ON SLIPPAGE RANK TESTS-(I) : THE DERIVATION OF OPTIMAL PROCEDURES

Kakiuchi, Itsuro
Department of Mathematics, Kyushu University

Kimura, Miyoshi
Department of Mathematics, Kyushu University

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ON SLIPPAGE RANK TESTS-(I) THE DERIVATION OF OPTIMAL PROCEDURES

By

Itsurô KAKIUCHI* and Miyoshi KIMURA*

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§ 1. Introduction.

Slippage problems were first introduced by Mosteller [8] as a problem of testing the null hypothesis that all populations are identical, against the alternatives that one of them slips to the right. Since Paulson [10] treated the slippage problem of normal mean satisfactory, there have been published a number of papers relative to slippage problems such as Truax [11], Doornbos and Prins [1] and [2], Karin and Truax [7], Hall and Kudô [4] and Hall, Kudô and Yeh [5] etc. In the parametric case, Hall and Kudô established a mathematical formulation of slippage problems by using the concept of invariance other than the ones with respect to location/or scale, and showed the optimum properties of test procedures proposed previously. On the other hand, in the nonparametric case, some rank procedures have been proposed, but their optimum properties, so far as the present authors are aware, do not seem to be established except for some special cases ([7]).

In this paper, we will consider the slippage problems along the line of the formulation of Hall and Kudô, and derive locally optimum rank procedures. Slippage problems we will be concerned with, include both two-sided ([A]) and one-sided ([B]) cases. In § 2, the formulations of two slippage problems ([A] and [B]) and the preliminary definitions are given. In § 3, the most powerful size α (MPS-α) rank tests are derived from the results of Hall and Kudô. In § 4, there will be derived the test locally most powerful in a weak sense, which we call as the extended locally most powerful size α (ELMPS-α) rank test. In § 5, we will treat the cases where the parameter is location and scale. It will be shown that there exist the MPS-α and the ELMPS-α rank tests for both [A] and [B] in the case of a location parameter, and for [B] in the case of scale. Further we will consider the general problems ([A′] and [B′]) of [A] and [B], and show that the same results hold still true. In § 6, we will notice that all the results of § 3, § 4 and § 5 also hold true in the slippage problems with a control population, which were discussed in [7]. In § 7, we will be concerned with the critical values of the k-sample slippage analogues of the Wilcoxon two-sample test, which are the ELMPS-α rank tests when the distribution contains a location parameter and its form is logistic.

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* Department of Mathematics, Kyushu University, Fukuoka.

§ 2. Definitions and Formulations.

Let $f(x|\delta)$ be the probability density function on the real line with respect to Lebesgue measure μ where δ is a real parameter. Let $X_{i1}, X_{i2}, \cdots, X_{in}$ $(i=1, 2, \cdots, k)$ be mutually independent random variables distributed with $f(x|\delta_i)$ respectively. Further let $\varDelta$ be a positive real parameter which expresses the slip.

Under these assumptions, we consider [A] the two-sided and [B] the one-sided slippage problems: [A] being the testing of the hypothesis H_0 that $\delta_l=0$ for $l=1, 2, \cdots, k$, against $2k$ $(k\geqq 3)$ alternatives $H_{ij}(\varDelta)$ $(i=1, 2, \cdots, k\,;\ j=1, 2)$ that $\delta_l=0$ for $l=1, 2, \cdots, k$, $l\neq i$ and $\delta_i=+\varDelta, -\varDelta$ if $j=1, 2$, and [B] being the testing of the hypothesis H_0 that $\delta_l=0$ for $l=1, 2, \cdots, k$, against k $(k\geqq 2)$ alternatives $H_i(\varDelta)$ $(i=1, 2, \cdots, k)$ that $\delta_l=0$ for $l=1, 2, \cdots, k$, $l\neq i$ and $\delta_i=\varDelta$. More explicitly,

[A]
$$\begin{aligned} H_0 \quad &: \delta_1=\delta_2=\cdots=\delta_k=0 \\ H_{i1}(\varDelta) &: \delta_1=\cdots=\delta_{i-1}=\delta_i-\varDelta=\delta_{i+1}=\cdots=\delta_k=0 \\ H_{i2}(\varDelta) &: \delta_1=\cdots=\delta_{i-1}=\delta_i+\varDelta=\delta_{i+1}=\cdots=\delta_k=0 \\ & \qquad i=1, 2, \cdots, k\,. \end{aligned}$$

[B]
$$\begin{aligned} H_0 \quad &: \delta_1=\delta_2=\cdots=\delta_k=0 \\ H_i(\varDelta) &: \delta_1=\cdots=\delta_{i-1}=\delta_i-\varDelta=\delta_{i+1}=\cdots=\delta_k=0\,, \\ & \qquad i=1, 2, \cdots, k\,. \end{aligned}$$

The rest of this section is offered to definitions and notations. A decision procedure is said to be a *rank test*, when it is based on ranks of all observations. Let D_θ denote the decision to accept H_θ and $P(D_\theta|H_{\theta'})$ denote the probability accepting H_θ under $H_{\theta'}$. A rank test is said to be of *size* α, if it satisfies

$$P(D_0|H_0)\geqq 1-\alpha\,, \qquad 0<\alpha<1\,. \tag{2.1}$$

A rank test is said to be the *most powerful size* α (MPS-α) rank test for $\varDelta$, if it maximizes

$$\sum_{i=1}^{k}\sum_{j=1}^{2} P(D_{ij}|H_{ij}(\varDelta)) \qquad \text{when [A]}, \tag{2.2}$$

and

$$\sum_{i=1}^{k} P(D_i|H_i(\varDelta)) \qquad \text{when [B]}, \tag{2.3}$$

in the family of size α rank tests. A rank test is said to be *symmetric in power*, if it satisfies

$$P(D_{ij}|H_{ij}(\varDelta))=P(D_{lm}|H_{lm}(\varDelta)) \qquad \text{when [A]}, \tag{2.4}$$

and

$$P(D_i|H_i(\varDelta))=P(D_l|H_l(\varDelta)) \qquad \text{when [B]}, \tag{2.5}$$

where i, $l=1, 2, \cdots, k$ and j, $m=1, 2$. A rank test is said to be the *most powerful* *symmetric size* α (MPSS-α) rank test for Δ, if it maximizes (2.2) (or (2.3)) in the family of symmetric size α rank tests.

