THE LIMIT PROCESS OF WEIGHTED SPECTRAL ESTIMATES FOR A MULTIPLE STATIONARY GAUSSIAN PROCESS

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THE LIMIT PROCESS OF WEIGHTED SPECTRAL ESTIMATES FOR A MULTIPLE STATIONARY GAUSSIAN PROCESS

By

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0. Summary.

The limit process for a sequence of stochastic processes $\xi_N(\lambda)$, $0 \leqq \lambda \leqq \pi$, defined by weighted spectral estimates of multiple stationary Gaussian process is found using the theory of weak convergence.

The limit process is shown to be Gaussian with independent increments and with the covariance function defined by (1).

1. Introduction and notations.

Let $X(t) = (X_1(t), X_2(t), \cdots, X_p(t))'$, $t = \cdots, -1, 0, 1, 2, \cdots$ be a real multiple stationary Gaussian process with zero mean and with covariance matrix $\Gamma(h) = E\{X(t)X(t+h)'\}$, where " $'$ " denote the transposes.

We assume that the spectral density matrix $f(\lambda) = \{f_{jk}(\lambda), j, k = 1, 2, \cdots, p\}$, $-\pi \leqq \lambda \leqq \pi$, exists with $f(\lambda) = (2\pi)^{-1} \sum_{v=-\infty}^{\infty} e^{-iv\lambda} \Gamma(v)$.

Let $\{X(1), X(2), \cdots, X(N)\}$ be N observables of the process $X(t)$ and $X_N = (X(1)', X(2)', \cdots, X(N)')'$ a pN-column vector obtained by rearranging $\{X(1), X(2), \cdots, X(N)\}$. We denote $\Gamma_N = E\{X_N X_N'\}$.

For $-\pi \leqq \lambda \leqq \pi$, we define $d_N(\lambda) = \sum_{t=1}^{N} e^{-it\lambda} X(t)$ the finite Fourier transform of $\{X(1), X(2), \cdots, X(N)\}$, denoting its entries by $d_{jN}(\lambda)$, $j = 1, 2, \cdots, p$, and $I_N(\lambda) = (2\pi N)^{-1} d_N(\lambda) d_N(\lambda)^*$ the matrix of periodograms.

We shall use the following notations. For every Hermitian matrix A, say $q \times q$, $\|A\| = \sup |y^* A y|$, where sup is taken for all q-vector y with $\|y\| = 1$. For a matrix valued function $A(\lambda) = \{a_{jk}(\lambda), j, k = 1, 2, \cdots, p\}$, $\mathrm{Var}\,(A) = \left[\sum_{j=1}^{p} \sum_{k=1}^{p} \{\mathrm{Var}\,(a_{jk})\}^2\right]^{1/2}$ where $\mathrm{Var}\,(a_{jk}) = \int_{-\pi}^{\pi} d|a_{jk}(\lambda)|$ is the total variation of a_{jk}.

For $p \times p$ Hermitian matrix-valued functions $\Theta(\lambda) = \{\theta_{jk}(\lambda), j, k = 1, 2, \cdots, p\}$, $-\pi \leqq \lambda \leqq \pi$, we consider a set of the following conditions:

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$[\theta_1]$: $\Theta(\lambda)=\Theta(\lambda)^*=\overline{\Theta(-\lambda)}$, where "*" denote the conjugate transposes.
$[\theta_2]$: $\sup_{\lambda}\|\Theta(\lambda)\|\leqq M<\infty$, with a constant M.
$[\theta_3]$: $\mathrm{Var}(\Theta)<\infty$.
$e(\lambda)=(e^{i\lambda}, e^{2i\lambda}, \cdots, e^{Ni\lambda})'$.
$\Delta_N(\lambda)=\sum_{t=1}^{N} e^{-it\lambda}$.
$K_N(\lambda)=(2\pi N)^{-1}\cdot|\Delta_N(\lambda)|^2$, the Fejer kernel.
$M(\lambda:\Theta)=\int_0^{\lambda} e(-l)e(-l)^*\otimes\Theta(l)dl$ where $\otimes$ denote the Kronecker product.
$C[a, b]$ the space of all continuous functions on the finite closed interval $[a, b]$.

