured in time interval $(T-\tau, T)$ is not included (the failures during this interval do not count)

$q_{k(i)}^j(u)$ denotes probability similar to the $q_2^j(u)$, except the system has just entered the state $E_k(i)$ $(k=0, 1)$ at $u=0$ and when $k=0$, the initial failed state is not included in j.

$Q_2^n(u)$ sum of $q_2^j(u)$ from $j=0$ to $j=n$

$Q_{k(i)}^n(u)$ sum of $q_{k(i)}^j(u)$ from $j=0$ to $j=n$

MTSF$_2^n$, MTSF$_{k(i)}^n$ notations are analogous to $Q_2^n(u)$ and $Q_{k(i)}^n$ respectively. They are easy to show that MTSF$_2^n = Q_2^{*n}(s)|_{s=0}$, MTSF$_{k(i)}^n = Q_{k(i)}^{*n}(s)|_{s=0}$.

For $u<\tau$, $q_2^0(u)=1$, $q_2^j(u)=0$ $(j\geqq 1)$; $q_{k(i)}^0(u)=1$, $q_{k(i)}^j(u)=0$ $(k=0,1;\ i=1,2;\ j\geqq 1)$, and by definition of $q_2^j(u)$ and $q_{k(i)}^j(u)$, the following set of integral equations of this system can be easily set up:

$$q_2^0(T)=\begin{cases} e^{-(\lambda_1+\lambda_2+\lambda_{12})(T-\tau)} + \int_0^{T-\tau} e^{-(\lambda_1+\lambda_2+\lambda_{12})x}[\lambda_1 q_{1(1)}^0(T-x)+\lambda_2 q_{1(2)}^0(T-x)]dx\,, & T\geqq\tau\,, \\ =1\,, & \text{otherwise}\,. \end{cases} \quad (1)$$

$$q_{1(i)}^0(T)=\begin{cases} e^{-\lambda'_{3-i}(T-\tau)}[1-F_i(T-\tau)]+\int_0^{T-\tau} e^{-\lambda'_{3-i}x} f_i(x) q_2^0(T-x)dx\,, & T\geqq\tau\,, \\ =1\,, & \text{otherwise}\,. \end{cases} \quad (2)$$

$$q_{0(i)}^0(T)=\begin{cases} \int_0^{\tau} f_i(x) q_{1(3-i)}^0(T-x)dx\,, & T\geqq\tau\,, \\ =1\,, & \text{otherwise}\,. \end{cases} \quad (3)$$

$$q_2^j(T)=\begin{cases} \int_0^{T-\tau} e^{-(\lambda_1+\lambda_2+\lambda_{12})x}\{\lambda_1 q_{1(1)}^j(T-x)+\lambda_2 q_{1(2)}^j(T-x) + \lambda_{12}[\alpha_1 q_{0(1)}^{j-1}(T-x)+\alpha_2 q_{0(2)}^{j-1}(T-x)]\}dx\,, & T\geqq\tau\,, \\ =0\,, & \text{otherwise}\,. \end{cases} \quad (4)$$

$$q_{1(i)}^j(T)=\begin{cases} \int_0^{T-\tau}\int_{y=0}^{\tau} \lambda'_{3-i} e^{-\lambda'_{3-i}x} f_i(x+y) q_{1(3-i)}^{j-1}(T-x-y)dydx + \int_0^{T-\tau} e^{-\lambda'_{3-i}x} f_i(x) q_2^j(T-x)dx\,, & T\geqq\tau\,, \\ =0\,, & \text{otherwise}\,. \end{cases} \quad (5)$$

$$q_{0(i)}^j(T)=\begin{cases} \int_0^{\tau} f_i(x) q_{1(3-i)}^j(T-x)dx\,, & T\geqq\tau\,, \\ =0\,, & \text{otherwise}\,. \end{cases} \quad (6)$$

§ 3. Solution of the problem.

We solve this set of equations by employing the Laplace transform technique. Taking Laplace transforms of both of (1)-(6), we have [1**]

$$q_2^{*0}(s)=\{s+\lambda_{12}(1-e^{-s\tau})+s[\lambda_1 q_{1(1)}^{*0}(s)+\lambda_2 q_{1(2)}^{*0}(s)]\}/[s(s+\lambda_1+\lambda_2+\lambda_{12})], \tag{7}$$

$$q_{1(i)}^{*0}(s)=[s+\lambda'_{3-i}-\lambda'_{3-i}e^{-s\tau}][1-f_i^*(s+\lambda'_{3-i})]/[s(s+\lambda'_{3-i})]+f_i^*(s+\lambda'_{3-i})q_2^{*0}(s), \tag{8}$$

$$q_{0(i)}^{*0}(s)=\{(1-e^{-s\tau})+[e^{-s\tau}F_i(\tau)-f_i^\dagger(s)]\}/s+f_i^\dagger(s)q_{1(3-i)}^{*0}(s), \tag{9}$$

$$q_2^{*j}(s)=\{\lambda_1 q_{1(1)}^{*j}(s)+\lambda_2 q_{1(2)}^{*j}(s)+\lambda_{12}[\alpha_1 q_{0(1)}^{*j-1}(s)+\alpha_2 q_{0(2)}^{*j-1}(s)-\delta_{1j}(1-e^{-s\tau})/s]\}/(s+\lambda_1+\lambda_2+\lambda_{12}) \quad (j\geqq 1), \tag{10}$$

$$q_{1(i)}^{*1}(s)=\beta_i^*(s;\tau)q_{1(3-i)}^{*0}(s)+f_i^*(s+\lambda'_{3-i})q_2^{*1}(s)+\delta_i^*(s;\tau), \tag{11}$$

$$q_{1(i)}^{*j}(s)=\beta_i^*(s;\tau)q_{1(3-i)}^{*j-1}(s)+f_i^*(s+\lambda'_{3-i})q_2^{*j}(s) \quad (j\geqq 2), \tag{12}$$

