# ON TESTS FOR EQUALITY OF COHERENCES OF TWO BIVARIATE STATIONARY TIME SERIESES

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# ON TESTS FOR EQUALITY OF COHERENCES OF TWO BIVARIATE STATIONARY TIME SERIESES

By

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## 1. Introduction.

In this paper we shall consider test procedures concerning equality of coherences of two bivariate stationary time serieses. They are the likelihood ratio test and its approximating chi-square test and a test based an asymptotic normality of sample coherences.

Let $\pi_1$ and $\pi_2$ be two populations whose characteristics are measured by real bivariate stationary time serieses, say, $\boldsymbol{x}^{(1)}(t)=[x_1^{(1)}(t), x_2^{(2)}(t)]'$, $t=0, \pm 1, \pm 2, \cdots$ and $\boldsymbol{x}^{(2)}(t)=[x_1^{(2)}(t), x_2^{(2)}(t)]'$, $t=0, \pm 1, \pm 2, \cdots$ with zero means and having spectral density matrices

$$\boldsymbol{f}^{(1)}(\lambda)=\begin{bmatrix} f_{11}^{(1)}(\lambda) & f_{12}^{(1)}(\lambda) \\ f_{21}^{(1)}(\lambda) & f_{22}^{(1)}(\lambda) \end{bmatrix} \qquad -\pi \leqq \lambda \leqq \pi \tag{1.1}$$
$$\boldsymbol{f}^{(2)}(\lambda)=\begin{bmatrix} f_{11}^{(2)}(\lambda) & f_{12}^{(2)}(\lambda) \\ f_{21}^{(2)}(\lambda) & f_{22}^{(2)}(\lambda) \end{bmatrix} \qquad -\pi \leqq \lambda \leqq \pi$$

respectively.

We shall denote coherences of $\boldsymbol{x}^{(j)}(t)$, $j=1, 2$ by

$$\omega^{(j)}(\lambda)=\frac{|f_{12}^{(j)}(\lambda)|}{\{f_{11}^{(j)}(\lambda)f_{22}^{(j)}(\lambda)\}^{\frac{1}{2}}} \qquad j=1, 2 \tag{1.2}$$

and phases of $\boldsymbol{x}^{(j)}(t)$, $j=1, 2$ by

$$\theta^{(j)}(\lambda)=\tan^{-1}\left\{-\frac{q_{12}^{(j)}(\lambda)}{c_{12}^{(j)}(\lambda)}\right\} \qquad j=1, 2 \tag{1.3}$$

where $c_{12}^{(j)}(\lambda)$ and $q_{12}^{(j)}(\lambda)$ are the co-spectral densities and the quadrature spectral densities of $\boldsymbol{x}^{(j)}(t)$ respectively, that is, $f_{12}^{(j)}(\lambda)=\frac{1}{2}\{c_{12}^{(j)}(\lambda)-iq_{12}^{(j)}(\lambda)\}$.

For a given frequency $\lambda$ ($\lambda \neq 0, \pm\pi$), we shall consider test procedures for a null hypothesis

$$H_0: \quad \omega^{(1)}(\lambda)=\omega^{(2)}(\lambda)=\omega(\lambda)$$

against an alternative hypothesis

$$H_1: \quad \omega^{(1)}(\lambda)\neq\omega^{(2)}(\lambda).$$

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Throughout this paper we shall fix the frequency $\lambda$, $(\lambda \neq 0, \pm\pi)$, and drop the argument variable $\lambda$ in $\boldsymbol{f}^{(j)}(\lambda)$, $f_{kl}^{(j)}(\lambda)$, $\theta^{(j)}(\lambda)$ and $\omega^{(j)}(\lambda)$.

## 2. F. F. T. data and their asymptotic normality.

Let $\{\boldsymbol{x}_k^{(j)}(t), t=1, 2, \cdots, T\}$, $k=1, 2, \cdots, N_j$, $j=1, 2$ be $N_1$ and $N_2$ independently observed bivariate stationary time serieses of length $T$ from $\pi_1$ and $\pi_2$ respectively.