2. Main results.

For a $p\times p$ matrix valued function $\Theta(\lambda)$ satisfying $[\theta_1]$, $[\theta_2]$ and $[\theta_3]$, we define the random processes

$$\xi_N(\lambda:\Theta)=\sqrt{N}\,\mathrm{tr}\left\{\int_0^{\lambda}\Theta(l)(I_N(l)-f(l))dl\right\}, \qquad 0\leqq\lambda\leqq\pi,$$

$$\zeta_N(\lambda:\Theta)=\sqrt{N}\,\mathrm{tr}\left\{\int_0^{\lambda}\Theta(l)(I_N(l)-E\{I_N(l)\})dl\right\}, \qquad 0\leqq\lambda\leqq\pi.$$

If no confusion arises, we simply write $\xi_N(\lambda)$ and $\zeta_N(\lambda)$ for $\xi_N(\lambda:\Theta)$ and $\zeta_N(\lambda:\Theta)$ respectively.

We shall prove the following

THEOREM. *If* $\mathrm{tr}\{f(\lambda)\}$ *is square integrable and* $\int_0^{\lambda}\mathrm{tr}\{(\Theta(l)f(l))^2\}dl$ *has no intervals of constancy with* $\Theta(l)$ *satisfying* $[\theta_1]$, $[\theta_2]$ *and* $[\theta_3]$, *then, as* $N\to\infty$, *the measure* P_N *generated in* $C[0, \pi]$ *by the process* $\xi_N(\lambda:\Theta)$ *converges weakly to the measure* P *generated by the Gaussian process* $\zeta(\lambda)$, $0\leqq\lambda\leqq\pi$, *with* $\zeta(0)=E\{\zeta(\lambda)\}=0$ *and with*

$$E\{\zeta(\lambda)\zeta(\mu)\}=2\pi\int_0^{\min(\lambda,\mu)}\mathrm{tr}\{(\Theta(l)f(l))^2\}dl. \tag{1}$$

For statistical applications, we have

COROLLARY. *Under the assumptions of theorem, as* $N\to\infty$,

$$P\left\{\max_{0\leqq\lambda\leqq\pi}\sqrt{N}\left|\mathrm{tr}\left\{\int_0^{\lambda}\Theta(l)(I_N(l)-f(l))dl\right\}\right|<z\right\}\to P\{\max_{0\leqq\lambda\leqq\pi}|\zeta(\lambda)|<z\}$$
$$=\sum_{k=-\infty}^{\infty}(-1)^k[\Phi((2k+1)z/\sqrt{2\pi G})-\Phi((2k-1)z/\sqrt{2\pi G})],$$

where

$$\Phi(z)=(2\pi)^{-1/2}\int_{-\infty}^{z}e^{-u^2/2}du,$$

$$G=\int_0^{\pi}\mathrm{tr}\{(\Theta(l)f(l))^2\}dl.$$

To prove this theorem, we shall show asymptotic unbiasedness of $\xi_N(\lambda)$ in proposition 1, weak convergence of finite dimensional distributions in proposition 3 and conditional compactness of P_N or equivalently of $\tilde{P}_N$ generated by $\zeta_N(\lambda)$ in proposion 4.

Grenander and Rosenblatt [3] obtained the limit process for the auto-spectral estimates of a linear process without normality assumption.

Ibragimov [4] obtained the limit process for the auto-spectra of a Gaussian process essentially under the conditions that the spectral density function $f(\lambda)\in L_{2+\delta}$, for some $\delta>0$. Malevich [6] relaxed the condition on the spectral density to that of square integrability. MacNeil [5] found the limit processes for co-spectral and quadrature-spectral distribution functions essentially under the conditions that $f_{jj}(\lambda)\in L_{2+\delta}$, for some $\delta>0$. His basic result (Theorem 4.6 in [5]) is an easy consequence of our theorem with $\Theta(\lambda)$ being constant matrices. Brillinger [2], under a different set of assumptions involving the near independence of widely separated values of strictly stationary time series, obtained the limit processes for the matrix of cross-spectral distribution functions.