$$q_{0(i)}^{*j}(s)=f_i^\dagger(s)q_{1(3-i)}^{*j}(s). \tag{13}$$

After considerable manipulation we have

$$Q_{1(i)}^{*0}(s)=q_{1(i)}^{*0}(s)=[\alpha_{3-i}^*(s)\gamma_i^*(s;\tau)+\lambda_{3-i}f_i^*(s+\lambda'_{3-i})\gamma_{3-i}^*(s;\tau)]/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)], \tag{14}$$

$$Q_{0(i)}^{*0}(s)=q_{0(i)}^{*0}(s)=[1-e^{-s\tau}-f_i^\dagger(s)+e^{-s\tau}F_i(\tau)]/s+f_i^\dagger(s)[\alpha_i^*(s)\gamma_{3-i}^*(s;\tau)+\lambda_i f_{3-i}^*(s+\lambda'_i)\gamma_i^*(s;\tau)]/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)], \tag{15}$$

$$Q_2^{*0}(s)=q_2^{*0}(s)=\{s+\lambda_{12}(1-e^{-s\tau})+\sum_{j=1}^{2}\lambda_j[s+\lambda'_{3-j}(1-e^{-s\tau})][1-f_j^*(s+\lambda'_{3-j})]/[s(s+\lambda'_{3-j})]\}/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)], \tag{16}$$

$$Q_{0(i)}^{*n}(s)=[1-f_i^\dagger(s)-e^{-s\tau}(1-F_i(\tau))]/s+f_i^\dagger(s)Q_{1(3-i)}^{*n}(s), \tag{17}$$

$$Q_2^{*n}(s)=\{1+\lambda_1 Q_{1(1)}^{*n}(s)+\lambda_2 Q_{1(2)}^{*n}(s)+\lambda_{12}[\alpha_1 Q_{0(1)}^{*n-1}(s)+\alpha_2 Q_{0(2)}^{*n-1}(s)]\}/(s+\lambda_1+\lambda_2+\lambda_{12}) \tag{18}$$

$$=\Big\{\sum_{j=1}^{2}[\lambda_{12}\alpha_j f_j^\dagger(s)+\lambda_j\beta_j^*(s;\tau]Q_{1(3-j)}^{*n-1}(s)+\sum_{j=1}^{2}\lambda_j\varepsilon_j^*(s;\tau)+\eta^*(s;\tau)\Big\}/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)], \tag{19}$$

$$Q_{1(i)}^{*n}(s)=\beta_i^*(s;\tau)Q_{1(3-i)}^{*n-1}(s)+f_i^*(s+\lambda'_{3-i})Q_2^{*n}(s)+\varepsilon_i^*(s;\tau) \tag{20}$$

$$=[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)]^{-1}\{f_i^*(s+\lambda'_{3-i})[\lambda_{12}\alpha_{3-i}f_{3-i}^\dagger(s)+\lambda_{3-i}\beta_i^*(s;\tau)]Q_{1(i)}^{*n-1}(s)+[\lambda_{12}\alpha_i f_i^\dagger(s)f_i^*(s+\lambda'_{3-i})+\beta_i^*(s;\tau)\alpha_{3-i}^*(s)]Q_{1(3-i)}^{*n-1}(s)+f_i^*(s+\lambda'_{3-i})\eta^*(s;\tau)+\alpha_{3-i}^*(s)\varepsilon_i^*(s;\tau)+\lambda_{3-i}f_i^*(s+\lambda'_{3-i})\varepsilon_{3-i}^*(s;\tau)\}, \text{(2**)} \tag{21}$$

We rewrite (21) as

$$\begin{aligned}Q_{1(1)}^{*n}(s)&=A_1^*(s;\tau)Q_{1(1)}^{*n-1}(s)+B_1^*(s;\tau)Q_{1(2)}^{*n-1}(s)+C_1^*(s;\tau)\\ Q_{1(2)}^{*n}(s)&=A_2^*(s;\tau)Q_{1(2)}^{*n-1}(s)+B_2^*(s;\tau)Q_{1(1)}^{*n-1}(s)+C_2^*(s;\tau)\end{aligned} \tag{22}$$

where

$$A_i^*(s;\tau)=f_i^*(s+\lambda'_{3-i})[\lambda_{12}\alpha_{3-i}f_{3-i}^\dagger(s)+\lambda_{3-i}\beta_{3-i}^*(s;\tau)]/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)]$$

$$B_i^*(s;\tau)=[\lambda_{12}\alpha_i f_i^\dagger(s)f_i^*(s+\lambda'_{3-i})+\alpha_{3-i}^*(s)\beta_i^*(s;\tau)]/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)]$$

$$C_i^*(s;\tau)=\{f_i^*(s+\lambda'_{3-i})[1+\lambda_{12}\sum_{j=1}^{2}\alpha_j[1-f_j^\dagger(s)-e^{-s\tau}(1-F_j(\tau))]/s]+\alpha_{3-i}^*(s)\varepsilon_i^*(s;\tau)+\lambda_{3-i}f_i^*(s+\lambda'_{3-i})\varepsilon_{3-i}^*(s;\tau)\}/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)]. \tag{23}$$