We shall define the F. F. T. data of the observed time serieses by

$$\boldsymbol{w}_k^{(j)}(u)=\frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T} e^{it\lambda_u}[\boldsymbol{x}_k^{(j)}(t)-\bar{\boldsymbol{x}}_k^{(j)}] \qquad k=1, 2, \cdots, N_j, \quad u=1, 2, \cdots, q, \quad j=1, 2 \tag{2.1}$$

where

$$\lambda_u=\lambda+(u-1-m)\frac{2\pi}{T}, \qquad u=1, 2, \cdots, q, \quad q=2m+1 \ (m \geqq 0 \text{ integer}).$$

and

$$\bar{\boldsymbol{x}}_k^{(j)}=\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_k^{(j)}(t).$$

In the case where values of the serieses, well-separated in time, are only weakly statistically dependent, the F. F. T. data $\{\boldsymbol{w}_k^{(j)}(u), u=1, 2, \cdots, q, k=1, 2, \cdots, N_j\}$, $j=1, 2$ are asymptotically independent bivariate complex normal with zero means and variance-covariance matrices $\boldsymbol{f}^{(j)}$ $j=1, 2$, respectively.

Sufficient conditions for the asymptotic normality of the F. F. T. data are demonstrated by many authors, for example; M. Rosenblatt [5], E. J. Hannan [4] and D. Brillinger ([1], [2]).

In this paper we assume that our F. F. T. data are asymptotically bivariate complex normal as stated above and we shall treat them as if they follow their limit distributions.

## 3. Likelihood ratio criterion.

Under the hypothesis $H_1$ there are eight unknown parameters

$$\boldsymbol{\theta}_1=\{\boldsymbol{\theta}_1^{(1)}, \boldsymbol{\theta}_1^{(2)}\}=\{(f_{11}^{(1)}, f_{22}^{(1)}, \omega^{(1)}, \theta^{(1)}), (f_{11}^{(2)}, f_{22}^{(2)}, \omega^{(2)}, \theta^{(2)})\}$$

for the joint distribution of the F. F. T. data

$$\boldsymbol{w}=\{\boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}\}=\{(\boldsymbol{w}_k^{(1)}(u), u=1, 2, \cdots, q, k=1, 2, \cdots, N_1), (\boldsymbol{w}_k^{(2)}(u), u=1, 2, \cdots, q, k=1, 2, \cdots, N_2)\}.$$

Let us denote the likelihood functions of $\boldsymbol{\theta}_1^{(j)}$ based on the F. F. T. data $\boldsymbol{w}^{(j)}$ by

$$L(\boldsymbol{\theta}_1^{(j)}; \boldsymbol{w}^{(j)})=\pi^{-2\nu_j}|\boldsymbol{f}^{(j)}|^{-\nu_j}\exp\{-\nu_j \operatorname{tr}(\boldsymbol{f}^{(j)-1}\hat{\boldsymbol{f}}^{(j)})\}$$

where $\nu_j=N_j q$.

Then, the likelihood function of $\boldsymbol{\theta}_1$ based on $\boldsymbol{w}$ is given by

$$\begin{aligned} L(\boldsymbol{\theta}_1; \boldsymbol{w}) &= L(\boldsymbol{\theta}_1^{(1)}; \boldsymbol{w}^{(1)})L(\boldsymbol{\theta}_1^{(2)}; \boldsymbol{w}^{(2)}) \\ &= \pi^{-2N}|\boldsymbol{f}^{(1)}|^{-\nu_1}|\boldsymbol{f}^{(2)}|^{-\nu_2}\exp[-\nu_1 \operatorname{tr}(\boldsymbol{f}^{(1)-1}\hat{\boldsymbol{f}}^{(1)})-\nu_2 \operatorname{tr}(\boldsymbol{f}^{(2)-1}\hat{\boldsymbol{f}}^{(2)})] \end{aligned} \tag{3.1}$$

where $N=\nu_1+\nu_2=(N_1+N_2)q$.

N. R. Goodman [3] showed that the maximum likelihood estimates (M. L. E.) of the variance-covariance matrices $\boldsymbol{f}^{(j)}$ of $\boldsymbol{w}^{(j)}$ are given by the sample spectral densities defined by

$$\hat{\boldsymbol{f}}^{(j)}=\{\hat{f}_{k,l}^{(j)}\}_{k,l=1,2} \tag{3.2}$$
$$=\frac{1}{\nu_j}\sum_{k=1}^{N_j}\sum_{u=1}^{a}\boldsymbol{w}_k^{(j)}(u)\boldsymbol{w}_k^{(j)}(u)^*\,.$$

