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NONPARAMETRIC SELECTION PROCEDURES IN TWO-WAY LAYOUTS

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1. Introduction.

The nonparametric selection problems in analysis of variance have been mainly developed for one-way layout models. For example, Lehmann [4], Puri and Puri [6] and Alam and Thompson [1] have respectively discussed the selection procedures based on the ranks of the observations. Randles [7] has also emphasized the use of the Hodges-Lehmann estimates for the same models to eliminate the difficulties concerning the least favorable configuration (cf. Rizvi and Woodworth [8]). Only a work for the two-way layouts is seen in Hollander [3]. We consider some selection problems under the more general two-way layout models.

Let $X_{i\alpha}$ be the random observation on the i-th treatment Π_i in the α-th block and suppose that

$$X_{i\alpha}=\nu+\theta_i+\mu_\alpha+\varepsilon_{i\alpha} \qquad (i=1, \cdots, c;\ \alpha=1, \cdots, n) \tag{1.1}$$

$$\sum_{i=1}^{c}\theta_i=0\,, \qquad \sum_{\alpha=1}^{n}\mu_\alpha=0$$

where the θ's are treatment effects, the μ's are block effects (nuisance parameters) and the ε's are residual error components. It is furthermore assumed that $\underset{\sim}{\varepsilon}_\alpha=(\varepsilon_{1\alpha}, \cdots, \varepsilon_{c\alpha})$, $\alpha=1, \cdots, n$ are independent with common continuous cdf $F(x_1, \cdots, x_c)$ which is symmetric in its c arguments.

We here deal with the problem of selecting a subset of size s, where $t\leqq s\leqq c-1$, which includes the t treatments having largest θ-values. When $s=t$, it will be reduced to the selection problem of the best t treatments. Now the ordered values of the θ's be $\theta_{[1]}\leqq \cdots \leqq\theta_{[c]}$ and denote $\underset{\sim}{\theta}=(\theta_{[1]}, \cdots, \theta_{[c]})$. Further denote

$$D(\Delta^*)=\{\underset{\sim}{\theta}:\ \theta_{[c-t+1]}-\theta_{[c-t]}\geqq\Delta^*\} \tag{1.2}$$

and $\binom{c}{s}^{-1}\binom{c-t}{s-t}<P^*<1$ and Δ^* are preassigned constants. Then our goal is to find the procedures such that

$$P[\text{correct selection}]\geqq P^* \qquad \text{for} \quad \underset{\sim}{\theta}\in D(\Delta^*)\,. \tag{1.3}$$

In section 2, the procedure R is defined in terms of the Hodges-Lehmann estimate and its asymptotic properties are discussed. The asymptotic relative efficiency of the

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procedure R with respect to the normal theory procedure M is obtained in section 3.

Denote the cdf of $\varepsilon_{ij\alpha}=\varepsilon_{i\alpha}-\varepsilon_{j\alpha}$ and $(\varepsilon_{ij\alpha}, \varepsilon_{ik\alpha})$ be $G(x)$ and $G^*(x, y)$ respectively and assume that G has a continuous density g satisfying the conditions of Theorem 1 of Lehmann [5]. We here note that $G(x)$ is symmetric about $x=0$.

2. The Procedure R.

Define the Hodges-Lehmann estimate by

$$\hat{Y}_i=c^{-1}\sum_{j=1}^{c} Y_{ij}, \qquad i=1, \cdots, c \tag{2.1}$$

$$\hat{Y}_{ij}=\underset{1\leqq\alpha\leqq\beta\leqq n}{\operatorname{med}}\{2^{-1}[(X_{i\alpha}-X_{j\alpha})+(X_{i\beta}-X_{j\beta})]\} \qquad i, j=1, \cdots, c \tag{2.2}$$

and denote $X_{i\alpha}$ or $\hat{Y}_i$ corresponding to the treatment with $\theta_{[i]}$ by $X_{(i)\alpha}$ or $\hat{Y}_{(i)}$ and the ordered values by $\hat{Y}_{[1]}<\cdots<\hat{Y}_{[c]}$.