3. Asymptotic unbiasedness of the process $\xi_N(\lambda)$.

PROPOSITION 1. *Let $B_N(\lambda)$ be the expectation of the process $\xi_N(\lambda)$. Then, under the assumptions of theorem,*

$$\lim_{N\to\infty}\sup_{\lambda}|B_N(\lambda)|=0\,.$$

PROOF. Let

$$f_N(\lambda)=E\{I_N(\lambda)\}\,.$$

Then, since

$$f_N(\lambda)=\int_{-\pi}^{\pi}K_N(\lambda-l)f(l)dl\,,$$

we can write

$$B_N(\lambda)=\sqrt{N}\,\mathrm{tr}\left\{\int_{-\pi}^{\pi}\chi_{[0,\lambda]}(l)\Theta(l)(f_N(l)-f(l))dl\right\},$$

where $\chi_{[0,\lambda]}(l)=1$ for $l\in[0,\lambda]$; $=0$ for $l\notin[0,\lambda]$.

Let $Q(j)$ and $R_N(j)$ be the Fourier coefficients of the functions $\chi_{[0,\lambda]}(l)\Theta(l)$ and $f_N(l)-f(l)$ respectively. Then,

$$\begin{aligned}R_N(j)&=\int_{-\pi}^{\pi}e^{ij\lambda}\cdot(f_N(\lambda)-f(\lambda))d\lambda\\&=-|j|\cdot N^{-1}\cdot\Gamma(j)\,,\qquad\text{for}\quad|j|\leqq N\,,\\&=-\Gamma(j)\,,\qquad\text{for}\quad|j|>N\,.\end{aligned}$$

Hence, using Young's theorem (c. f. [7], p. 91), we can write

$$|B_N(\lambda)|=(2\pi)^{-1}\sqrt{N}\,|\mathrm{tr}\{N^{-1}\sum_{|j|\leqq N}|j|Q(j)\Gamma(j)+\sum_{|j|>N}Q(j)\Gamma(j)\}|\,.$$

From the condition $[\theta_3]$ and the theorem 2, p. 213 in [7], we have the following inequalities:

$$\|Q(j)\|\leqq\mathrm{Var}\,(\chi_{[0,\lambda]}\Theta)/|j|\leqq C_0/|j|\,,$$

where C_0 is a constant independent of λ.

Thus, we have (c. f. [4], p. 370)

$$|B_N(\lambda)|\leqq(\sqrt{2}\pi)^{-1}\cdot p\cdot C_0\cdot\left[N^{-1/4}\cdot\left\{\sum_{j=-\infty}^{\infty}\|\Gamma(j)\|^2\right\}^{1/2}+2\left\{\sum_{|j|\geqq N}\|\Gamma(j)\|^2\right\}^{1/2}\right].$$

Since $\sum_{j=-\infty}^{\infty}\|\Gamma(j)\|^2 \leqq 2\pi\cdot\int_{-\pi}^{\pi}\mathrm{tr}\{(f(l))^2\}dl<\infty$, we see that proposition 1 holds. Q. E. D.

From Proposition 1, it is sufficient to prove theorem replacing $\xi_N(\lambda)$ by $\zeta_N(\lambda)$.

4. The second moments of the process $\zeta_N(\lambda)$.

Let $\Theta^S(\lambda)$ be piecewise constant satisfying $[\theta_1]$, written in the following form for $0\leqq l\leqq\pi$,

$$\Theta^S(l)=\sum_{j=1}^{K}\chi_{I_j}(l)\cdot\Theta_j \tag{2}$$

where $\chi_I(l)=1$ for $l\in I$; $=0$ for $l\notin I$, and $\{I_j\}_{j=1}^{K}$ are disjoint intervals in $[0,\pi]$.

We denote $R_N^S(\lambda,\mu)=E\{\zeta_N(\lambda:\Theta^S)\zeta_N(\mu:\Theta^S)\}$. Then, we have

LEMMA 1. *As $N\to\infty$,*

$$R_N^S(\lambda,\mu)=2\pi\int_0^{\min(\lambda,\mu)}\mathrm{tr}\{(\Theta^S(l)f(l))^2\}dl+o(1).$$