Hence, the M. L. E.'s of coherences $\omega^{(j)}$ and phases $\theta^{(j)}$ which are uniquely determined by the spectral densities $f_{k,l}^{(j)}$ are given by the sample coherences $\hat{\omega}^{(j)}$ and the sample phases $\hat{\theta}^{(j)}$ defined by

$$\hat{\omega}^{(j)}=\frac{|\hat{f}_{12}^{(j)}|}{\{\hat{f}_{11}^{(j)}\hat{f}_{22}^{(j)}\}^{\frac{1}{2}}} \tag{3.3}$$

and

$$\hat{\theta}^{(j)}=\tan^{-1}\left\{-\frac{\hat{q}_{12}^{(j)}}{\hat{c}_{12}^{(j)}}\right\} \tag{3.4}$$

where

$$\hat{f}_{12}^{(j)}=\frac{1}{2}\{\hat{c}_{12}^{(j)}-i\hat{q}_{12}^{(j)}\}\,.$$

Thus the M. L. E. of $\boldsymbol{\Theta}_1$ is given by

$$\hat{\boldsymbol{\Theta}}_1=\{\hat{\boldsymbol{\Theta}}_1^{(1)},\hat{\boldsymbol{\Theta}}_1^{(2)}\}$$
$$=\{(\hat{f}_{11}^{(j)},\hat{f}_{22}^{(j)},\hat{\omega}^{(j)},\hat{\theta}^{(j)})\,;\ j=1,2\}$$

and the maximum value of the likelihood function $L(\boldsymbol{\Theta}_1;\boldsymbol{w})$ is given by

$$L(\hat{\boldsymbol{\Theta}}_1;\boldsymbol{w})=L(\hat{\boldsymbol{\Theta}}_1^{(1)};\boldsymbol{w}^{(1)})L(\hat{\boldsymbol{\Theta}}_1^{(2)};\boldsymbol{w}^{(2)}) \tag{3.5}$$
$$=\frac{\pi^{-2N}\exp(-2N)}{\{\hat{f}_{11}^{(1)}\hat{f}_{22}^{(1)}(1-\hat{\omega}^{(1)2})\}^{\nu_1}\{\hat{f}_{11}^{(2)}\hat{f}_{22}^{(2)}(1-\hat{\omega}^{(2)2})\}^{\nu_2}}\,.$$

Under the hypothesis $H_0:\omega^{(1)}=\omega^{(2)}=\omega$ there are seven unknown parameters $\boldsymbol{\Theta}_0=\{f_{11}^{(1)},f_{22}^{(1)},\theta^{(1)},f_{11}^{(2)},f_{22}^{(2)},\theta^{(2)},\omega\}$. The likelihood function of $\boldsymbol{\Theta}_0$ based on $\boldsymbol{w}$ is given by

$$L(\boldsymbol{\Theta}_0;\boldsymbol{w})=L(\boldsymbol{\Theta}_0;\boldsymbol{w}^{(1)})L(\boldsymbol{\Theta}_0;\boldsymbol{w}^{(2)}) \tag{3.6}$$
$$=\pi^{-2N}|\boldsymbol{f}_0^{(1)}|^{-\nu_1}|\boldsymbol{f}_0^{(2)}|^{-\nu_2}\exp[-\nu_1\operatorname{tr}(\boldsymbol{f}_0^{(1)-1}\hat{\boldsymbol{f}}^{(1)})-\nu_2\operatorname{tr}(\hat{\boldsymbol{f}}_0^{(2)-1}\boldsymbol{f}^{(2)})]$$
$$=\pi^{-2N}[f_{11}^{(1)}f_{22}^{(1)}]^{-\nu_1}[f_{11}^{(2)}f_{22}^{(2)}]^{-\nu_2}(1-\omega^2)^{-N}\exp\left\{-\frac{1}{1-\omega^2}[H-R]\right\}$$

where we have put

$$\boldsymbol{f}_0^{(j)}=\begin{bmatrix} f^{(j)} & (f_{11}^{(j)}f_{22}^{(j)})^{\frac{1}{2}}\omega e^{i\theta^{(j)}} \\ (f_{11}^{(j)}f_{22}^{(j)})^{\frac{1}{2}}\omega e^{-i\theta^{(j)}} & f_{22}^{(j)} \end{bmatrix}. \tag{3.7}$$

$$H=\nu_1\left\{\frac{\hat{f}_{11}^{(1)}}{f_{11}^{(1)}}+\frac{\hat{f}_{22}^{(1)}}{f_{22}^{(1)}}\right\}+\nu_2\left\{\frac{\hat{f}_{11}^{(2)}}{f_{11}^{(2)}}+\frac{\hat{f}_{22}^{(2)}}{f_{22}^{(2)}}\right\} \tag{3.8}$$

and

$$R=2\omega\sum_{j=1}^{2}\nu_j\hat{\omega}^{(j)}\left\{\frac{\hat{f}_{11}^{(j)}\hat{f}_{22}^{(j)}}{f_{11}^{(j)}f_{22}^{(j)}}\right\}^{\frac{1}{2}}\cos(\theta^{(j)}-\hat{\theta}^{(j)})\,. \tag{3.9}$$