In case when $\tau \neq 0$, we get $A_1^*(s;\tau)A_2^*(s;\tau) \neq B_1^*(s;\tau)B_2^*(s;\tau)$ and $[1-A_1^*(s;\tau]\cdot[1-A_2^*(s;\tau)]-B_1^*(s;\tau)B_2^*(s;\tau)>0$.[3**] Therefore we obtain general solution

$$Q_{1(1)}^{*n}(s)=E^*(s;\tau)\rho^*(s;\tau)^n+F^*(s;\tau)\sigma^*(s;\tau)^n+D_1^*(s;\tau), \tag{24}$$

$$Q_{1(2)}^{*n}(s)=E^*(s;\tau)\{[\rho^*(s;\tau)-A_1^*(s;\tau)]/B_1^*(s;\tau)\}^n$$
$$+F^*(s;\tau)\{[\sigma^*(s;\tau)-A_1^*(s;\tau)]/B_1^*(s;\tau)\}\sigma^*(s;\tau)^n+D_2^*(s;\tau), \tag{25}$$

where

$$\rho^*(s;\tau)=\{A_1^*(s;\tau)+A_2^*(s;\tau)$$
$$+\sqrt{[A_1^*(s;\tau)+A_2^*(s;\tau)]^2-4[A_1^*(s;\tau)A_2^*(s;\tau)-B_1^*(s;\tau)B_2^*(s;\tau)]}\}/2,$$
$$\sigma^*(s;\tau)=A_1^*(s;\tau)+A_2^*(s;\tau)-\rho^*(s;\tau),$$
$$D_i^*(s;\tau)=\{B_i^*(s;\tau)C_{3-i}^*(s;\tau)+C_i^*(s;\tau)[1-A_{3-i}^*(s;\tau)]\}$$
$$/\{[1-A_1^*(s;\tau)][1-A_2^*(s;\tau)]-B_1^*(s;\tau)B_2^*(s;\tau)\},$$
$$E^*(s;\tau)=\{[Q_{1(1)}^{*0}(s)-D_1^*(s;\tau)][\sigma^*(s;\tau)-A_1^*(s;\tau)]$$
$$-[Q_{1(2)}^{*0}(s)-D_2^*(s;\tau)]B_1^*(s;\tau)\}/[\sigma^*(s;\tau)-\rho^*(s;\tau)],$$
$$F^*(s;\tau)=Q_{1(1)}^{*0}(s)-D_1^*(s;\tau)-E^*(s;\tau). \tag{26}$$

Since we get $|\rho^*(s;\tau)|<1$ and $|\sigma^*(s;\tau)|<1$[4**], we have for limiting case as $n\to\infty$

$$Q_{1(i)}^{*\infty}(s)=D_i^*(s;\tau). \tag{27}$$

Using (17) and (19), we obtain $Q_{k(i)}^n(s)$ and $Q_{k(i)}^\infty(s)$ ($k=2,0,\ i=1,2$). Moreover we obtain the mean time to system failure, $\mathrm{MTSF}_{k(i)}^n$ and $\mathrm{MTSF}_{k(i)}^\infty$. For the limiting case as $n\to\infty$, we obtain after considerable manipulation[5**]

$$Q_2^{*\infty}(s)=p^*(s;\tau)\Big\{1+\sum_{j=1}^2\frac{\lambda_j}{s+\lambda_j'}+\lambda_{12}\cdot\sum_{j=1}^2\alpha_j\Big[\frac{1-e^{-s\tau}}{s}+\frac{f_j^\dagger(s)}{s+\lambda_{3-j}'}-F_j^\dagger(s)\Big]$$
$$-\sum_{j=1}^2\frac{N_j^*(s;\tau)}{(s+\lambda_{3-j}')[1-\beta_1^*(s;\tau)\beta_2^*(s;\tau)]}\Big\} \tag{28}$$

where

$$p^*(s;\tau)=\Big\{s+\lambda_1+\lambda_2+\lambda_{12}-\frac{\sum_{j=1}^2 f_j(s+\lambda_{3-j}')[k_j^*(s;\tau)\beta_{3-j}^*(s;\tau)+k_{3-j}^*(s;\tau)]}{1-\beta_1^*(s;\tau)\beta_2^*(s;\tau)}\Big\}^{-1},$$

$$N_j^*(s;\tau)=\frac{\lambda_{3-j}'}{s}\{k_{3-j}^*(s;\tau)-k_j^*(s;\tau)+[k_j^*(s;\tau)\beta_{3-j}^*(s;\tau)-k_{3-j}^*(s;\tau)\beta_j^*(s;\tau)]\}$$
$$+[k_j^*(s;\tau)\beta_{3-j}^*(s;\tau)+k_{3-j}^*(s;\tau)]\Big\{f_j^*(s+\lambda_{3-j}')+\lambda_{3-j}'\Big[\frac{e^{-s\tau}-1}{s}+F_j^\dagger(s)\Big]\Big\} \tag{29}$$

(28) coincides with formula (22) in Fukuta and Kodama [1]. Therefore all the results in this paper for the limiting case coincide with the results in [1]. Hence we cite only the finite case.

Especially when $\tau=0$, we have $\beta_i^*(s;0)=f_i^\dagger(s)=0$ and $A_i^*(s;0)=B_i^*(s;0)=0$. Hence we have

$$Q_{1(i)}^{*n}(s)=D_i^*(s;\tau)=C_i^*(s;\tau)=\{f_i^*(s+\lambda_{3-i}')+\alpha_{3-i}^*(s)[1-f_i^*(s+\lambda_{3-i}')]/(s+\lambda_{3-i}')$$
$$+\lambda_{3-i}f_i^*(s+\lambda_{3-i}')[1-f_{3-i}^*(s+\lambda_i')]/(s+\lambda_i')\}/[\alpha_1^*(s)-\lambda_2f_2^*(s+\lambda_1')]$$
$$\text{for all } n, \tag{30}$$

$$Q_2^{*n}(s)=\Big\{1+\sum_{j=1}^{2}\lambda_j[1-f_j(s+\lambda'_{3-j})]/(s+\lambda'_{3-j})\Big\}/[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)] \qquad \text{for all } n. \tag{31}$$

Above formulae coincide the Laplace transform of probability that the major breakdown does not occur during the time interval $(0, t)$, given that the system starts in $E_1(i)$ or E_2 at $t=0$.