PROOF. Since we can write

$$\zeta_N(\lambda:\Theta^S)=(2\pi\sqrt{N})^{-1}\cdot\{X_N'\cdot M(\lambda:\Theta^S)\cdot X_N-E\{X_N'\cdot M(\lambda:\Theta^S)\cdot X_N\}\},$$

it is easily seen that

$$\begin{aligned}R_N^S(\lambda,\mu)&=(4\pi^2N)^{-1}\cdot[\mathrm{tr}\{\Gamma_N M(\lambda:\Theta^S)\Gamma_N M(\mu:\Theta^S)\}+\mathrm{tr}\{\Gamma_N M(\lambda:\Theta^S)\Gamma_N\overline{M(\lambda:\Theta^S)}\}]\\&=(4\pi^2N)^{-1}\sum_{j=1}^{K}\sum_{m=1}^{K}\int_{-\pi}^{\pi}dl\int_{-\pi}^{\pi}dq\,\Psi_{11N}^{[I_j^\lambda,I_m^\mu]}(q,l)\,\mathrm{tr}\{f(l)\cdot\Theta_j\cdot f(q)\cdot\Theta_m\}\\&\quad+(4\pi^2N)^{-1}\sum_{j=1}^{K}\sum_{m=1}^{K}\int_{-\pi}^{\pi}dl\int_{-\pi}^{\pi}dq\,\Psi_{12N}^{[I_j^\lambda,I_m^\mu]}(q,l)\,\mathrm{tr}\{f(l)\cdot\Theta_j\cdot f(q)\cdot\bar{\Theta}_m\}\end{aligned}$$

where $I_j^\lambda=[0,\lambda]\cap I_j$, $I_m^\mu=[0,\mu]\cap I_m$ and

$$\Psi_{11N}^{[I_1,I_2]}(q,l)=\int_{I_1}d\alpha\int_{I_2}d\beta\,\frac{\sin\frac{N(q-\alpha)}{2}\cdot\sin\frac{N(l-\alpha)}{2}}{\sin\frac{q-\alpha}{2}\cdot\sin\frac{l-\alpha}{2}}\cdot\frac{\sin\frac{N(q-\beta)}{2}\cdot\sin\frac{N(l-\beta)}{2}}{\sin\frac{q-\beta}{2}\cdot\sin\frac{l-\beta}{2}},$$

$$\Psi_{12N}^{[I_1,I_2]}(q,l)=\int_{I_1}d\alpha\int_{I_2}d\beta\,\frac{\sin\frac{N(q-\alpha)}{2}\cdot\sin\frac{N(l-\alpha)}{2}}{\sin\frac{q-\alpha}{2}\cdot\sin\frac{l-\alpha}{2}}\cdot\frac{\sin\frac{N(q+\beta)}{2}\cdot\sin\frac{N(l+\beta)}{2}}{\sin\frac{q+\beta}{2}\cdot\sin\frac{l+\beta}{2}}.$$

These two functions $\Psi_{11N}^{[I_1,I_2]}$ and $\Psi_{12N}^{[I_1,I_2]}$ enjoy similar properties to those of the Fejer kernel (c. f. Theorem A and B in [5]), and hence we have

$$\begin{aligned}&\lim_{N\to\infty}(4\pi^2N)^{-1}\int_{-\pi}^{\pi}dl\int_{-\pi}^{\pi}dq\,\Psi_{11N}^{[I_1,I_2]}(q,l)\,\mathrm{tr}\{f(l)\Theta_j f(q)\Theta_m\}\\&\quad=2\pi\int_{I_1\cap I_2}\mathrm{tr}\{f(l)\Theta_j f(l)\Theta_m\}dl\end{aligned}$$

and for $I_1\cup I_2\subset[0,\pi]$,

$$\lim_{N\to\infty}(4\pi^2N)^{-1}\int_{-\pi}^{\pi}dl\int_{-\pi}^{\pi}dq\,\Psi_{12N}^{[I_1,I_2]}(q,l)\,\mathrm{tr}\{f(l)\Theta_j f(q)\bar{\Theta}_m\}=0.$$

From these, Lemma 1 follows. Q. E. D.