We prepare the following proposition

PROPOSITION 1. *The M. L. E.* $\hat{\hat{\boldsymbol{\Theta}}}_0=\{\hat{\hat{f}}_{11}^{(1)}, \hat{\hat{f}}_{22}^{(1)}, \hat{\hat{\theta}}^{(1)}, \hat{\hat{f}}_{11}^{(2)}, \hat{\hat{f}}_{22}^{(2)}, \hat{\hat{\theta}}^{(2)}, \hat{\omega}\}$ *of* $\boldsymbol{\Theta}_0$ *based on* $\boldsymbol{w}$ *under* $H_0$ *are given by*

$$\hat{\hat{\theta}}^{(j)}=\hat{\theta}^{(j)}; \quad \text{mod } (2\pi) \qquad j=1, 2, \tag{3.10}$$

$$\hat{\omega}=\frac{(1+\hat{\omega}^{(1)}\hat{\omega}^{(2)})-\{(1-\hat{\omega}^{(1)}\hat{\omega}^{(2)})^2-4p(1-p)(\hat{\omega}^{(1)}-\hat{\omega}^{(2)})^2\}^{\frac{1}{2}}}{2\{p\hat{\omega}^{(2)}+(1-p)\hat{\omega}^{(1)}\}}, \tag{3.11}$$

$$\hat{\hat{f}}_{kk}^{(j)}=\hat{f}_{kk}^{(j)}\left(\frac{1-\hat{\omega}^{(j)}\hat{\omega}}{1-\hat{\omega}^2}\right) \tag{3.12}$$

*where* $p=\dfrac{\nu_1}{N}$.

Proof of this proposition 1 is given in Appendix.

The maximum value of the likelihood function under $H_0$ is thus given by

$$L(\hat{\hat{\boldsymbol{\Theta}}}_0; \boldsymbol{w})=\frac{\pi^{-2N}e^{-2N}(1-\hat{\omega}^2)^N}{\{\hat{f}_{11}^{(1)}\hat{f}_{22}^{(1)}(1-\hat{\omega}\hat{\omega}^{(1)})^2\}^{\nu_1}\{\hat{f}_{11}^{(2)}\hat{f}_{22}^{(2)}(1-\hat{\omega}\hat{\omega}^{(2)})^2\}^{\nu_2}}. \tag{3.13}$$

The likelihood ratio criterion for testing $H_0$ against $H_1$ is now given be

$$\begin{aligned}\Lambda(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})&=\frac{L(\hat{\hat{\boldsymbol{\Theta}}}_0; \boldsymbol{w})}{L(\hat{\boldsymbol{\Theta}}_1; \boldsymbol{w})}\\ &=[\Phi(\hat{\omega}^{(1)}, \hat{\omega})]^{\nu_1}[\Phi(\hat{\omega}^{(2)}, \hat{\omega})]^{\nu_2}\end{aligned} \tag{3.14}$$

where

$$\Phi(x, y)=(1-x^2)(1-y^2)(1-xy)^{-2}.$$

We notice that the likelihood ratio criterion $\Lambda=\Lambda(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$ depends only on the sample coherences $(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$ and $(\nu_1, \nu_2)$, since $\hat{\omega}$ is completely determined by $(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$.

## 4. Likelihood ratio test for $H_0$ against $H_1$.

From the well-known results of N. R. Goodman [3], the joint p. d. f. of the sample coherences $(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$ is given by

$$\varphi_{\nu_1,\nu_2}(y_1, y_2; \omega^{(1)}, \omega^{(2)})=h_{\nu_1}(y_1; \omega^{(1)})h_{\nu_2}(y_2; \omega^{(2)}), \qquad 0\leqq y_1, y_2<1 \tag{4.1}$$

where

$$h_\nu(y; \omega)=\frac{2(\nu-1)(1-\omega^2)^\nu y}{[\Gamma(\nu)]^2}\sum_{l=0}^{\infty}\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+1)}\right\}^2(\omega y)^{2l}(1-y^2)^{\nu-2}, \qquad 0\leqq y<1.$$