For two identical unit model, the formula do not depend on i, so we define $q_{1(i)}^{*j}(s)=q_1^{*j}(s)$, $Q_{1(i)}^{*j}(s)=Q_1^{*j}(s), \cdots$, so on. Noting that $\rho^*(s;\tau)=A^*(s;\tau)+B^*(s;\tau)$, $\sigma^*(s;\tau)=A^*(s;\tau)-B^*(s;\tau)$, $E^*(s;\tau)=Q_1^{*0}(s)-D^*(s;\tau)$ and $F^*(s;\tau)=0$, we have

$$\begin{aligned}Q_1^{*n}(s)=&\Big\{\frac{\lambda_{12}f^{\dagger}(s)f^*(s+\lambda')+(s+2\lambda+\lambda_{12})\beta^*(s;\tau)}{\alpha^*(s)-\lambda f^*(s+\lambda')}\Big\}^n\Big\{\frac{(s+2\lambda+\lambda_{12})\gamma^*(s;\tau)}{\alpha^*(s)-\lambda f^*(s+\lambda')}\Big\}\\&+\Big\{1-\Big[\frac{\lambda_{12}f^{\dagger}(s)f^*(s+\lambda')+(s+2\lambda+\lambda_{12})\beta^*(s;\tau)}{\alpha^*(s)-\lambda f^*(s+\lambda')}\Big]^n\Big\}\\&\cdot\Big\{\frac{f^*(s+\lambda')[1+\lambda_{12}[1-f^{\dagger}(s)-e^{-s\tau}(1-F(\tau))]/s]+(s+2\lambda+\lambda_{12})\varepsilon(s;\tau)}{(s+2\lambda+\lambda_{12})[1-\beta^*(s;\tau)]-[2\lambda+\lambda_{12}f^{\dagger}(s)]f^*(s+\lambda')}\Big\},\end{aligned} \tag{32}$$

$$Q_2^{*n}(s)=\frac{1+[\lambda_{12}f^{\dagger}(s)+2\lambda\beta^*(s;\tau)]Q_1^{*n-1}(s)+2\lambda\varepsilon^*(s;\tau)+\lambda_{12}[1-f^{\dagger}(s)-e^{-s\tau}(1-F(\tau))]/s}{\alpha^*(s)-\lambda f^*(s+\lambda')}, \tag{33}$$

$$\begin{aligned}\mathrm{MTSF}_1^n=&\Big\{\frac{\lambda_{12}F(\tau)f^*(\lambda')+(2\lambda+\lambda_{12})\beta^*(0;\tau)}{\lambda_{12}+2\lambda[1-f^*(\lambda')]}\Big\}^n\Big\{\frac{(2\lambda+\lambda_{12})\gamma^*(0;\tau)}{\lambda_{12}+2\lambda[1-f^*(\lambda')]}\Big\}\\&+\Big\{1-\Big[\frac{\lambda_{12}F(\tau)f^*(\lambda')+(2\lambda+\lambda_{12})\beta^*(0;\tau)}{\lambda_{12}+2\lambda[1-f^*(\lambda')]}\Big]^n\Big\}\\&\cdot\Big\{\frac{f^*(\lambda')\Big[1+\lambda_{12}\Big(\tau-\int_0^{\tau}F(t)dt\Big)\Big]+(2\lambda+\lambda_{12})\varepsilon^*(0;\tau)}{(2\lambda+\lambda_{12})[1-\beta^*(0;\tau)]-[2\lambda+\lambda_{12}F(\tau)]f^*(\lambda')}\Big\},\end{aligned} \tag{34}$$

$$\mathrm{MTSF}_2^n=\frac{1+[\lambda_{12}F(\tau)+2\lambda\beta^*(0;\tau)]\,\mathrm{MTSF}_1^{n-1}+2\lambda\varepsilon^*(0;\tau)+\lambda_{12}\Big[\tau-\int_0^{\tau}F(t)dt\Big]}{\lambda_{12}+2\lambda[1-f^*(\lambda)]}. \tag{35}$$

Especially when $F(t)=1-\exp(-\mu t)$, we have[6**]

$$\begin{aligned}\mathrm{MTSF}_1^n=&[1-e^{-\mu\tau}]^n\Big[\tau+\frac{2\lambda+\lambda_{12}+\mu}{2\lambda\lambda'+\lambda_{12}(\lambda'+\mu)}\Big]\\&+[1-(1-e^{-\mu\tau})^n]\Big\{\Big[\frac{1}{\mu}+\frac{2\lambda+\lambda_{12}+\mu}{2\lambda\lambda'+\lambda_{12}(\lambda'+\mu)}\Big]e^{-\mu\tau}-\frac{1}{\mu}\Big\},\end{aligned} \tag{36}$$

$$\mathrm{MTSF}_2^n=\frac{(2\lambda+\lambda'+\mu)+[2\lambda\lambda'+\lambda_{12}(\lambda'+\mu)](1-e^{-\mu\tau})\Big[\mathrm{MTSF}_1^{n-1}+\frac{1}{\mu}\Big]}{2\lambda\lambda'+\lambda_{12}(\lambda'+\mu)}. \tag{37}$$

The improvement factor $I^n=\lambda\,\mathrm{MTSF}_2^n$ (the ratio of the MTSF for this system to one unit system without repair) are show in Table 1 for the case $\lambda_{12}=0$ and $\lambda'=\lambda$.