The procedure R: Select the s treatments associated with $\hat{Y}_{[c-s+1]}, \cdots, \hat{Y}_{[c]}$. We first prove a lemma concerning the least favorable configuration of $\underset{\sim}{\theta}$ analogous to Randles [7].

LEMMA 2.1. *It holds for any* $\underset{\sim}{\theta}\in D(\Delta^*)$ *that*

$$P[CS\,|\,R, \underset{\sim}{\theta}]\geqq P[CS\,|\,R, \underset{\sim}{\theta}_0] \tag{2.3}$$

where $\underset{\sim}{\theta}_0$ *is defined by*

$$\theta_{[1]}=\cdots=\theta_{[c-t]}=\theta_{[c-t+1]}-\Delta^*=\cdots=\theta_{[c]}-\Delta^*. \tag{2.4}$$

PROOF. The probability of correct selection under the procedure R is given by

$$\begin{aligned} P[CS\,|\,R, \underset{\sim}{\theta}]=P[&(s-t+1)st \text{ largest of } (\hat{Y}_{(1)}, \cdots, \hat{Y}_{(c-t)}) \\ &<\min(\hat{Y}_{(c-t+1)}, \cdots, Y_{(c)}\,|\,\underset{\sim}{\theta}]. \end{aligned} \tag{2.5}$$

Consider a configuration $\underset{\sim}{\theta}_i=(\theta_{[1]}, \cdots, \theta_{[i-1]}, \theta_{[i]}-\delta, \theta_{[i+1]}, \cdots, \theta_{[c]})$. Let

$$X^*_{(i)\alpha}=X_{(i)\alpha}-\delta, \qquad X^*_{(j)\alpha}=X_{(j)\alpha} \qquad \text{for } j\neq i \tag{2.6}$$

and define $\hat{Y}^*_{(i)}$ by the same manner as (2.1), then it is easy to show that

$$\hat{Y}^*_{(i)}=\hat{Y}_{(i)}-c^{-1}(c-1)\delta, \qquad \hat{Y}^*_{(j)}=\hat{Y}_{(j)}+c^{-1}\delta \qquad \text{for } j\neq i. \tag{2.7}$$

If taking $(c-t+1)\leqq i\leqq c$, $\delta>0$, it follows that

$$\begin{aligned} P[CS\,|\,R, \underset{\sim}{\theta}_i]&=P[(s-t+1)st \text{ largest of } (\hat{Y}^*_{(1)}, \cdots, \hat{Y}^*_{(c-t)}) \\ &\qquad <\min(\hat{Y}^*_{(c-t+1)}, \cdots, \hat{Y}^*_{(c)})\,|\,\underset{\sim}{\theta}] \\ &=P[(s-t+1)st \text{ largest of } (\hat{Y}_{(1)}, \cdots, \hat{Y}_{(c-t)}) \\ &\qquad <\min(\hat{Y}_{(c-t+1)}, \cdots, \hat{Y}_{(i-1)}, \hat{Y}_{(i)}-\delta, \hat{Y}_{(i+1)}, \cdots, \hat{Y}_{(c)}\,|\,\underset{\sim}{\theta}] \\ &\leqq P[CS\,|\,R, \underset{\sim}{\theta}]. \end{aligned} \tag{2.8}$$

By decreasing the value of $\theta_{[c-t+2]}$ to that of $\theta_{[c-t+1]}$, we get the following

$$P[CS\,|\,R, \underset{\sim}{\theta}]\geqq P[CS\,|\,R, \underset{\sim}{\theta}_{c-t+2}]. \tag{2.9}$$

Repeat this process in order of $\theta_{[c-t+2]}, \cdots, \theta_{[c]}$ and we obtain the following

$$P[CS \mid R, \underset{\sim}{\theta}] \geqq P[CS \mid R, \underset{\sim}{\theta}^*] \tag{2.10}$$

where

$$\theta^*: \quad \theta_{[1]} \leqq \cdots \leqq \theta_{[c-t]} \leqq \theta_{[c-t+1]} - \Delta^* = \cdots = \theta_{[c]} - \Delta^* .$$

The similar considerations for $1 \leqq i \leqq c-t$ also show that

$$P[CS \mid R, \underset{\sim}{\theta}^*] \geqq P[CS \mid R, \underset{\sim}{\theta}_0] . \tag{2.11}$$

The result (2.3) is obtained from the inequalities (2.10) and (2.11).