In particular, under $H_0: \omega^{(1)}=\omega^{(2)}=\omega$ the joint p. d. f. of $(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$ is given by

$$\varphi_{\nu_1,\nu_2}(y_1, y_2; \omega)=h_{\nu_1}(y_1; \omega)h_{\nu_2}(y_2; \omega), \qquad 0\leqq y_1, y_2\leqq 1.$$

For the likelihood ratio test, we can make use of the following decision rule; Let us put

$$\begin{aligned}c_0&\equiv c_0(\hat{\omega}^{(1)}, \hat{\omega}^{(2)}, \nu_1, \nu_2)\\ &=\Lambda(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})\end{aligned} \tag{4.2}$$

the realized value of the likelihood criterion based on the sample coherences $(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$.

We consider a region $D$ in $[0, 1]^2$ defined by

$$D=\{(y_1, y_2) \mid \Lambda(y_1, y_2) \leqq c_0\} . \tag{4.3}$$

The region $D$, of course, depends on $(\nu_1, \nu_2)$.

Under the hypothesis $H_0: \omega^{(1)}=\omega^{(2)}=\omega$, the probability that the sample coherences $(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$ fall into the region $D$ is given by

$$\begin{aligned} \alpha_1(\omega) &= P\{(\hat{\omega}^{(1)}, \hat{\omega}^{(2)}) \in D \mid H_0\} \\ &= \iint_D \varphi_{\nu_1,\nu_2}(y_1, y_2 ; \omega) dy_1 dy_2 . \end{aligned} \tag{4.4}$$

Since the coherence $\omega$ is unknown, we have to estimate the probability $\alpha_1(\omega)$.

It is natural to estimate $\alpha_1(\omega)$ by substituting the M.L.E. $\hat{\hat{\omega}}$ defined in (3.11) in the place of $\omega$ in (4.4).

Thus, we can estimate the probability $\alpha_1(\omega)$ by $\alpha_1(\hat{\hat{\omega}})$.

With a given level $\alpha_0$, say $\alpha_0=0.05$, we adopt the following decision rule:

$$\begin{aligned} &\text{To reject } H_0 \quad \text{if} \quad \alpha_1(\hat{\hat{\omega}}) \leqq \alpha_0 , \\ &\text{to accept } H_0 \quad \text{otherwise} . \end{aligned}$$

This is an approximating likelihood ratio test based on the F. F. T. data $H_0$ against $H_1$ with the level $\alpha_0$.

For a fixed $c_0$, $0<c_0<1$, the equation

$$\Lambda(y_1, y_2)=c_0 \tag{4.5}$$

has two solutions

$$y_2=\phi_1(y_1) \geqq y_1$$

and

$$y_2=\phi_2(y_2) \geqq y_1 .$$

These two functions $\phi_1(y_1)$ and $\phi_2(y_2)$ are smooth and their numerical values can be easily computed since the likelihood criterion (3.14) is comparatively simple.

The region $D$ is thus written as

$$D=\{(y_1, y_2) \mid \phi_1(y_1) \leqq y_2<1\} \cup \{(y_1, y_2) \mid 0 \leqq y_2 \leqq \phi_2(y_1)\}$$

and the integral (4.4) is written as

$$\begin{aligned} \alpha_1(\omega) &= \int_0^1 dy_1 \int_{\phi_1(y_1)}^1 \varphi_{\nu_1,\nu_2}(y_1, y_2 ; \omega) dy_2 \\ &+ \int_0^1 dy_1 \int_0^{\phi_2(y_1)} \varphi_{\nu_1,\nu_2}(y_1, y_2 ; \omega) dy_2 . \end{aligned}$$

### 5. Numerical table for critical region.

Numerical table of critical value is needed for practice of the likelihood ratio test. Table 1 shows the probability level in cases of

$$\begin{aligned} &\omega=0.2 \ (0.2) \ 0.8 , \\ &N=10, 50 , \quad p=0.5 , \\ &c_0=0.1 \ (0.1) \ 0.9 . \end{aligned}$$

Practical application of this test is prepared in our subsequent paper [6], to the "analytical evaluation of blood pressure of the aged".

Fig. 1 shows the monograph of critical region for $\alpha_0=0.05$ for any value $(y_1, y_2)$ in cases as Table 1.

Table 1. Probability level of likelihood ratio test in case of $\omega=0.1(0.1)0.8$, $c_0=0.1(0.1)0.9$, $p=0.5$.