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Table 1 The improvement factor I^n

$\rho(=\mu/\lambda)$	n	$m(=\mu\tau)$			
		10^{-2}	10^{-1}	1	10
100	0	0.5150E 02	0.5150E 02	0.5151E 02	0.5160E 02
	1	0.5201E 02	0.5635E 02	0.8375E 02	0.1026E 03
	3	0.5201E 02	0.5686E 02	0.1170E 03	0.2046E 03
	5	0.5201E 02	0.5686E 02	0.1303E 03	0.3066E 03
	10	0.5201E 02	0.5686E 02	0.1383E 03	0.5616E 03
	100	0.5201E 02	0.5686E 02	0.1391E 03	0.5141E 04
	∞	0.5201E 02	0.5686E 02	0.1391E 03	0.1124E 07

Appendix

DERIYATION OF (1**)

$$q^{*j}_{1(i)}(s)=\int_0^\infty e^{-st}q^j_{1(i)}(t)dt, \qquad j\geqq 1$$

$$=\int_\tau^\infty e^{-sT}dT\int_0^{T-\tau}dx\int_0^\tau \lambda'_{3-i}e^{-\lambda'_{3-i}x}f_i(x+y)q^{j-1}_{1(3-i)}(T-x-y)dy$$

$$+\int_\tau^\infty e^{-sT}dT\int_0^{T-\tau}e^{-\lambda'_{3-i}x}f_i(x)q^j_2(T-x)dx$$

$$=\int_0^\tau dy\int_0^\infty \lambda'_{3-i}e^{-\lambda'_{3-i}x}f_i(x+y)dx\int_{x+\tau}^\infty e^{-sT}q^{j-1}_{1(3-i)}(T-x-y)dT$$

$$+\int_0^\infty e^{-\lambda'_{3-i}x}f_i(x)\,dx\int_{x+\tau}^\infty e^{-sT}q^j_2(T-x)dT$$

$$=\int_0^\tau e^{-sy}dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)dx\int_{\tau-y}^\infty e^{-su}q^{j-1}_{1(3-i)}(u)du$$

$$+\int_0^\infty e^{-\lambda'_{3-i}x}f_i(x)dx\int_\tau^\infty e^{-s(x+u)}q^j_2(u)du$$

$$=\int_0^\tau e^{-sy}dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)dx\Big[q^{*j-1}_{1(3-i)}(s)-\int_0^{\tau-y}e^{-su}q^{j-1}_{1(3-i)}(u)du\Big]$$

$$+\int_0^\infty e^{-(s+\lambda'_{3-i})x}f_i(x)dx\Big[\int_0^\infty e^{-su}q^j_2(u)du-\int_0^\tau e^{-su}q^j_2(u)du\Big]$$

$$=\int_0^\tau e^{-sy}dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)dx\{q^{*j-1}_{1(3-i)}(s)-\delta_{1j}[1-e^{-s(\tau-y)}]/s\}$$

$$+f^*_i(s+\lambda'_{3-i})q^{*j}_2(s)\,.$$

(i) $j=1$

$$\int_0^\tau e^{-sy}dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)\Big[q^{*j-1}_{1(3-i)}(s)-\frac{1}{s}+\frac{1}{s}e^{-s(\tau-y)}\Big]dx$$

$$=\beta^*_i(s\,;\,\tau)\Big[q^{*j-1}_{1(3-i)}(s)-\frac{1}{s}\Big]+\frac{1}{s}e^{-s\tau}\int_0^\tau dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)dx$$

where

$$\beta_i^*(s;\tau)=\int_0^\tau e^{-sy}dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)dx$$

$$=\int_0^\tau e^{-sy}dy\int_y^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})(v-y)}f_i(v)dv$$

$$=\int_0^\tau \lambda'_{3-i}e^{-\lambda'_{3-i}y}dy\Big[\int_0^\infty e^{-(s+\lambda'_{3-i})v}f_i(v)dv-\int_0^y e^{-(s+\lambda'_{3-i})v}f_i(v)\,dv\Big]$$

$$=(e^{\lambda'_{3-i}\tau}-1)f_i^*(s+\lambda'_{3-i})-\int_0^\tau e^{-(s+\lambda'_{3-i})v}f_i(v)\,dv\int_v^\tau \lambda'_{3-1}e^{\lambda'_{3-i}y}dy$$

$$=(e^{\lambda'_{3-i}\tau}-1)f_i^*(s+\lambda'_{3-i})-e^{\lambda'_{3-i}\tau}f_i^\dagger(s+\lambda'_{3-i})+f_i^\dagger(s),$$

$$\int_0^\tau dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)dx$$

$$=\int_0^\tau dy\int_y^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})(v-y)}f_i(v)dv$$

$$=\int_0^\tau \lambda'_{3-i}e^{(s+\lambda'_{3-i})y}dy\Big[\int_0^\infty e^{-(s+\lambda'_{3-i})v}f_i(v)dv-\int_0^y e^{-(s+\lambda'_{3-i})v}f_i(v)\,dv\Big]$$

$$=\lambda'_{3-i}f_i^*(s+\lambda'_{3-i})[e^{(s+\lambda'_{3-i})\tau}-1]/(s+\lambda'_{3-i})$$

$$-\int_0^\tau e^{-(s+\lambda'_{3-i})v}f_i(v)\,dv\int_v^\tau \lambda'_{3-i}e^{(s+\lambda'_{3-i})y}dy$$

$$=\frac{\lambda'_{3-i}}{s+\lambda'_{3-i}}\Big\{f_i^*(s+\lambda'_{3-i})[e^{(s+\lambda'_{3-i})\tau}-1]-f_i^\dagger(s+\lambda'_{3-i})e^{(s+\lambda'_{3-i})\tau}+F_i(\tau)\Big\}.$$

Hence we have

$$q^{*1}_{1(i)}(s)=\beta_i(s;\tau)\Big[q^{*0}_{1(3-i)}(s)-\frac{1}{s}\Big]$$

$$+\frac{\lambda'_{3-i}e^{-s\tau}}{s(s+\lambda'_{3-i})}\Big\{f_i^*(s+\lambda'_{3-i})[e^{(s+\lambda'_{3-i})\tau}-1]-f_i^\dagger(s+\lambda'_{3-i})e^{(s+\lambda'_{3-i})\tau}+F_i(\tau)\Big\}$$

$$+f_i^*(s+\lambda'_{3-i})q_2^{*1}(s)$$

$$=\beta_i(s;\tau)q^{*0}_{1(3-i)}(s)+f_i^*(s+\lambda'_{3-i})q_2^{*1}(s)+\delta_i^*(s;\tau).$$

(ii) $j\geqq 2$

$$q^{*j}_{1(i)}(s)=\int_0^\tau e^{-sy}dy\int_0^\infty \lambda'_{3-i}e^{-(s+\lambda'_{3-i})x}f_i(x+y)dx\cdot q^{*j-1}_{1(3-i)}(s)+f_i^*(s+\lambda'_{3-i})q_2^{*j}(s)$$

$$=\beta_i^*(s;\tau)q^{*j-1}_{1(3-i)}(s)+f_i^*(s+\lambda'_{3-i})q_2^{*j}(s).$$

Similary we have the formulae (7)-(10) and (13).