When it is needed to clarify the test procedure d, we use the expressions of $P_d(D_{ij}|H_{ij}(\Delta))$ and $P_d(D_i|H_i(\Delta))$. A size α rank test d^* is said to be the *extended locally most powerful size* α (ELMPS-α) rank test, if it satisfies that for any positive real number ε there exists a positive real number Δ_ε such that for all $0<\Delta<\Delta_\varepsilon$,

$$\sum_{i=1}^{k}\sum_{j=1}^{2} P_{d^*}(D_{ij}|H_{ij}(\Delta)) > \sup_{d\in\mathscr{D}} \sum_{i=1}^{k}\sum_{j=1}^{2} P_d(D_{ij}|H_{ij}(\Delta))-\varepsilon \quad \text{when [A]}, \tag{2.6}$$

and

$$\sum_{i=1}^{k} P_{d^*}(d_i|H_i(\Delta)) > \sup_{d\in\mathscr{D}} \sum_{i=1}^{k} P_d(D_i|H_i(\Delta))-\varepsilon \quad \text{when [B]}, \tag{2.7}$$

where $\mathscr{D}$ denotes the family of size α rank tests.

The similar definition can be given in the family of symmetric rank tests. A symmetric size α rank test $d^{*\prime}$ is said to be the *extended locally most powerful symmetric size* α (ELMPSS-α) rank test, if for any positive real number ε there exists a positive real number Δ_ε such that for all $0<\Delta<\Delta_\varepsilon$,

$$P_{d^{*\prime}}(D_{ij}|H_{ij}(\Delta)) > \sup_{d\in\mathscr{D}'} P_d(D_{ij}|H_{ij}(\Delta))-\varepsilon \quad \text{when [A]}, \tag{2.8}$$

and

$$P_{d^{*\prime}}(D_i|H_i(\Delta)) > \sup_{d\in\mathscr{D}'} P_d(D_i|H_i(\Delta))-\varepsilon \quad \text{when [B]}, \tag{2.9}$$

where $\mathscr{D}'$ denotes the family of symmetric size α rank tests.

§ 3. Most powerful rank tests.

Let $\mathfrak{X}$, $\mathfrak{B}$ and μ be $N\,(=nk)$-dimensional Euclidean space, the N-dimensional Borel field and N-dimensional Lebesgue measure, respectively. For $\boldsymbol{X}=(X_{11}, X_{12}, \cdots, X_{kn})\in\mathfrak{X}$, let $X_{(i)}$ and $R_{ij}(\boldsymbol{X})$ be the i-th order statistic and the rank of X_{ij} in pooled samples $X_{11}, X_{12}, \cdots, X_{kn}$ of size N, respectively, and $R(\boldsymbol{X})=(R_{11}(\boldsymbol{X}), R_{12}(\boldsymbol{X}), \cdots, R_{kn}(\boldsymbol{X}))$. Let $\mathfrak{R}$ be the space of all permutations on $(1, 2, \cdots, N)$. Let $P_\theta(\cdot|\Delta)$ and $Q_\theta(\cdot|\Delta)$ be the distributions of $\boldsymbol{X}$ and $R(\boldsymbol{X})$ under $H_\theta(\Delta)$, respectively. Further let $f_\theta(\boldsymbol{x}|\Delta)$ be the probability density function of $P_\theta(\cdot|\Delta)$ with respect to μ, and $h_\theta(r|\Delta)=Q_\theta(r|\Delta)$ for $r\in\mathfrak{R}$. Here we should understand that θ is the element of $\Theta=\{0, (1,1), (1,2), \cdots, (k,2)\}$ when [A], and of $\Theta=\{0, 1, 2, \cdots, k\}$ when [B], and that $H_0(\Delta)=H_\theta(0)=H_0$.

Now, throughout this paper, we assume that $\theta\neq\theta'$ implies $Q_\theta(\cdot|\Delta)\neq Q_{\theta'}(\cdot|\Delta)$, because we will intend to test the slippage problems [A] and [B] on the basis of $R(\boldsymbol{X})$.

The slippage problem [A] (or [B]) is said to be *invariant* under the transformation group G on $\mathfrak{X}$, if G induces the transformation group Π_G on Θ leaving 0 invariant (i.e. $\pi_g 0=0$ for all $\pi_g\in\Pi_G$), that is,

$$f_{\pi_g\theta}(g\boldsymbol{x}|\Delta)=f_\theta(\boldsymbol{x}|\Delta) \quad \text{for all } \boldsymbol{x}\in\mathfrak{X},\ g\in G,\ \theta\in\Theta, \tag{3.1}$$

$$\mu(g\boldsymbol{A})=\mu(\boldsymbol{A}) \quad \text{for all } \boldsymbol{A}\in\mathfrak{B},\ g\in G. \tag{3.2}$$

The slippage problem [A] (or [B]) is said to be *invariant in* $\mathfrak{R}$ under the transformation group $\bar{G}$ on $\mathfrak{R}$, if $\bar{G}$ induces the transformation group $\Pi_{\bar{G}}$ on Θ leaving 0 invariant (i.e. $\pi_{\bar{g}}0=0$ for all $\pi_{\bar{g}}\in\Pi_{\bar{G}}$), that is,

$$h_{\pi_{\bar{g}}\theta}(\bar{g}r|\Delta)=h_\theta(r|\Delta) \qquad \text{for all} \quad r\in\mathfrak{R},\ \bar{g}\in\bar{G},\ \theta\in\Theta\,. \tag{3.3}$$

We now give a condition and two lemmas applied in the sequel.

CONDITION I. For all $\boldsymbol{x}_1,\ \boldsymbol{x}_2\in\mathfrak{X}$, $R(g\boldsymbol{x}_1)=R(g\boldsymbol{x}_2)$ for all $g\in G$ if and only if $R(\boldsymbol{x}_1)=R(\boldsymbol{x}_2)$.

LEMMA 1. *If Condition I is satisfied, then the transformation group G on* $\mathfrak{X}$ *induces the transformation group* $\bar{G}$ *on* $\mathfrak{R}$ *which is defined as* $\bar{G}=\{\bar{g}\,|\,\bar{g}: R(\boldsymbol{x})\to R(g\boldsymbol{x}),\ g\in G\}$.

As the above lemma is well known, the proof is ommited.

LEMMA 2. *If the slippage problem* [A] (*or* [B]) *is invariant under G, then it is also invariant in* $\mathfrak{R}$ *under* $\bar{G}$ *defined as in Lemma* 1. *Furthermore, the transformation group* Π_G *on* Θ *induced from G is equal to the transformation group* $\Pi_{\bar{G}}$ *on* Θ *induced from* $\bar{G}$, *and* $\pi_g=\pi_{\bar{g}}$.

PROOF. Since [A] (or [B]) is invariant under G, there exists the transformation group Π_G on Θ ($\pi_g 0=0$ for all $\pi_g\in\Pi_G$) from (3.1) and (3.2) such that

$$P_{\pi_g\theta}(A|\Delta)=P_\theta(g^{-1}A|\Delta) \qquad \text{for all} \quad A\in\mathfrak{B},\ g\in G,\ \theta\in\Theta\,. \tag{3.4}$$

According to the defiinition of $\bar{G}$ and (3.4), we have

$$\begin{aligned} h_\theta(r|\Delta) &= P_\theta(R(\boldsymbol{X})=r|\Delta)=P_\theta(R(g\boldsymbol{X})=\bar{g}r|\Delta) \\ &= P_{\pi_g\theta}(R(\boldsymbol{X})=\bar{g}r|\Delta)=h_{\pi_g\theta}(\bar{g}r|\Delta) \qquad \text{for all} \quad r\in\mathfrak{R},\ g\in G,\ \theta\in\Theta\,. \end{aligned}$$

This shows that [A] (or [B]) is invariant in $\mathfrak{R}$ under $\bar{G}$ and $\pi_{\bar{g}}$ consists with π_g.