$N=10$

| $c_0$ \ $\omega$ | 0.100 | 0.200 | 0.300 | 0.400 | 0.500 | 0.600 | 0.700 | 0.800 |
|---|---|---|---|---|---|---|---|---|
| 0.100 | 0.0101 | 0.0129 | 0.0176 | 0.0239 | 0.0314 | 0.0391 | 0.0457 | 0.0524 |
| 0.200 | 0.0286 | 0.0351 | 0.0454 | 0.0586 | 0.0728 | 0.0858 | 0.0960 | 0.1057 |
| 0.300 | 0.0553 | 0.0659 | 0.0821 | 0.1017 | 0.1215 | 0.1385 | 0.1507 | 0.1621 |
| 0.400 | 0.0912 | 0.1060 | 0.1279 | 0.1532 | 0.1776 | 0.1973 | 0.2108 | 0.2234 |
| 0.500 | 0.1384 | 0.1570 | 0.1840 | 0.2140 | 0.2417 | 0.2630 | 0.2768 | 0.2907 |
| 0.600 | 0.1996 | 0.2216 | 0.2526 | 0.2861 | 0.3156 | 0.3373 | 0.3511 | 0.3660 |
| 0.700 | 0.2800 | 0.3042 | 0.3377 | 0.3726 | 0.4023 | 0.4234 | 0.4361 | 0.4558 |
| 0.800 | 0.3890 | 0.4135 | 0.4468 | 0.4805 | 0.5082 | 0.5271 | 0.5384 | 0.5583 |
| 0.900 | 0.5504 | 0.5717 | 0.5999 | 0.6277 | 0.6497 | 0.6642 | 0.6729 | 0.6905 |

$N=50$

| $c_0$ \ $\omega$ | 0.100 | 0.200 | 0.300 | 0.400 | 0.500 | 0.600 | 0.700 | 0.800 |
|---|---|---|---|---|---|---|---|---|
| 0.100 | 0.0053 | 0.0133 | 0.0231 | 0.0296 | 0.0325 | 0.0338 | 0.0345 | 0.0351 |
| 0.200 | 0.0191 | 0.0392 | 0.0588 | 0.0696 | 0.0741 | 0.0761 | 0.0772 | 0.0778 |
| 0.300 | 0.0418 | 0.0751 | 0.1033 | 0.1170 | 0.1225 | 0.1250 | 0.1264 | 0.1271 |
| 0.400 | 0.0745 | 0.1210 | 0.1562 | 0.1719 | 0.1780 | 0.1809 | 0.1824 | 0.1833 |
| 0.500 | 0.1194 | 0.1780 | 0.2183 | 0.2350 | 0.2415 | 0.2445 | 0.2461 | 0.2471 |
| 0.600 | 0.1795 | 0.2480 | 0.2913 | 0.3082 | 0.3147 | 0.3178 | 0.3195 | 0.3205 |
| 0.700 | 0.2598 | 0.3347 | 0.3785 | 0.3947 | 0.4009 | 0.4039 | 0.4055 | 0.4065 |
| 0.800 | 0.3707 | 0.4458 | 0.4866 | 0.5010 | 0.5065 | 0.5091 | 0.5105 | 0.5117 |
| 0.900 | 0.5357 | 0.6003 | 0.6329 | 0.6439 | 0.6481 | 0.6501 | 0.6512 | 0.6528 |

## 6. An approximating chi-square test.

Let us write $\delta=\omega^{(2)}-\omega^{(1)}$. Then the hypothesis $H_0$ and $H_1$ are rewritten as

$$H_0:\ \delta=0$$

and

$$H_1:\ \delta\neq 0.$$

Under the hypothesis $H_1$, there are eight unknown parameters for the joint distribution of the F. F. T. data $\boldsymbol{w}$.