DERIVATIONS OF (2**)

Substituting (17) and (18) into (20) and arranging, we have

$$Q^{*n}_{1(1)}(s)[s+\lambda_1+\lambda_2+\lambda_{12}-\lambda_1 f_1^*(s+\lambda'_2)]$$

$$=\lambda_2 f_1^*(s+\lambda'_2)Q^{*n}_{1(2)}(s)+f_1^*(s+\lambda'_2)\Big\{1+\lambda_{12}\cdot\sum_{j=1}^{2}\alpha_j[[1-f_j^\dagger(s)-e^{-s\tau}(1-F_j(\tau))]/s$$

$$+f_j(s)Q^{*n-1}_{1(3-j)}(s)]\Big\},$$

$$Q^{*n}_{1(2)}(s)[s+\lambda_1+\lambda_2+\lambda_{12}-\lambda_2 f^*_2(s+\lambda'_1)]$$
$$=\lambda_1 f^*_2(s+\lambda'_1)Q^{*n}_{1(1)}(s)+f^*_2(s+\lambda'_1)\Big\{1+\lambda_{12}\cdot\sum_{j=1}^{2}\alpha_j[[1-f^\dagger_j(s)-e^{-s\tau}(1-F_j(\tau))]/s$$
$$+f^\dagger_j(s)Q^{*n-1}_{1(3-j)}(s)]\Big\}.$$

From the above equations we have (21) and (22).

DERIVATION OF (3**)

After simple calculation, we have the following inequalities

$$\beta^*_i(s;\tau)\geqq(e^{\lambda_{3-i}\tau}-1)[f^*_i(s+\lambda_{3-i})-f^\dagger_i(s+\lambda_{3-i})]$$
$$\geqq 0, \tag{38}$$

the equality is true when $\tau=0$

$$1-f^*_i(s+\lambda'_{3-i})-\beta^*_i(s;\tau)$$
$$=1-e^{\lambda'_{3-i}\tau}\int_\tau^\infty e^{-(s+\lambda'_{3-i})x}f_i(x)dx-f^\dagger_i(s)$$
$$>1-e^{\lambda'_{3-i}\tau}\cdot e^{-(s+\lambda'_{3-i})\tau}[1-F_i(\tau)]-F_i(\tau)$$
$$\geqq 1-[1-F_i(\tau)]-F_i(\tau)$$
$$=0. \tag{39}$$

$$1-f^*_i(s+\lambda'_{3-i})-\beta^*_i(s;\tau)[\beta^*_{3-i}(s;\tau)+f^*_{3-i}(s+\lambda'_3)]$$
$$>1-f^*_i(s+\lambda'_{3-i})-\beta_i(s;\tau) \qquad \text{by (39)}$$
$$>0. \qquad \text{by (39)} \tag{40}$$

$$1-\beta^*_i(s;\tau)\beta^*_{3-i}(s;\tau)-\alpha_{3-i}f^\dagger_{3-i}(s)[f^*_i(s+\lambda'_{3-i})+f^*_{3-i}(s+\lambda'_i)\beta^*_i(s;\tau)]$$
$$-\alpha_i f^\dagger_i(s)[f^*_{3-i}(s+\lambda'_i)+f^*_i(s+\lambda'_{3-i})\beta^*_{3-i}(s;\tau)]$$
$$>1-\alpha_i\{f^*_{3-i}(s+\lambda'_i)+\beta^*_{3-i}(s;\tau)[f^*_i(s+\lambda'_{3-i})+\beta^*_i(s;\tau)]\}$$
$$-\alpha_{3-i}\{f^*_i(s+\lambda'_{3-i})+\beta^*_i(s;\tau)[f^*_{3-i}(s+\lambda'_i)+\beta^*_{3-i}(s;\tau)]\}$$
$$>1-\alpha_i-\alpha_{3-i} \qquad \text{by (40)}$$
$$=0. \tag{41}$$

It is clear that

$$1-\beta^*_i(s;\tau)\beta^*_{3-i}(s;\tau)>0, \qquad \alpha^*_1(s)-\lambda_2 f^*_2(s+\lambda'_1)>0. \tag{42}$$

Also from (23) we have

$$[1-A^*_1(s;\tau)][1-A^*_2(s;\tau)]-B^*_1(s;\tau)B^*_2(s;\tau)$$
$$=[\alpha^*_1(s)-\lambda_2 f^*_2(s+\lambda'_1)]^{-1}\{s[1-\beta^*_1(s;\tau)\beta^*_2(s;\tau)]$$
$$+\lambda_1[1-\beta^*_1(s;\tau)\beta^*_2(s;\tau)-f^*_1(s+\lambda'_2)-f^*_2(s+\lambda'_1)\beta^*_1(s;\tau)]$$
$$+\lambda_2[1-\beta^*_1(s;\tau)\beta^*_2(s;\tau)-f^*_2(s+\lambda_1)-f^*_1(s+\lambda'_2)\beta^*_2(s;\tau)]$$
$$+\lambda_{12}[1-\beta^*_1(s;\tau)\beta^*_2(s;\tau)-\alpha_2 f^\dagger_2(s)[f^*_1(s+\lambda'_2)+f^*_2(s+\lambda'_1)\beta^*_1(s;\tau)]$$
$$-\alpha_1 f^\dagger_1(s)[f^*_2(s+\lambda'_1)+f^*_1(s+\lambda'_2)\beta^*_2(s;\tau)]]\}.$$