(Q. E. D.)

The following condition is essentially the same as the one given in [4].

CONDITION II. There exists an transformation group $\bar{G}$ on $\mathfrak{R}$ which satisfies (i) and (ii).

(i) Slippage problem [A] (or [B]) is invariant in $\mathfrak{R}$ under $\bar{G}$.

(ii) $\Pi_{\bar{G}}$, which is induced from $\bar{G}$, is transitive on the subspace $\{(1,1), (1,2), \cdots, (k,2)\}$ (or $\{1, 2, \cdots, k\}$) of Θ.

According to [4], it is found that the MPS-α rank tests and the MPSS-α rank test for [A] (or [B]) can be derived under Condition II.

THEOREM 1A. *If* $f(x|0)=f(-x|0)$ *and* $f(x|\Delta)=f(-x|-\Delta)$ *are satisfied, then for any* $\alpha\in(0,1)$ *there exist the* MPS-α *rank tests for* [A], *which are given by the following form.*

$$d_0(r)=\begin{cases} 1 & \textit{if} \quad \max\limits_{i,j} h_{ij}(r|\Delta)<\lambda\cdot h_0(r) \\ \xi(r) & \textit{if} \qquad\qquad = \\ 0 & \textit{if} \qquad\qquad > \end{cases} \tag{3.5}$$

$$d_{lm}(r)=\begin{cases} \eta_{lm}(r) & \textit{if} \quad h_{lm}(r|\Delta)=\max\limits_{i,j} h_{ij}(r|\Delta)\geqq\lambda\cdot h_0(r) \\ 0 & \textit{if} \quad \textit{otherwise} \end{cases}$$

$$i,\ l=1, 2, \cdots, k\,;\ j,\ m=1, 2$$

where $\xi(r)$ *and* $\eta_{lm}(r)$ *are arbitrary and* λ *is constant, such that* $P(D_0|H_0)=1-\alpha$.

PROOF. We will give a transformation group G on $\mathfrak{X}$ under which [A] is invariant and which induces the transformation group Π_G being transitive on $\{(1, 1), (1, 2), \cdots, (k, 2)\}$ and also satisfies Condition I. Then, by Lemma 1, $\bar{G}$ is defined on $\mathfrak{R}$ and according to Lemma 2, [A] is invariant under $\bar{G}$ and $\Pi_G = \Pi_{\bar{G}}$. These facts imply that Condition II is satisfied. Therefore the proof follows from [4]. Thus we need only to construct the above G. Roughly speaking, this G is the composition of the sign change and the permutation group on k populations. Though troublesome, we explicitly describe the construction of G.

Let Π_1 be the symmetric group on $\{1, 2, \cdots, k\}$ and $G_{\Pi_1} = \{g_{\pi_1} | g_{\pi_1} : x_{ij} \to x_{\pi_1^{-1} ij}, \pi_1 \in \Pi_1\}$. G_{Π_1} is a group on $\mathfrak{X}$ with the operation defined as $g_{\pi_1} \circ g_{\pi_1'} = g_{\pi_1 \circ \pi_1'}$. Since the correspondence $\pi_1 \to g_{\pi_1}$ is an isomorphism from Π_1 onto G_{Π_1}, the inverse correspondence $g_{\pi_1} \to \pi_1$ is also an isomorphism from G_{Π_1} onto Π_1. Accordingly, $g_{\pi_1} \to \pi_1$ can be written as $g_1 \to \pi_{g_1}$ and let $G_{\Pi_1} = G_1$.

Let $G_2 = \{e_2, a\}$ be the transformation group on $\mathfrak{X}$, where e_2 is the identity and a is the sign change $x_{ij} \to -x_{ij}$. In the same way, let $\Pi_2 = \{e_2^*, a^*\}$ be the symmetric group on $\{1, 2\}$, where e_2^* is the identity and a^* is another. Then the correspondence $e_2 \to e_2^*$, $a \to a^*$ from G onto Π_2 is isomorphic and is written as $g_2 \to \pi_{g_2}$.

Let G be the transformation group on $\mathfrak{X}$ defined as the direct product group of G_1 and G_2, and Π be the transformation group of $\{(1, 1), (1, 2), \cdots, (k, 2)\}$ defined as the direct product group of Π_1 and Π_2. That is, the element $g = (g_1, g_2)$ of $G = G_1 \times G_2$ is the transformation $x_{ij} \to x_{\pi_{g_1} i, j}$ if $g_2 = e_2$ and $x_{ij} \to -x_{\pi_{g_1} i, j}$ if $g_2 = a$, and G has the usual operation defined as $(g_1, g_2) \circ (g_1', g_2') = (g_1 \circ g_1', g_2 \circ g_2')$. Similarly, the element $\pi = (\pi_1, \pi_2)$ of $\Pi = \Pi_1 \times \Pi_2$ is the transformation $(i, j) \to (\pi_1 i, \pi_2 j)$ and the operation of Π is defined as $(\pi_1, \pi_2) \circ (\pi_1', \pi_2') = (\pi_1 \circ \pi_1', \pi_2 \circ \pi_2')$. Then it is easily seen that Π is transitive on $\{(1, 1), (1, 2), \cdots, (k, 2)\}$.

Let us consider the correspondence $(g_1, g_2) \to (\pi_{g_1}, \pi_{g_2})$ from G onto Π and denote it as $g \to \pi_g$. Further, in order to include 0, add $\pi 0 = 0$ for all $\pi \in \Pi$. Then this correspondence is also isomorphic.

Now, about such g and π_g, we can obtain (3.1) without difficulty. This means that [A] is invariant under G, and Π consists with Π_G induced from G. It is also obvious that G satisfies Condition I. Thus it is G that we would like to seek.

(Q. E. D.)

COROLLARY 1A. *Under the assumptions of Theorem* 1A, *for any* $\alpha \in (0, 1)$ *there exists the* MPSS-α *rank test for* [A], *which is given by the following form. Furthermore it is also the* MPS-α *rank test for* [A].

$$
(3.6) \quad
\begin{aligned}
d_0(r) &= \begin{cases} 1 & \text{if} \quad \max\limits_{i,j} h_{ij}(r|\Delta) < \lambda \cdot h_0(r) \\ \xi_\alpha & \text{if} \quad\quad = \\ 0 & \text{if} \quad\quad > \end{cases} \\
d_{lm}(r) &= \begin{cases} 1/m(r) & \text{if} \quad h_{lm}(r|\Delta) = \max\limits_{i,j} h_{ij}(r|\Delta) > \lambda \cdot h_0(r) \\ (1-\xi_\alpha)/m(r) & \text{if} \quad\quad = \\ 0 & \text{if} \quad \text{otherwise} \end{cases}
\end{aligned}
$$

$$i, l = 1, 2, \cdots, k; j, m = 1, 2.$$

Where $m(r)$ *is the number of times* $\max_{i,j} h_{ij}(r|\varDelta)$ *is attained, and* λ *and* ξ_α *are constants such that* $P(D_0|H_0)=1-\alpha$.