We prepare the following

LEMMA 2. i) *For any* $\varepsilon>0$,

$$\|\Gamma_N\| \leqq \sqrt{N}\Big\{2\pi\varepsilon+\varepsilon^{-1}\int_{[\|f\|>\varepsilon\sqrt{N}]}\|f(l)\|^2 dl\Big\}.$$

ii) *If*

$$\sup_{-\pi\leqq\lambda\leqq\pi}\|\Theta(\lambda)\|\leqq M<\infty,$$

$$\sup_{0\leqq\lambda\leqq\pi}\|M(\lambda:\Theta)\|\leqq 2\pi M.$$

PROOF. Let $y=(y_1', y_2', \cdots, y_N')'$, $y_t'=(y_{t1}, y_{t2}, \cdots, y_{tp})$, $t=1, 2, \cdots, N$, be pN-vectors with

$$\|y\|^2=\sum_{t=1}^{N}\|y_t\|^2=1,$$

and let

$$\phi(y, \lambda)=\sum_{t=1}^{N} y_t e^{it\lambda}.$$

Then,

$$\int_{-\pi}^{\pi}\|\phi(y, \lambda)\|^2 d\lambda=2\pi.$$

Since

$$\Gamma_N=\int_{-\pi}^{\pi} e(-l)e(-l)^*\otimes f(l)dl,$$

we have

$$\begin{aligned}
\|\Gamma_N\| &= \sup_{\|y\|=1}|y^*\Gamma_N y| \\
&= \sup_{\|y\|=1}\left|\int_{-\pi}^{\pi}\phi(y, \lambda)^* f(\lambda)\phi(y, \lambda)d\lambda\right| \\
&\leqq \sup_{\|y\|=1}\Big\{\int_{[\|f\|>\varepsilon\sqrt{N}]}\|f(l)\|\,\|\phi(y, l)\|^2 dl+\varepsilon\sqrt{N}\int_{[\|f\|>\varepsilon\sqrt{N}]}\|\phi(y, l)\|^2 dl\Big\} \\
&\leqq \varepsilon^{-1}\sqrt{N}\int_{[\|f\|>\varepsilon\sqrt{N}]}\|f(l)\|^2 dl+2\pi\varepsilon\sqrt{N},
\end{aligned}$$

and also we have

$$\begin{aligned}
\|M(\lambda:\Theta)\| &= \sup_{\|y\|=1}\left|\int_0^{\lambda}\phi(y, l)^*\Theta(l)\phi(y, l)dl\right| \\
&\leqq \sup_{-\pi\leqq l\leqq\pi}\|\Theta(l)\|\cdot\sup_{\|y\|=1}\left|\int_0^{\lambda}\|\phi(y, l)\|^2 dl\right| \\
&\leqq 2\pi M.
\end{aligned}$$

Q. E. D.

Let $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ be satisfying $[\theta_1]$, $[\theta_2]$ and $[\theta_3]$.

Let us write

$$R_N^i(\lambda, \mu)=E\{\zeta_N(\lambda:\Theta_i)\cdot\zeta_N(\lambda:\Theta_i)\} \qquad i=1, 2.$$

Then, we have

LEMMA 3. *If*

$$\sup_{-\pi\leqq\lambda\leqq\pi}\|\Theta_1(\lambda)-\Theta_2(\lambda)\|<\varepsilon,$$

then

$$\sup_{0\leqq\lambda, \mu\leqq\pi}|R_N^1(\lambda, \mu)-R_N^2(\lambda, \mu)|<C_1\cdot\varepsilon,$$

where C_1 is a constant independent of N.

PROOF.

$$M(\lambda_1 : \Theta_1) - M(\lambda : \Theta_2) = M(\lambda : \Theta_1 - \Theta_2)$$

and hence

$$\|M(\lambda : \Theta_1) - M(\lambda : \Theta_2)\| \leqq 2\pi\varepsilon .$$

Thus, we have from lemma 2,

$$\begin{aligned} |R_N^1(\lambda, \mu) - R_N^2(\lambda, \mu)| &\leqq (2\pi^2 N)^{-1} \cdot p \|\Gamma_N\|^2 \cdot \{\|M(\lambda, \Theta_1)\| \cdot \|M(\mu : \Theta_1 - \Theta_2)\| \\ &\quad + \|M(\mu : \Theta_2)\| \cdot \|M(\lambda : \Theta_1 - \Theta_2)\|\} \\ &\leqq C_1 \cdot \varepsilon . \end{aligned}$$