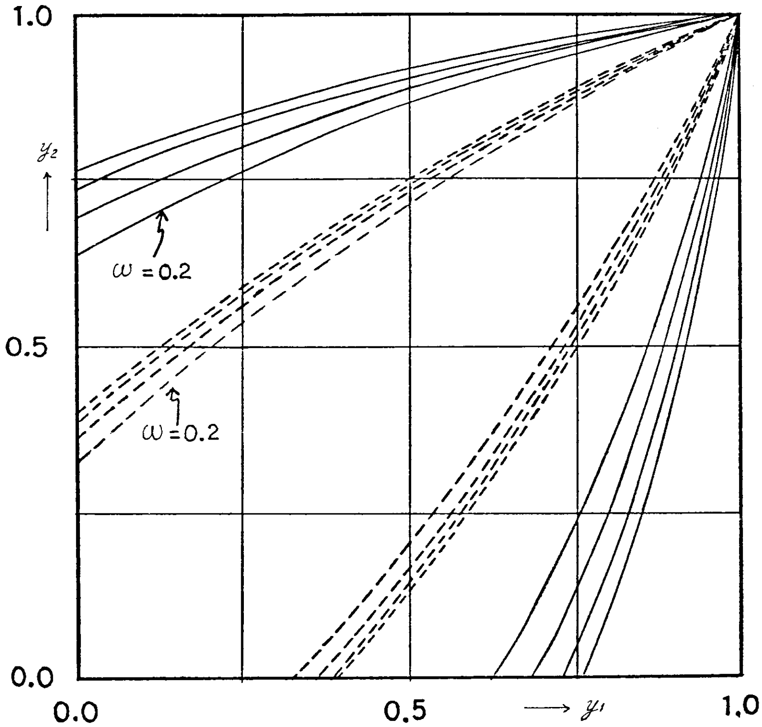


Fig. 1. Critical region of $(y_1, y_2)$ at the level 0.05 in case of $\omega=0.2\,(0.2)\,0.8$, where solid line and dashed line show the case of $N=10$ and $N=50$ respectively.

Hence the distribution of the familiar statistic $-2\log \Lambda(\hat{\omega}^{(1)}, \hat{\omega}^{(2)})$ is asymptotically chi-square distribution with 7 degrees of freedom.

For a given level $\alpha_0$ let $\chi_7^2(\alpha_0)$ be the $100\alpha_0\%$ point of chi-square distribution with 7 degrees of freedom, that is, $P\{\chi_7^2 \geqq \chi_7^2(\alpha_0)\}=\alpha_0$.

Then, the approximating critical region for the test of $H_0$ against $H_1$ of the level $\alpha_0$ is given by

$$D^* = \{(y_1, y_2) \mid -2\log \Lambda(y_1, y_2) \geqq \chi_7^2(\alpha_0)\}\,.$$

## 7. A test based on asymptotic normality.

From the result of E. J. Hannan [4, p. 291],

$$\frac{\sqrt{2\nu_j}\,(\hat{\omega}^{(j)}-\omega^{(j)})}{1-\hat{\omega}^{(j)2}} \qquad j=1, 2$$

are asymptotically normal with zero mean and unit variance.

Hence, under the hypothesis $H_0: \omega^{(1)}=\omega^{(2)}=\omega$, the following statistic:

$$\frac{\sqrt{2\nu_1\nu_2}\,(\hat{\omega}^{(1)}-\hat{\omega}^{(2)})}{\sqrt{\nu_1+\nu_2}\,(1-\hat{\omega}^2)}$$

is asymptotically normal with zero mean and unit variance.

Thus, the approximating critical region for the test of the hypothesis $H_0$ against

$H_1$ of the level $\alpha_0$ is given by

$$D^{**}=\left\{(y_1, y_2)\middle|\sqrt{\frac{2\nu_1\nu_2}{\nu_1+\nu_2}}\frac{|y_1-y_2|}{\{1-\hat{\omega}(y_1, y_2)^2\}}\geqq a(\alpha_0)\right\}$$

where $a(\alpha_0)$ is $100\frac{\alpha_0}{2}\%$ point of the standard normal distribution, and

$$\hat{\omega}(y_1, y_2)=\frac{(1+y_1y_2)-\{(1-y_1y_2)^2-4p(1-p)(y_1-y_2)^2\}^{\frac{1}{2}}}{2\{py_2+(1-p)y_1\}}$$

with $p=\frac{\nu_1}{\nu_1+\nu_2}$.

## 8. Appendix.

PROOF OF PROPOSITION 1.

From (3.9), it is easily seen that M. L. E.'s of $\theta^{(j)}$, $j=1, 2$, are given by (3.10). By substituting $\hat{\hat{\theta}}^{(j)}$, $j=1, 2$, into (3.6), we obtain

$$L(\boldsymbol{\Theta}_0\,;\boldsymbol{w})=L(\boldsymbol{\Theta}_0^*\,;\boldsymbol{w})$$
$$=\pi^{-2N}L^{(1)}(f_{11}^{(1)}, f_{22}^{(1)}, \omega)L^{(2)}(f_{11}^{(2)}, f_{22}^{(2)}, \omega)$$