PROOF. The test of (3.6) is obviously one of the MPS-α rank tests because of the special form of (3.5). We will show that it is symmetric in power. Let $\bar{G}$ and $\Pi_{\bar{G}}$ be the ones constructed in the proof of Theorem 1A. Then we can have easily,

$$d_{\pi_{\bar{g}}\theta}(\bar{g}r)=d_\theta(r) \qquad \text{for all } r\in\mathfrak{R},\ \bar{g}\in\bar{G},\ \theta\in\Theta, \tag{3.7}$$

while we previously had (3.3). Since $\Pi_{\bar{G}}$ is transitive on $\{(1,1),(1,2),\cdots,(k,2)\}$, for any (i,j) and (l,m) there is the element $\bar{g}$ of $\bar{G}$ such that $\pi_{\bar{g}}(i,j)=(l,m)$. Therefore we have from (3.3) and (3.7)

$$\begin{aligned}P(D_{ij}|H_{ij}(\varDelta))&=\sum_{r\in\mathfrak{R}} d_{ij}(r)h_{ij}(r|\varDelta)=\sum_{r\in\mathfrak{R}} d_{\pi_{\bar{g}}(i,j)}(\bar{g}r)h_{\pi_{\bar{g}}(i,j)}(\bar{g}r|\varDelta)\\&=\sum_{r\in\mathfrak{R}} d_{lm}(\bar{g}r)h_{lm}(\bar{g}r|\varDelta)=\sum_{r\in\mathfrak{R}} d_{lm}(r)h_{lm}(r|\varDelta)\\&=P(D_{lm}|H_{lm}(\varDelta)).\end{aligned}$$

This implies that the test is symmetric in power. Finally, notice that in the class of tests symmetric in power, the sum of power $\sum_{i=1}^{k}\sum_{j=1}^{2} P(D_{ij}|H_{ij}(\varDelta))$ is maximized if and only if the common power $P(D_{ij}|H_{ij}(\varDelta))$ is maximized. These facts lead us to the conclusion of the proof. (Q. E. D.)

As for slippage problem [B], we can derive the MPS-α rank tests and the MPSS-α rank test for [B] in the same manner as in [A], so we will only present the results.

THEOREM 1B. *For any* $\alpha\in(0,1)$ *there exist the* MPS-α *rank tests for* [B], *which are given by the following form.*

$$d_0(r)=\begin{cases}1 & \text{if } \max_{1\le i\le k} h_i(r|\varDelta)<\lambda\cdot h_0(r)\\ \xi(r) & \text{if } \qquad\qquad = \\ 0 & \text{if } \qquad\qquad > \end{cases} \tag{3.8}$$

$$d_j(r)=\begin{cases}\eta_j(r) & \text{if } h_j(r|\varDelta)=\max_{1\le i\le k} h_i(r|\varDelta)\ge\lambda\cdot h_0(r)\\ 0 & \text{if otherwise}\end{cases} \qquad j=1,2,\cdots,k.$$

where λ, $\xi(r)$ *and* $\eta_j(r)$ *are defined in the same way as in Theorem* 1A.

COROLLARY 1B. *For any* $\alpha\in(0,1)$ *there exists the* MPSS-α *rank test for* [B], *which is given by the following from. Furthermore, it is also the* MPS-α *rank test.*

$$d_0(r)=\begin{cases}1 & \text{if } \max_{1\le i\le k} h_i(r|\varDelta)<\lambda\cdot h_0(r)\\ \xi_\alpha & \text{if } \qquad\qquad = \\ 0 & \text{if } \qquad\qquad > \end{cases} \tag{3.9}$$

$$d_j(r)=\begin{cases}1/m(r) & \text{if } h_j(r|\varDelta)=\max_{1\le i\le k} h_i(r|\varDelta)>\lambda\cdot h_0(r)\\ (1-\xi_\alpha)/m(r) & \text{if } \qquad\qquad\qquad = \\ 0 & \text{if otherwise}\end{cases} \qquad j=1,2,\cdots,k,$$

where λ, ξ_α *and* $m(r)$ *are defined in the same way as in Corollary* 1A.

REMARK 1. In the proof of Theorem 1B and Corollary 1B, G and Π in the proof of Theorem 1A are to be replaced by G_1 and Π_1.

§ 4. Extended locally most powerful rank tests.

We proceed to the derivation of the ELMPS-α and the ELMPSS-α rank tests. The following conditions are due to Hájek and Šidák [3].

Let J be an open interval containning 0.

CONDITION III.

(i) $f(x|\delta)$ is absolutely continuous on finite intervals in $\delta\in J$ for almost every x.

(ii) The limit

$$\dot{f}(x|0)=\lim_{\delta\to 0}\frac{1}{\delta}[f(x|\delta)-f(x|0)]$$

exists for almost every x.

(iii) It holds that

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty}|\dot{f}(x|\delta)|\,d\mu(x)=\int_{-\infty}^{\infty}|\dot{f}(x|0)|\,d\mu(x)<\infty$$

with $\dot{f}(x|\delta)$ denoting the partial derivative with respect to δ.

THEOREM 2A. *If* $f(x|\delta)=f(-x|-\delta)$ *for any* $\delta\in J$ *and Condition III are satisfied, then for any* $\alpha\in(0,1)$ *there exist the* ELMPS-α *rank tests for* [A], *which are given by the following form.*

$$(4.1)\qquad \begin{gathered} d_0(r)=\begin{cases}1 & \textit{if}\quad \max_{i,j} T_{ij}(r)<\lambda\\ \xi(r) & \textit{if}\quad\quad = \\ 0 & \textit{if}\quad\quad >\end{cases}\\ d_{lm}(r)=\begin{cases}\eta_{lm}(r) & \textit{if}\quad T_{lm}(r)=\max_{i,j}T_{ij}(r)\geqq\lambda\\ 0 & \textit{if}\quad \textit{otherwise}\end{cases}\\ i,\ l=1,2,\cdots,k\,;\ j,\ m=1,2\,,\end{gathered}$$

where

$$(4.2)\qquad T_{i1}(r)=\sum_{j=1}^{n}E_0\Big[\frac{\dot{f}(X_{(r_{ij})}|0)}{f(X_{(r_{ij})}|0)}\Big]\quad \textit{and}\quad T_{i2}(r)=\sum_{j=1}^{n}E_0\Big[\frac{\dot{f}(X_{(N+1-r_{ij})}|0)}{f(X_{(N+1-r_{ij})}|0)}\Big]$$

with E_0 *denoting the expectation under* H_0, *and* λ, $\xi(r)$ *and* $\eta_{lm}(r)$ *are defined in the same way as in Theorem* 1A.