Q. E. D.

We shall show

LEMMA 4. *Let $\Theta(\lambda)$ be satisfying $[\theta_1]$, $[\theta_2]$ and $[\theta_3]$. Then for every $\varepsilon > 0$ we can find an Hermitian matrix valued function $\Theta^\varepsilon(\lambda)$ of the form of (2) with*

$$\sup_{-\pi \leqq \lambda \leqq \pi} \|\Theta(\lambda) - \Theta^\varepsilon(\lambda)\| < \varepsilon .$$

PROOF. From $[\theta_3]$,[*)] for every $\lambda \in [0, \pi]$ and every $\varepsilon > 0$, there always exists an interval $\Gamma_\lambda \subset [0, \pi]$ such that

$$\|\Theta(l) - \Theta(\lambda - 0)\| < \varepsilon \qquad \text{for} \quad l \in [0, \lambda] \cap \Gamma_\lambda ,$$
$$\|\Theta(l) - \Theta(\lambda + 0)\| < \varepsilon \qquad \text{for} \quad l \in [\lambda, \pi] \cap \Gamma_\lambda .$$

we can choose from $\{\Gamma_\lambda\}_{\lambda \in [0, \pi]}$ a finite covering $\{\Gamma_{\lambda_j}\}_{j=1}^M$ of $[0, \pi]$ with $0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_M \leqq \pi$. Thus, if we define

$$\begin{aligned} \Theta^\varepsilon(\lambda) &= \Theta(\lambda_i - 0) \qquad \text{for} \quad \lambda \in [0, \lambda_i] \cap \Gamma_{\lambda_i} \cap \Gamma_{\lambda_{i-1}}^c \\ &= \Theta(\lambda_i + 0) \qquad \text{for} \quad \lambda \in [\lambda_i, \pi] \cap \Gamma_{\lambda_i} \cap \Gamma_{\lambda_{i-1}}^c , \end{aligned}$$

then $\Theta^\varepsilon(\lambda)$ satisfies Lemma 4. Q. E. D.

Combining Lemma 1, Lemma 3 and Lemma 4, we have proved

PROPOSITION 2. *For any $\Theta(\lambda)$ satisfying $[\theta_1]$, $[\theta_2]$ and $[\theta_3]$, as $N \to \infty$,*

$$E\{\zeta_N(\lambda) \cdot \zeta_N(\mu)\} = 2\pi \int_0^{\min(\lambda, \mu)} \operatorname{tr}\{(\Theta(l) \cdot f(l))^2\} dl + o(1) .$$

5. Asymptotic normality of the finite dimensional distributions of the process $\zeta_N(\lambda)$.

PROPOSITION 3. *Under the assumptions of theorem, the distribution of any k-vector $(\zeta_N(\lambda_1), \zeta_N(\lambda_2), \cdots, \zeta_N(\lambda_k))$ is asymptotically normal with mean zero and covariance matrix*

$$\Sigma_k = \left\{2\pi \int_0^{\min(\lambda_i, \lambda_j)} \operatorname{tr}\{(\Theta(l) \cdot f(l))^2\} dl\right\} .$$

PROOF. Let $t = (t_1, t_2, \cdots, t_k)'$ be a real k-vector. Then, we can write

$$\sum_{j=1}^k t_j \zeta_N(\lambda_j) = X_N' A X_N - E\{X_N' A X_N\} ,$$

*) $[\theta_3]$ implies that every discontinuities of $\Theta(\lambda)$ is at most of the first kind. (c.f. T. Kawata: Fourier Analysis in Probability Theory, 1972, Academic P., p. 17).

where

$$A=(2\pi\sqrt{N})^{-1}\sum_{j=1}^{k}t_j M(\lambda_j:\Theta).$$

From Ibragimov [4]'s appendix, it is sufficient to prove the follows:

(i) $\lim_{N\to\infty} D(X_N' A X_N)=t'\sum_k t>0$,

(ii) $\lim_{N\to\infty} \|\Gamma_N\|\cdot\|A\|=0$,

where D denotes variances. From Proposition 2, we see that (i) holds. Using Lemma 2, we have for any $\varepsilon>0$

$$\|\Gamma_N\|\|A\|\leqq M\cdot\Big(\sum_{j=1}^{k}|t_j|\Big)\cdot\Big(2\pi\varepsilon+\varepsilon^{-1}\int_{[\|f\|>\varepsilon\sqrt{N}]}\|f(l)\|^2dl\Big).$$

From this, Proposition 3 follows. Q. E. D.