Therefore from (35)-(40) we have

$$[1-A_1^*(s;\tau)][1-A_2^*(s;\tau)]-B_1^*(s;\tau)B_2^*(s;\tau)>0. \tag{43}$$

DERIVATION OF (4**)

From the definition and (35) we have

$$f_i^*(s+\lambda'_{3-i})[\lambda_{12}\alpha_{3-i}f_{3-i}^\dagger(s)+\lambda_{3-i}\beta_{3-i}^*(s;\tau)]\geqq 0, \tag{44}$$

$$[\lambda_{12}\alpha_i f_i^\dagger(s)f_i^*(s+\lambda'_{3-i})+\alpha_{3-i}^*(s)\beta_i^*(s;\tau)]\geqq 0, \tag{45}$$

the equalities are true when $\tau=0$.

Also we have

$$\begin{aligned}&\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)-f_i^*(s+\lambda'_{3-i})[\lambda_{12}\alpha_{3-i}f_{3-i}^\dagger(s)+\lambda_{3-i}\beta_{3-i}^*(s;\tau)]\\&\quad>\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)-[\lambda_{12}+\lambda_{3-i}\beta_{3-i}^*(s;\tau)]\\&\quad=s+\lambda_1[1-f_1^*(s+\lambda'_2)-\delta_{i2}\beta_{3-i}^*(s;\tau)]+\lambda_2[1-f_1^*(s+\lambda'_1)-\delta_{i1}\beta_{3-i}^*(s;\tau)]\\&\quad>0,\qquad \text{by}\ (40)\end{aligned} \tag{46}$$

$$\begin{aligned}&\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)-\lambda_{12}\alpha_i f_i^\dagger(s)f_i^*(s+\lambda'_{3-i})-\alpha_{3-i}^*(s)\beta_i^*(s;\tau)\\&\quad\geqq s[1-\beta_i^*(s;\tau)]+\lambda_1[1-f_1^*(s+\lambda'_2)-\beta_i^*(s;\tau)+\delta_{i2}\beta_{3-i}^*(s;\tau)]\\&\qquad+\lambda_2[1-f_2^*(s+\lambda'_1)-\beta_i^*(s;\tau)+\delta_{i1}\beta_{3-i}^*(s;\tau)]+\lambda_{12}[1-f_i^*(s+\lambda'_{3-i})-\beta_i^*(s;\tau)]\\&\quad>0.\qquad \text{by}\ (40)\end{aligned} \tag{47}$$

From (44)-(47) we have

$$1>A_i^*(s;\tau)\geqq 0,\qquad 1>B_i^*(s;\tau)\geqq 0. \tag{48}$$

Since we have $A_1^*(s;\tau)+A_2^*(s;\tau)>0$, $(\tau\neq 0)$, to prove $|\rho^*(s;\tau)|<1$ and $|\sigma^*(s;\tau)|<1$ it suffices to show the following inequalities

$$1>A_1^*(s;\tau)A_2^*(s;\tau)-B_1^*(s;\tau)B_2^*(s;\tau)>A_1^*(s;\tau)+A_2^*(s;\tau)-1. \tag{49}$$

The inequality on the right becomes

$$[1-A_1^*(s;\tau)][1-A_2^*(s;\tau)]-B_1^*(s;\tau)B_2^*(s;\tau)>0$$

which is true by (43).

On the other hand, the inequality on the left is clear from (48), $(\tau\neq 0)$. Hence we have $|\rho^*(s;\tau)|<1$ and $|\sigma^*(s;\tau)|<1$.

DERIVATION OF (5**).

We obtain after considerable manipulation

$$\begin{aligned}&[1-A_1^*(s;\tau)][1-A_2^*(s;\tau)]-B_1^*(s;\tau)B_2^*(s;\tau)\\&\quad=[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda'_1)]^{-1}\Big\{(s+\lambda_1+\lambda_2+\lambda_{12})[1-\beta_1^*(s;\tau)\beta_2^*(s;\tau)]\\&\qquad-\sum_{j=1}^{2}f_j^*(s+\lambda'_{3-j})\cdot[k_{3-j}^*(s;\tau)+\beta_{3-j}^*(s;\tau)k_j^*(s;\tau)]\Big\}\end{aligned} \tag{50}$$

$$B_i^*(s\,;\tau)C_{3-i}^*(s\,;\tau)+C_i^*(s\,;\tau)[1-A_{3-i}^*(s\,;\tau)]$$
$$=[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda_1')]^{-1}\{\eta^*(s\,;\tau)[f_1^*(s+\lambda_{3-i}')+\beta_i^*(s\,;\tau)f_{3-i}^*(s+\lambda_i')]$$
$$+\varepsilon_{3-i}^*(s\,;\tau)\cdot[(s+\lambda_1+\lambda_2+\lambda_{12})\beta_i^*(s\,;\tau)+k_i^*(s\,;\tau)f_i^*(s+\lambda_{3-i}')]$$
$$+\varepsilon_i^*(s\,;\tau)[s+\lambda_1+\lambda_2+\lambda_{12}-f_{3-i}^*(s+\lambda_i')k_i^*(s\,;\tau)]\}\,. \tag{50}$$