PROOF. At first, we notice that, as in the proof of Theorem in [3] (on page 71), we have

$$(4.3)\qquad h_{ij}(r|\Delta)=\frac{1}{N!}+\Delta\cdot T_{ij}(r)+o(\Delta)\qquad \text{for}\quad i=1,2,\cdots,k\,;\ j=1,2\,.$$

Since $\mathfrak{R}$ has $N!$ elements and the index set $\{(i,j)\}$ of alternative hypotheses consists of $2k$ elements, we have uniformly, for sufficiently small Δ, the relation that $T_{ij}(r)<,>T_{lm}(r)$ implies $h_{ij}(r|\Delta)<,>h_{lm}(r|\Delta)$, respectively. On the other hand, if $T_{ij}(r)=T_{lm}(r)$, we cannot distinct which of three cases $h_{ij}(r|\Delta)<,=,>h_{lm}(r|\Delta)$ is true, but the difference of $h_{ij}(r|\Delta)$ and $h_{lm}(r|\Delta)$ is at most within the order $o(\Delta)$.

Taking account of the above facts, the replacement of $h_{ij}(r|\Delta)$ by $T_{ij}(r)$ in (3.5) leads us to that the tests given by (4.1) are the ELMPS-α rank tests. (Q. E. D.)

COROLLARY 2A. *Under the assumptions of Theorem* 2A, *for any* $\alpha\in(0,1)$ *there exists the* ELMPSS-α *rank test for* [A], *which is given by the following form. Furthermore, it is also* ELMPS-α *rank test for* [A].

$$
(4.4)\qquad
\begin{aligned}
d_0(r)&=\begin{cases}1 & \text{if } \max_{i,j} T_{ij}(r)<\lambda\\ \xi_\alpha & \text{if } \qquad = \\ 0 & \text{if } \qquad >\end{cases}\\
d_{lm}(r)&=\begin{cases}1/m(r) & \text{if } T_{lm}(r)=\max_{i,j} T_{ij}(r)>\lambda\\ (1-\xi_\alpha)/m(r) & \text{if } \qquad = \\ 0 & \text{if otherwise}\end{cases}\\
& i,\, l=1,2,\cdots,k\,;\ j,\, m=1,2\,.
\end{aligned}
$$

where λ, ξ_α, *and* $m(r)$ *are defined in the same way as in Corollary* 1A.

PROOF. We see easily that under $\bar{G}$ and $\Pi_{\bar{G}}$ in the proof of THEOREM 1A, the rank test (4.4) satisfies (3.7). The remainder of the proof is the same as in Corollary 1A. (Q. E. D.)

As to the case [B], let us describe only the results as before.

THEOREM 2B. *If Condition III is satisfied, then for any* $\alpha\in(0,1)$ *there exist the* ELMPS-α *rank tests for* [B], *which are given by the following form.*

$$
(4.5)\qquad
\begin{aligned}
d_0(r)&=\begin{cases}1 & \text{if } \max_{1\le i\le k} T_i(r)<\lambda\\ \xi(r) & \text{if } \qquad = \\ 0 & \text{if } \qquad >\end{cases}\\
d_j(r)&=\begin{cases}\eta_j(r) & \text{if } T_j(r)=\max_{1\le i\le k} T_i(r)\ge\lambda\\ 0 & \text{if otherwise}\end{cases}\\
& j=1,2,\cdots,k
\end{aligned}
$$

where λ, $\xi(r)$ *and* $\eta_j(r)$ *are defined in the same way as in Theorem* 1A, *and* T_i *is* T_{i1} *of* (4.2).

COROLLARY 2B. *If Condition III is satisfied, then for any* $\alpha\in(0,1)$ *there exists the* ELMPSS-α *rank test for* [B], *which is given by the following form. Furthermore, it is also the* ELMPS-α *rank test.*

$$
(4.6)\qquad
\begin{aligned}
d_0(r)&=\begin{cases}1 & \text{if } \max_{1\le i\le k} T_i(r)<\lambda\\ \xi_\alpha & \text{if } \qquad = \\ 0 & \text{if } \qquad >\end{cases}\\
d_j(r)&=\begin{cases}1/m(r) & \text{if } T_j(r)=\max_{1\le i\le k} T_i(r)>\lambda\\ (1-\xi_\alpha)/m(r) & \text{if } \qquad = \\ 0 & \text{if otherwise}\end{cases}\\
& j=1,2,\cdots,k
\end{aligned}
$$

where λ, ξ_α *and* $m(r)$ *are defined in the same way as in Corollary* 1A.

REMARK 2. It should be especially noticed that the tests of (4.1) are neither the MPS-α rank test nor the locally most powerful rank test. Also the test of (4.4) is neither the MPSS-α rank test nor the locally most powerful rank test. This is an interesting feature of the slippage tests differing from two decision problems such as two sample problems, where for some proper level α there exists the locally most powerfull rank test which is preferable to the ELMPS-α rank test. The reason for this difference can be formed in the following considerations. $T_{ij}(r)=T_{lm}(r)$ does not reflect $h_{ij}(r|\Delta)=h_{lm}(r|\Delta)$, and it may cause an disturbance on the power of the slippage test, as there is a positive probability of accepting $H_{ij}(\Delta)$ in some cases of $h_{ij}(r|\Delta)<h_{lm}(r|\Delta)$.

§ 5. Location and Scale problems.

In this section, we particularly consider the location problems with $f(x|\delta)=f(x-\delta)$ and the scale problems with $f(x|\delta)=e^{-\delta}\cdot f(x\cdot e^{-\delta})$, where $f(x)$ is absolutely continuous.