6. Convergence of the measures $\tilde{P}_N$.

Lastly, we shall show that the measures $\tilde{P}_N$ in $C[0,\pi]$ generated by $\zeta_N(\lambda)$ are conditionally compact. In order to do this, we evaluate the moments of $\zeta_N(\lambda_1)-\zeta_N(\lambda_2)$.

Let us put

$$\eta_{jk}(\lambda)=\int_0^{\lambda}\theta_{jk}(l)[d_{kN}(l)\overline{d_{jN}(l)}-E\{d_{kN}(l)\overline{d_{jN}(l)}\}]dl.$$

Then, we have for an integer $r\geqq 0$

$$E\{|\zeta_N(\lambda_1)-\zeta_N(\lambda_2)|^{2r}\}\leqq C_1(r)N^{-r}\sum_{j=1}^{p}\sum_{k=1}^{p}E\{|\eta_{jk}(\lambda_1)-\eta_{jk}(\lambda_2)|^{2r}\},$$

where $C_1(r), C_2(r), \cdots$ hereafter denote constants independent of λ_1, λ_2 and N.

For notational convenience, we write $\tilde{d}_j(l_t)=\overline{d_{jN}(l_t)}$ for $t=$even; $=d_{jN}(l_t)$ for $t=$odd, $\tilde{d}_k(l_t)=d_{kN}(l_t)$, for $t=$even; $=\overline{d_{kN}(l_t)}$ for $t=$odd, we can then write, for $0\leqq\lambda_2<\lambda_1\leqq\pi$,

$$\begin{aligned}&E\{|\eta_{jk}(\lambda_1)-\eta_{jk}(\lambda_2)|^{2r}\}\\&\qquad\leqq p^r\cdot M^{2r}\cdot\int_{\lambda_2}^{\lambda_1}\int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_2}^{\lambda_1}E\Big\{\prod_{t=1}^{2r}[\tilde{d}_j(l_t)\tilde{d}_k(l_t)-E\{\tilde{d}_j(l_t)\tilde{d}_k(l_t)\}]\Big\}\cdot dl_1dl_2\cdots dl_{2r},\end{aligned}$$

since $\sup_l|\theta_{jk}(l)|\leqq\sqrt{p}M$.

From the normality of the random vector

$$(d_{jN}(l),\ d_{jN}(l_2),\ \cdots,\ d_{jN}(l_{2r}),\ d_{kN}(l_1),\ \cdots,\ d_{kN}(l_{2r})),$$

we obtain

$$\begin{aligned}&E\Big\{\prod_{t=1}^{2r}[\tilde{d}_j(l_t)\tilde{d}_k(l_t)-E\{\tilde{d}_j(l_t)\tilde{d}_k(l_t)\}]\Big\}\\&\qquad\leqq C_2(r)\sum_{(a_t)}\sum_{(i_t)}\prod_{t=1}^{r}|E\{\tilde{d}_{a_{2t-1}}(l_{i_{2t-1}})\tilde{d}_{a_{2t}}(l_{i_{2t}})\}|^2\end{aligned}$$

where $\sum_{(i_t)}$ denotes the summation running over all permutations $(i_1, i_2, \cdots, i_{2r})$ of $(1, 2, \cdots, 2r)$ and $\sum_{(a_t)}$ the summation over all possible assignments such that $a_t=j$ or k.

We write $m_{ab}(\lambda,\mu)=E\{d_{aN}(\lambda)d_{bN}(\mu)\}$ and

$$S_{ab}(\lambda, \mu)=|m_{aa}(\lambda, \mu)|^2+|m_{aa}(\lambda, -\mu)|^2+|m_{ab}(\lambda, \mu)|^2+|m_{ab}(\lambda, -\mu)|^2$$
$$+|m_{ab}(\mu, \lambda)|^2+|m_{ab}(\mu, -\lambda)|^2+|m_{bb}(\lambda, \mu)|^2+|m_{bb}(\lambda, -\mu)|^2 .$$