where

$$\boldsymbol{\Theta}_0^*=\{f_{11}^{(1)}, f_{22}^{(1)}, \hat{\hat{\theta}}^{(1)}, f_{11}^{(2)}, f_{22}^{(2)}, \hat{\hat{\theta}}^{(2)}, \omega\}$$

and

$$L^{(j)}(f_{11}^{(j)}, f_{22}^{(j)}, \omega)$$
$$=(f_{11}^{(j)}f_{22}^{(j)})^{-\nu_j}(1-\omega^2)^{-\nu_j}\exp\left[-\frac{\nu_j}{1-\omega^2}\{X_j^2-2\hat{\omega}^{(j)}\omega X_jY_j+Y_j^2\}\right]$$

with

$$X_j=\left\{\frac{\hat{f}_{11}^{(j)}}{f_{11}^{(j)}}\right\}^{\frac{1}{2}}, \qquad Y_j=\left\{\frac{\hat{f}_{22}^{(j)}}{f_{22}^{(j)}}\right\}^{\frac{1}{2}}.$$

From the following inequalities:

(A.1) For every $x, y$,

$$x^2-2\hat{\omega}^{(j)}\omega xy+y^2\geqq 2(1-\hat{\omega}^{(j)}\omega)xy\,.$$

(A.2) For any $a>0$ and any $z>0$

$$\log z-az\leqq -\{\log a+1\}$$

we have

(A.3) $$L^{(j)}(f_{11}^{(j)}, f_{22}^{(j)}, \omega)$$
$$\leqq(\hat{f}_{11}^{(j)}\hat{f}_{22}^{(j)})^{-\nu_j}(1-\omega^2)^{-\nu_j}\exp\left[2\nu_j\left\{\log\frac{1-\omega^2}{1-\hat{\omega}^{(j)}\omega}-1\right\}\right].$$

The R. H. S. of (A.3) can be attained by taking $f_{kk}^{(j)}$, $k=1, 2$, to be

(A.4) $$\tilde{f}_{kk}^{(j)}=\left(\frac{1-\hat{\omega}^{(j)}\omega}{1-\omega^2}\right)\hat{f}_{kk}^{(j)}, \qquad k=1, 2\,.$$

Thus, we have

$$L(\boldsymbol{\Theta}_0^*\,;\boldsymbol{w})\leqq L(\boldsymbol{\Theta}_0^{**}\,;\boldsymbol{w})$$
$$=\frac{\{\hat{f}_{11}^{(1)}\hat{f}_{22}^{(1)}\}^{-\nu_1}\{\hat{f}_{11}^{(2)}\hat{f}_{22}^{(2)}\}^{-\nu_2}(1-\omega^2)^N}{\{\pi e\}^{2N}(1-\hat{\omega}^{(1)}\omega)^{2\nu_1}(1-\hat{\omega}^{(2)}\omega)^{2\nu_2}}$$

where

$$\boldsymbol{\Theta}_0^{**}=\{\tilde{f}_{11}^{(1)}, \tilde{f}_{22}^{(1)}, \hat{\hat{\theta}}^{(1)}, \tilde{f}_{11}^{(2)}, \tilde{f}_{22}^{(2)}, \hat{\hat{\theta}}^{(2)}, \omega\}.$$

Let $\hat{\omega}$ be the smaller solution of the equation of $\omega$:

$$\{p\hat{\omega}^{(2)}+(1-p)\hat{\omega}^{(1)}\}\omega^2-(1+\hat{\omega}^{(1)}\hat{\omega}^{(2)})\omega+\{p\hat{\omega}^{(1)}+(1-p)\hat{\omega}^{(2)}\}=0$$

where $p=\dfrac{\nu_1}{\nu_1+\nu_2}$.

Then, the solution $\hat{\omega}$ is given by (3.11) which satisfies that $0\leqq\hat{\omega}\leqq1$ and it is earily seen that for any $\omega$, $0\leqq\omega\leqq1$

$$\frac{(1-\hat{\hat{\omega}}^2)^N}{(1-\hat{\omega}^{(1)}\hat{\hat{\omega}})^{2\nu_1}(1-\hat{\omega}^{(2)}\hat{\hat{\omega}})^{2\nu_2}}\geqq\frac{(1-\omega^2)^N}{(1-\hat{\omega}^{(1)}\omega)^{2\nu_1}(1-\hat{\omega}^{(2)}\omega)^{2\nu_2}}.$$

Thus, we proved that the M. L. E.'s of $\boldsymbol{\Theta}_0$ are given by (3.10), (3.11) and (3.12).

Q. E. D.

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