At first, we are concerned with the location problems. In this case, if f is symmetric i. e. $f(x)=f(-x)$, then $f(x|\delta)=f(-x|-\delta)$ is satisfied for any δ. Hence Theorem 1A and Corollary 1A hold. However Theorem 1B and Corollary 1B hold without the assumption of the symmetry. Furthermore, if $\int_{-\infty}^{\infty}|f'(x)|d\mu<\infty$ is satisfied, then Condition III is satisfied, and so Theorem 2A, 2B, Corollary 2A and 2B hold. Here, we note

$$T_{i1}(r)=\sum_{j=1}^{n}E_0\Big[-\frac{f'(X_{(r_{ij})})}{f(X_{(r_{ij})})}\Big] \quad\text{and}\quad T_{i2}(r)=\sum_{j=1}^{n}E_0\Big[-\frac{f'(X_{(N+1-r_{ij})})}{f(X_{(N+1-r_{ij})})}\Big]. \tag{5.1}$$

It is well known that when f is normal, the tests of (4.1) are the k-sample slippage analogues of normal scores test with

$$T_{i1}(r)=\sum_{j=1}^{n}E_0[X_{(r_{ij})}] \quad\text{and}\quad T_{i2}(r)=\sum_{j=1}^{n}E_0[X_{(N+1-r_{ij})}] \tag{5.2}$$

and that when f is logistic, the tests of (4.1) are equivalent to the k-samples slippage analogues of the two-sample Wilcoxon test which are of the form (4.1) with

$$T_{i1}(r)=\sum_{j=1}^{n}r_{ij} \quad\text{and}\quad T_{i2}(r)=n(nk+1)-\sum_{j=1}^{n}r_{ij}. \tag{5.3}$$

Now, let us consider the generalized slippage problems [A′] and [B′] of [A] and [B], that is,

[A′]

$$\begin{aligned}
H_0' \quad &: \delta_1=\delta_2=\cdots=\delta_k=\delta\\
H_{i1}'(\Delta) &: \delta_1=\cdots=\delta_{i-1}=\delta_i-\Delta=\delta_{i+1}=\cdots=\delta_k=\delta\\
H_{i2}'(\Delta) &: \delta_1=\cdots=\delta_{i-1}=\delta_i+\Delta=\delta_{i+1}=\cdots=\delta_k=\delta\\
&\qquad i=1,2,\cdots,k.
\end{aligned}$$

[B′]

$$\begin{aligned}
H_0' \quad &: \delta_1=\delta_2=\cdots=\delta_k=\delta\\
H_i'(\Delta) &: \delta_1=\cdots=\delta_{i-1}=\delta_i-\Delta=\delta_{i+1}=\cdots=\delta_k=\delta\\
&\qquad i=1,2,\cdots,k.
\end{aligned}$$

where δ is unknown.

We should point out that Theorem 1A, 2A, 1B, 2B, Corollary 1A, 2A, 1B and 2B all still hold true for [A′] and [B′]. Because the distribution $Q_\theta(\cdot\,|\,\Delta,\,\delta)$ of $R(X)$ under $H'_\theta(\Delta)$ does not depend on δ, [A′] and [B′] is reduced to [A] and [B] respectively.

Secondly, we briefly refere to the scale problems. In this case, if $\int_{-\infty}^{\infty}|x\cdot f'(x)|\,d\mu<\infty$ is satisfied, then Condition III is satisfied. Therefore Theorem 1B, 2B, Corollary 1B and 2B hold. Here notice that

$$T_i(r)=\sum_{j=1}^{n}E_0\Big[-1-X_{(rij)}\cdot\frac{f'(X_{(rij)})}{f(X_{(rij)})}\Big]. \tag{5.4}$$

We would like to state again that in the scale problem [B′], Theorem 1B, 2B, Corollary 1B and 2B still hold true owing to the same reason as in the location problem [A′].

§ 6. Complementary notes.

Let us consider the slippage problems with a control population which were discussed in [7], that is, the problems of testing the hypothesis that all k populations are equal to the control population against k alternatives that all $k-1$ populations but the i-th one are equal to the control population. The formulations to the problems of this type are obtained by slightly modifying the formulations of § 2: Let X_{01}, X_{02}, $\cdots$, X_{0a} and X_{i1}, X_{i2}, $\cdots$, X_{in} $(i=1,2,\cdots,k)$ be mutually independent random variables distributed according to $f(x\,|\,0)$ and $f(x\,|\,\delta_i)$ respectively, and let all the other parts be the same as those of § 2.

Now, we can state that all the results of § 3, § 4 and § 5 still hold true in the case with a controled population, however in location and scale problems ([A′] and [B′]) of § 5, $f(x\,|\,0)$ is replaced by $f(x\,|\,\delta)$. We should notice that the total sample size is $N=nk+a$ and that for $\boldsymbol{X}=(X_{01},\cdots,X_{0a},X_{11},\cdots,X_{kn})\in\mathfrak{X}$ $X_{(i)}$ and $R_{ij}(\boldsymbol{X})$ are the i-th order statistic and the rank of X_{ij} in pooled samples $X_{01},\cdots,X_{0a},X_{11},\cdots,X_{kn}$, respectively and $R(\boldsymbol{X})=(R_{01}(\boldsymbol{X}),\cdots,R_{0a}(\boldsymbol{X}),R_{11}(\boldsymbol{X}),\cdots,R_{kn}(\boldsymbol{X}))$.

§ 7. Critical values of the k-sample slippage analogues of the Wilcoxon two-sample test.

The purpose of this section is to present the critical values of the one-sided and the two-sided k-sample slippage analogues of the Wilcoxon two-sample test, which are the ELMPS-α rank tests ((4.1) and (4.5)) when f is logistic with a location parameter.

Odeh [9] gives the distribution of the maximum rank sum $\max_{1\le i\le k}T_i=\max_{1\le i\le k}\sum_{j=1}^{n}R_{ij}$ for $k=2(1)5$, $n=2(1)5$, and the critical values of the one-sided k-sample slippage analogue of the Wilcoxon two-sample test for $k=2(1)5$, $n=2(1)8$, and $\alpha=0.001$, 0.005, 0.01, 0.025, 0.05, 0.10 and 0.20.

In our tables, the approximated critical values for $n=2(1)20$, $\alpha=0.01$, 0.025, 0.05, and $k=2(1)10$ for one-sided case [B], and $k=3(1)10$ for two-sided case [A], are presented.

The following is the brief description of the computed method. In order to clarify the sample size, we use the expression of $T_{ij}^{(n)}$ and λ_n. The expectation and the variance of $T_{ij}^{(n)}$ under H_0 are given by

$$E_0(T_{ij}^{(n)})=\frac{n(nk+1)}{2} \quad \text{and} \quad V_0(T_{ij}^{(n)})=\frac{n^2(k-1)(nk+1)}{12}$$

$$\text{for} \quad i=1, 2, \cdots, k;\ j=1, 2.$$

Normalizing these statistics, let

$$S_i^{(n)}=\Big(T_{i1}^{(n)}-\frac{n(nk+1)}{2}\Big)\Big/\sqrt{\frac{n^2(k-1)(nk+1)}{12}}$$

$$S_{k+i}^{(n)}=\Big(T_{i2}^{(n)}-\frac{n(nk+1)}{2}\Big)\Big/\sqrt{\frac{n^2(k-1)(nk+1)}{12}}$$

$$\text{for} \quad i=1, 2, \cdots, k,$$

then we have $S_i^{(n)}=-S_{k+i}^{(n)}$ and $\sum_{i=1}^{k} S_i^{(n)}=0$, and furthermore, under H_0 the asymptotic joint distribution of $(S_1^{(n)}, S_2^{(n)}, \cdots, S_k^{(n)})$ is the singular normal distribution with mean vector $(0, 0, \cdots, 0)$ and coxariance matrix (σ_{ij}) where $\sigma_{ij}=1$ or $-\frac{1}{k-1}$ as $i=j$ or not. For the sake of approximation, let $(S_1, S_2, \cdots, S_k)$ be distributed with the above singular normal distribution and let $S_{k+i}=-S_i$ $(i=1, 2, \cdots, k)$.