Since

$$m_{ab}(\lambda, \mu)=\int_{-\pi}^{\pi} \Delta_N(\lambda-l)\overline{\Delta_N(\mu-l)}f_{ab}(l)dl ,$$

we have for every $a, b=1, 2, \cdots, p$,

$$\int_{\lambda_2}^{\lambda_1}\int_{\lambda_2}^{\lambda_1}|m_{ab}(q, l)|^2 dq\, dl \leqq 4\pi^2 p^2 \int_{\lambda_2}^{\lambda_1} dl \int_{-\pi}^{\pi} |\Delta_N(l-\alpha)|^2 \|f(\alpha)\|^2 d\alpha$$
$$\leqq 4\pi^2 p^2 \int_{-\pi}^{\pi} |\Delta_N(l)|^2 dl \Big(\int_{-\lambda_1}^{-\lambda_2}+\int_{+\lambda_2}^{+\lambda_1}\Big)\|f(l-\alpha)\|^2 d\alpha ,$$

and similarly

$$\int_{\lambda_2}^{\lambda_1}\int_{\lambda_2}^{\lambda_1}|m_{ab}(q, -l)|^2 dq\, dl \leqq 4\pi^2 p^2 \int_{-\pi}^{\pi} |\Delta_N(l)|^2 dl \Big(\int_{-\lambda_1}^{-\lambda_2}+\int_{+\lambda_2}^{+\lambda_1}\Big)\|f(l-\alpha)\|^2 d\alpha .$$

Let us write

$$W_N(\lambda)=\int_{-\pi}^{\pi} K_N(l)dl \int_{-\lambda}^{\lambda} \|f(l-\alpha)\|^2 d\alpha \qquad 0\leqq \lambda \leqq \pi .$$

Then, we have for $0\leqq \lambda_2<\lambda_1\leqq \pi$,

$$\int_{\lambda_2}^{\lambda_1}\int_{\lambda_2}^{\lambda_1} S_{jk}(q, l)dq\, dl \leqq C\cdot N\cdot\{W_N(\lambda_1)-W_N(\lambda_2)\} ,$$

where C is a constant. Thus, we obtain for $0\leqq \lambda_2<\lambda_1\leqq \pi$,

$$E\{|\zeta_N(\lambda_1)-\zeta_N(\lambda_2)|^{2r}\}\leqq C_4(r)\cdot\{W_N(\lambda_1)-W_N(\lambda_2)\}^r . \tag{3}$$

$W_N(\lambda)$, $0\leqq \lambda\leqq \pi$, is continuous and strictly increasing with $W_N(0)=0$ and $W_N(\pi)=\int_{-\pi}^{\pi}\|f(l)\|^2 dl=G_2$, say. Hence, the inverse function $W_N^{-1}(\mu)$ from $[0, G_2]$ to $[0, \pi]$ is one-one and continuous.

If we make the substitution

$$\lambda=W_N^{-1}(\mu) \qquad 0\leqq \mu\leqq G_2 ,$$

the process $\zeta_N(\lambda)$ transformed into the process

$$\hat{\zeta}_N(\mu)=\zeta_N(W_N^{-1}(\mu)) \qquad 0\leqq \mu \leqq G_2 .$$

From (3), we obtain for $0\leqq \mu_2<\mu_1\leqq G_2$,

$$E\{|\hat{\zeta}_N(\mu_1)-\hat{\zeta}_N(\mu_2)|^{2r}\}\leqq C_4(r)|\mu_1-\mu_2|^r .$$

We have, thus seen that the family of measures $\hat{P}_N$ in $C[0, G_2]$ generated by $\hat{\zeta}_N(\mu)$ is conditionally compact. (c. f. Billingsley [1], p. 95)

The precompactness of the family of the functions $W_N(\lambda)$ in $C[0, \pi]$ and the continuity of the inverse function $W_N^{-1}(\mu)$ guarantee that the family of the measures $\tilde{P}_N$ is conditionally compact.

Thus we have proved

PROPOSITION 4. *Under the assumptions of theorem, the family of the measures* $\tilde{P}_N$ *in* $C[0, \pi]$ *generated by* $\zeta_N(\lambda)$ *is conditionally compact.*

By combining Propositions 1, 2, 3 and 4, our theorem is thus proved.

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