Hence we have

$$Q_{1(i)}^{*\infty}(s)=D_i^*(s\,;\tau)=\{\eta^*(s\,;\tau)[f_i^*(s+\lambda_{3-i}')+\beta_i^*(s\,;\tau)f_{3-i}^*(s+\lambda_i')]$$
$$+\varepsilon_{3-i}^*(s\,;\tau)[(s+\lambda_1+\lambda_2+\lambda_{12})\cdot\beta_i^*(s\,;\tau)+k_i^*(s\,;\tau)f_i^*(s+\lambda_{3-i}')]$$
$$+\varepsilon_i^*(s\,;\tau)[(s+\lambda_1+\lambda_2+\lambda_{12})-f_{3-i}^*(s+\lambda_i')k_i^*(s\,;\tau)]\}$$
$$/\{(s+\lambda_1+\lambda_2+\lambda_{12})[1-\beta_1^*(s\,;\tau)\beta_2^*(s\,;\tau)]$$
$$-\sum_{j=1}^{2}f_j^*(s+\lambda_{3-j}')[k_{3-j}^*(s\,;\tau)+\beta_{3-j}^*(s\,;\tau)k_j^*(s\,;\tau)]\}\,. \tag{52}$$

Noting that $k_i^*(s\,;\tau)=\lambda_{3-i}+\lambda_{12}\alpha_i f_i^\dagger(s)$, and from (19) and (52) we have

$$Q_2^{*\infty}(s)=[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda_1')]^{-1}\Big\{\sum_{j=1}^{2}[\lambda_{12}\alpha_j f_j^\dagger(s)+\lambda_j\beta_j^*(s\,;\tau)]Q_{1(3-j)}^{*\infty}(s)$$
$$+\sum_{j=1}^{2}\lambda_j\varepsilon_j^*(s\,;\tau)+\eta^*(s\,;\tau)\Big\}$$
$$=[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda_1')]^{-1}\Big\{(s+\lambda_1+\lambda_2+\lambda_{12})[1-\beta_1^*(s\,;\tau)\beta_2^*(s\,;\tau)]$$
$$-\sum_{j=1}^{2}f_j(s+\lambda_{3-j}')[k_{3-j}^*(s\,;\tau)+\beta_{3-j}^*(s\,;\tau)k_j^*(s\,;\tau)]\Big\}^{-1}$$
$$\cdot\{\eta^*(s\,;\tau)\{[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda_1')][1-\beta_1^*(s\,;\tau)\beta_2^*(s\,;\tau)]\}\}$$
$$+\sum_{j=1}^{2}\varepsilon_j^*(s\,;\tau)\{[\alpha_1^*(s)-\lambda_2 f_2^*(s+\lambda_1')][k_j^*(s\,;\tau)\beta_{3-i}^*(s\,;\tau)+k_{3-j}^*(s\,;\tau)]\} \tag{53}$$

where

$$\sum_{j=1}^{2}\varepsilon_j^*(s\,;\tau)[k_j^*(s\,;\tau)\beta_{3-j}^*(s\,;\tau)+k_{3-j}^*(s\,;\tau)]$$
$$=[1-\beta_1^*(s\,;\tau)\beta_2^*(s\,;\tau)]\sum_{j=1}^{2}[\lambda_{3-j}+\lambda_{12}\alpha_j f_j^\dagger(s)]/(s+\lambda_{3-j}')$$
$$-\sum_{j=1}^{2}\Big\{\frac{\lambda_{3-j}'}{s}\{k_{3-j}^*(s\,;\tau)-k_j^*(s\,;\tau)+[k_j^*(s\,;\tau)\beta_{3-j}^*(s\,;\tau)-k_{3-j}^*(s\,;\tau)\beta_j^*(s\,;\tau)]\}$$
$$+[k_j^*(s\,;\tau)\beta_{3-j}^*(s\,;\tau)+k_{3-j}^*(s\,;\tau)]$$
$$\cdot\Big[f_j^*(s+\lambda_{3-j}')+\lambda_{3-j}'\Big(\frac{e^{-s\tau}-1}{s}+F_j^\dagger(s)\Big)\Big]\Big\}/(s+\lambda_{3-j}') \tag{54}$$

Hence we have (5**) from (53) and (54).

DERIVATION OF (6**)

After simple calculations we have

$$\lambda_{12}+2\lambda[1-f^*(\lambda')]=\frac{2\lambda\lambda'+\lambda_{12}(\lambda'+\mu)}{\lambda'+\mu},\ (2\lambda+\lambda_{12})\gamma^*(0,\tau)=\frac{(2\lambda+\lambda_{12})(1+\lambda'\tau)+\mu(1+\lambda_{12}\tau)}{\lambda'+\mu},$$

$$\lambda_{12}F(\tau)f^*(\lambda')+(2\lambda+\lambda_{12})\beta^*(0\,;\tau)=\frac{[2\lambda\lambda'+\lambda_{12}(\lambda'+\mu)][1-e^{-\mu\tau}]}{\lambda'+\mu},$$

$$f^*(\lambda')\Big[1+\lambda_{12}\Big(\tau-\int_0^\tau F(t)dt\Big)\Big]+(2\lambda+\lambda_{12})\varepsilon^*(0\,;\tau)$$

$$=\frac{\mu[\mu+\lambda_{12}(1-e^{-\mu\tau})]+(2\lambda+\lambda_{12})[\mu+\lambda'(1-e^{\mu\tau})]}{\mu(\lambda'+\mu)},$$

$$(2\lambda+\lambda_{12})[1-\beta^*(0\,;\tau)]-[2\lambda+\lambda_{12}F(\tau)]f^*(\lambda')=\frac{[\mu\lambda_{12}+\lambda'(2\lambda+\lambda_{12})]e^{-\mu\tau}}{\lambda'+\mu},$$

$$1+2\lambda\varepsilon^*(0\,;\tau)+\lambda_{12}\Big[\tau-\int_0^\tau F(t)dt\Big]=\frac{\mu(2\lambda+\lambda'+\mu)+[2\lambda\lambda'+\lambda_{12}(\lambda'+\mu)](1-e^{-\mu\tau})}{\mu(\lambda'+\mu)}, \quad (55)$$

From (34), (35) and (55) we have (6**).

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