In the two-sided case, the approximation is

$$\begin{aligned} P(\max_{i,j} T_{ij}^{(n)} \leqq \lambda_n) &\doteqdot P(\max_{1\leqq i\leqq 2k} S_i \leqq C_n) \\ &\doteqdot 1-\sum_{i=1}^{2k} P(S_i > C_n)+\sum_{1\leqq i<j\leqq 2k} P(S_i > C_n, S_j > C_n) \\ &= 1-2k\cdot\Phi(C_n)+k(k-1)\{L(C_n, C_n;\rho)+L(C_n, C_n;-\rho)\} \end{aligned}$$

where $\Phi(C_n)=\int_{C_n}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}t^2\Big)dt$

$$\begin{aligned} L(C_n, C_n;\rho) &= \int_{C_n}^{\infty}\int_{C_n}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\Big[-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\Big]dxdy \\ &= \int_0^{\rho}\frac{1}{2\pi\sqrt{1-t^2}}\exp\Big(-\frac{C_n^2}{1+t}\Big)dt+\Phi^2(C_n) \end{aligned}$$

$$C_n=\Big(\lambda_n-\frac{n(nk+1)}{2}\Big)\Big/\sqrt{\frac{n^2(k-1)(nk+1)}{12}} \quad \text{and} \quad \rho=-\frac{1}{k-1}.$$

The first approximation is based on the asymptotic normality and the second is Bonferroni's inequality. The formula of $L(C_n, C_n:\rho)$ can be obtained from [6].

Since the approximation for one-sided case is similarly obtained, we do not mention it.

The tables of the critical values computed by the above methods are given in the last part of this paper. In the tables, we print the values over the first decimal place, after counting fractions of .5 and over as units and cut away the rest at the second decimal place. The maximum error of the critical values in the approximation of

Bonferroni's inequality is at most 0.1. The approximation under the asymptotic distribution is sufficiently accurate for large n. Concerning small n, we can compare them with [9].

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Table

The critical values of *k*-sample slippage analogues of the Wilcoxon two-sample test

One-sided

$k=2$ [B]

n \ α	0.05	0.025	0.01
2	7.5	7.9	8.3
3	15.0	15.6	16.4
4	24.8	25.8	26.9
5	36.9	38.2	39.8
6	51.2	53.0	55.1
7	67.8	70.0	72.6
8	86.7	89.3	92.5
9	107.7	110.9	114.7
10	130.9	134.7	139.1
11	156.4	160.6	165.7
12	184.0	188.8	194.6
13	213.7	219.2	225.7
14	245.7	251.8	259.1
15	279.8	286.5	294.6
16	316.0	323.5	332.4
17	354.4	362.6	372.3
18	395.0	403.9	414.4
19	437.6	447.3	458.8
20	482.5	492.9	505.3

Two-sided

$k=3$

n \ α	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	12.1	12.6	13.3	11.6	12.2	12.9
3	24.1	25.1	26.3	23.2	24.3	25.5
4	39.8	41.3	43.2	38.5	40.1	42.0
5	59.1	61.3	63.8	57.4	59.6	62.2
6	82.0	84.8	88.1	79.7	82.6	86.0
7	108.4	111.9	116.1	105.5	109.1	113.4
8	138.3	142.5	147.6	134.8	139.1	144.3
9	171.6	176.6	182.7	167.4	172.6	178.8
10	208.3	214.2	221.3	203.4	209.4	216.7
11	248.4	255.2	263.4	242.7	249.7	258.1
12	291.9	299.6	308.9	285.4	293.4	302.9
13	338.7	347.4	357.9	331.4	340.4	351.1
14	388.9	398.6	410.3	380.8	390.7	402.7
15	442.4	453.2	466.1	433.4	444.5	457.7
16	499.2	511.1	525.4	489.3	501.5	516.1
17	559.3	572.4	588.0	548.5	561.8	577.9
18	622.8	637.0	654.0	611.0	625.5	642.9
19	689.5	704.9	723.3	676.7	692.5	711.4
20	759.5	776.1	796.0	745.7	762.7	783.1

$k=4$

n \ α	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	16.4	17.2	18.1	15.7	16.5	17.4
3	32.9	34.2	35.8	31.6	33.0	34.7
4	54.4	56.4	58.9	52.5	54.6	57.2
5	80.8	83.6	87.1	78.2	81.1	84.9
6	112.0	115.8	120.3	108.6	112.5	117.1
7	148.0	152.7	158.4	143.7	148.6	154.4
8	188.7	194.5	201.3	183.5	189.4	196.5
9	234.1	240.9	249.1	227.9	234.9	243.4
10	284.1	292.0	301.6	276.8	285.0	294.9
11	338.6	347.8	358.8	330.2	339.7	351.1
12	397.7	408.2	420.7	388.1	398.9	412.0
13	461.3	473.1	487.3	450.6	462.7	477.4
14	529.5	542.6	558.5	517.5	531.0	547.5
15	602.1	616.7	634.3	588.8	603.9	622.0
16	679.3	695.3	714.6	664.6	681.1	701.2
17	760.8	778.4	799.6	744.8	762.9	784.8
18	846.9	866.0	889.0	829.4	849.1	873.0
19	937.3	958.1	983.1	918.4	939.8	965.7
20	1032.2	1054.6	1081.6	1011.7	1034.9	1062.8

$k=5$

n \ α	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	20.8	21.7	22.8	19.9	20.9	22.0
3	41.7	43.4	45.4	40.1	41.9	44.0
4	69.0	71.6	74.7	66.6	69.3	82.5
5	102.6	106.2	110.5	99.2	102.9	107.4
6	142.3	146.9	152.6	137.9	142.7	148.6
7	188.0	193.8	200.9	182.4	188.5	195.9
8	239.6	246.7	255.4	232.8	240.2	249.2
9	297.1	305.6	315.9	289.0	297.8	308.5
10	360.4	370.3	382.4	350.9	361.2	373.8
11	429.5	440.9	454.8	418.5	430.5	444.9
12	504.3	517.3	533.2	491.9	505.4	521.9
13	584.8	599.5	617.4	570.8	586.1	604.7
14	671.1	687.5	707.4	655.4	672.5	693.2
15	762.9	781.2	803.2	745.6	764.5	787.5
16	860.5	880.5	904.8	841.4	862.2	887.5
17	963.6	985.6	1012.2	942.7	965.5	993.2
18	1072.4	1096.3	1125.3	1049.6	1074.4	1104.6
19	1186.7	1212.6	1244.0	1162.0	1188.9	1221.6
20	1306.6	1334.6	1368.5	1279.9	1309.0	1344.3

$k=6$

n \ α	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	25.2	26.3	27.6	24.1	25.3	26.7
3	50.6	52.6	55.1	48.7	50.8	53.3
4	83.9	86.9	90.6	80.9	84.1	87.9
5	124.6	128.9	134.0	120.5	124.9	130.3
6	172.8	178.3	185.1	167.4	173.2	180.2
7	228.2	235.2	243.7	221.4	228.7	237.6
8	290.8	299.3	309.7	282.5	291.4	302.2
9	360.5	370.6	383.0	350.6	361.2	374.1
10	437.2	449.1	463.6	425.7	438.0	453.1
11	520.9	534.6	551.3	507.6	521.9	539.3
12	611.6	627.2	646.2	596.4	612.7	632.5
13	709.1	726.7	748.1	692.0	710.3	732.6
14	813.5	833.1	857.1	794.4	814.9	839.8
15	924.7	946.5	973.1	903.6	926.3	953.9
16	1042.7	1066.7	1096.0	1019.4	1044.4	1074.9
17	1167.6	1193.8	1225.8	1142.0	1169.4	1202.7
18	1299.1	1327.7	1362.6	1271.3	1301.1	1337.4
19	1437.4	1468.4	1506.2	1407.3	1439.6	1478.9
20	1582.4	1615.9	1656.7	1549.9	1584.8	1627.3

$k=7$

n	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	29.7	30.9	32.5	28.4	29.7	31.4
3	59.6	61.9	64.8	57.3	59.8	62.7
4	98.8	102.3	106.6	95.2	99.0	103.5
5	146.8	151.7	157.7	141.8	147.1	153.3
6	203.5	209.9	217.8	196.9	203.9	212.1
7	268.7	276.8	286.7	260.4	269.2	279.5
8	342.3	352.2	364.4	332.3	342.9	355.5
9	424.3	436.1	450.5	412.3	425.0	440.0
10	514.5	528.3	545.2	500.4	515.3	532.9
11	612.9	628.8	648.3	596.7	613.9	634.1
12	719.4	737.6	759.8	701.0	720.5	743.6
13	834.0	854.5	879.5	813.2	835.3	861.3
14	956.7	979.5	1007.5	933.4	958.1	987.1
15	1087.3	1112.6	1143.6	1061.6	1088.9	1121.1
16	1225.9	1253.8	1288.0	1197.6	1227.6	1263.1
17	1372.5	1403.0	1440.4	1341.4	1374.3	1413.2
18	1527.0	1560.2	1600.9	1493.1	1529.0	1571.3
19	1689.3	1725.4	1769.5	1652.6	1691.5	1737.4
20	1859.5	1898.5	1946.1	1819.9	1861.9	1911.5

$k=8$

n \ α	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	34.1	35.6	37.4	32.7	34.2	36.1
3	68.7	71.3	74.5	66.1	68.8	72.2
4	113.8	117.8	122.7	109.8	114.0	119.1
5	169.1	174.7	181.5	163.5	169.4	176.5
6	234.3	241.7	250.7	227.1	234.7	244.1
7	309.4	318.6	330.0	300.3	309.9	321.7
8	394.1	405.4	419.2	383.0	394.7	409.1
9	488.4	501.8	518.4	475.1	489.1	506.3
10	592.2	607.9	627.2	576.6	593.0	613.1
11	705.3	723.4	745.7	687.4	706.3	729.4
12	827.8	848.4	873.8	807.3	828.9	855.3
13	959.5	982.8	1011.4	936.5	960.8	990.5
14	1100.5	1126.5	1158.4	1074.8	1101.9	1135.1
15	1250.6	1279.5	1314.9	1222.1	1252.2	1289.0
16	1409.9	1441.7	1480.7	1378.5	1411.6	1452.2
17	1578.3	1613.1	1655.8	1543.9	1580.2	1624.6
18	1755.8	1793.6	1840.2	1718.3	1757.8	1806.2
19	1942.3	1983.3	2033.8	1901.6	1944.5	1996.9
20	2137.8	2182.1	2236.6	2093.9	2140.1	2196.8

$k=9$

n \ α	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	38.7	40.3	42.3	37.1	38.7	40.8
3	77.8	80.7	84.4	74.9	78.0	81.7
4	128.9	133.4	138.9	124.4	129.1	134.9
5	191.5	197.8	205.5	185.3	191.8	199.8
6	265.4	273.6	283.7	257.2	265.8	276.3
7	350.3	360.7	373.4	340.0	350.8	364.1
8	446.2	458.8	474.7	433.6	446.8	463.0
9	552.8	567.9	586.4	537.9	553.6	572.8
10	670.2	687.8	709.5	652.7	671.1	693.6
11	798.1	818.4	843.5	777.9	799.1	825.1
12	936.6	959.7	988.3	913.6	937.7	967.4
13	1085.6	1111.6	1143.8	1059.6	1086.8	1120.2
14	1244.9	1274.0	1309.9	1215.9	1246.3	1283.6
15	1414.6	1446.9	1486.7	1382.4	1416.2	1457.5
16	1594.6	1630.2	1674.0	1559.2	1596.4	1641.9
17	1784.9	1823.8	1871.9	1746.1	1786.8	1836.7
18	1985.4	2027.8	2080.1	1943.2	1987.5	2041.8
19	2196.1	2242.1	2298.8	2150.3	2198.4	2257.2
20	2417.0	2466.6	2527.9	2367.5	2419.4	2483.0

$k=10$

n \ α	[A] 0.05	[A] 0.025	[A] 0.01	[B] 0.05	[B] 0.025	[B] 0.01
2	43.2	45.0	47.2	41.4	43.3	45.6
3	87.0	90.2	94.2	83.7	87.1	91.3
4	144.1	149.0	155.2	139.1	144.3	150.7
5	214.0	220.9	229.5	207.1	214.3	223.2
6	296.5	305.6	316.9	287.5	297.0	308.6
7	391.4	402.9	417.0	380.0	391.9	406.6
8	498.5	512.4	529.7	484.5	499.1	517.0
9	617.5	634.2	654.8	600.9	618.3	639.6
10	748.5	768.0	792.1	729.0	749.4	774.4
11	891.3	913.8	941.6	868.8	892.3	921.2
12	1045.8	1071.4	1103.1	1020.3	1047.0	1079.8
13	1212.0	1240.9	1276.6	1183.2	1213.3	1250.3
14	1389.8	1422.0	1461.9	1357.6	1391.3	1432.6
15	1579.1	1614.9	1659.1	1543.4	1580.7	1626.6
16	1779.9	1819.3	1868.0	1740.6	1781.7	1832.2
17	1992.1	2035.3	2088.6	1949.1	1994.1	2049.4
18	2215.8	2262.7	2320.8	2168.8	2217.9	2278.1
19	2450.7	2501.7	2564.7	2399.9	2453.1	2518.4
20	2697.0	2752.0	2820.1	2642.1	2699.5	2770.1