Under the assumption that the functional form of F is unknown, we can apply the large-sample methods for the problem. To the end, consider the following situation for increasing n,

$$D(\Delta^{(n)}) = \{\underset{\sim}{\theta}^{(n)} : \theta_{[c-t+1]} - \theta_{[c-t]} \geqq \Delta^{(n)}] \tag{2.12}$$

where $\Delta^{(n)}$ will be determined below.

The following theorem gives a large-sample solution for the problem.

THEOREM 2.1. *For fixed P^*, let n be determined by*

$$\inf_{D(\Delta^{(n)})} P[CS \mid R, \underset{\sim}{\theta}^{(n)}] = P^* . \tag{2.13}$$

Then $n \to \infty$,

$$\Delta^{(n)} = AB^{-1}\Delta(2n^{-1})^{\frac{1}{2}} + o(n^{-\frac{1}{2}}) \tag{2.14}$$

where

$$\begin{aligned} A^2 &= c^{-1}[1+(c-2)\{12\lambda(G)-3\}] \\ B^2 &= 12\Big[\int_{-\infty}^{\infty} g^2(x)dx\Big]^2 \\ \lambda(G) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x)G(y)dG^*(x, y) \end{aligned} \tag{2.15}$$

and Δ is determined by the condition

$$P^* = (c-s)\binom{c-t}{s-t} Q_{c-1}(\underbrace{\Delta, \cdots, \Delta}_{t}, \underbrace{0, \cdots, 0}_{c-t-1}) \tag{2.16}$$

where Q_{c-1} is the cdf of a normally distributed random vector $(U_1, \cdots, U_t, V_{t+1}, \cdots, V_s, W_{s+1}, \cdots, W_{c-1})$ with

$$\begin{aligned} &E(U_i) = E(V_j) = E(W_k) = 0 , && \mathrm{Cov}(U_i, U_{i'}) = (1+\delta_{ii'})/2 \\ &\mathrm{Cov}(V_j, V_{j'}) = (1+\delta_{jj'})/2 , && \mathrm{Cov}(W_k, W_{k'}) = (1+\delta_{kk'})/2 \\ &\mathrm{Cov}(V_j, U_i) = 1/2 , && \mathrm{Cov}(U_i, W_k) = \mathrm{Cov}(V_j, W_k) = -1/2 \\ &i, i' = 1, \cdots, t; \quad j, j' = t+1, \cdots, s; && k, k' = s+1, \cdots, c-1 \end{aligned} \tag{2.17}$$

and δ's are the Kronecker deltas.

PROOF. From Lemma 2.1, we have

$$\inf_{D(\Delta^{(n)})} P[CS \mid R, \underset{\sim}{\theta}^{(n)}] = P[CS \mid R, \underset{\sim}{\theta}_0^{(n)}]$$

where

$$\underset{\sim}{\theta}_0^{(n)} : \quad \theta_{[1]} = \cdots = \theta_{[c-t]} = \theta_{[c-t+1]} - \Delta^{(n)}, \ \theta_{[c]} = \theta_{[c-t+1]} .$$

Hence the sample size n is determined by the condition,

$$P^* = P[(s-t+1)st \text{ largest of } (\hat{Y}_{(1)}, \cdots, \hat{Y}_{(c-t)}) < \min(\hat{Y}_{(c-t+1)}, \cdots, \hat{Y}_{(c)}) \mid \underset{\sim}{\theta}_0^{(n)}]$$
$$= (c-s)\binom{c-t}{s-t} < P[Z_{1i} < 0, Z_{j1} < 0, i = c-s+1, \cdots, c;\ j = 2, \cdots, c-s \mid \underset{\sim}{\theta}_0^{(n)}]$$

where $Z_{ij} = \hat{Y}_{(i)} - \hat{Y}_{(j)}$. Applying the results of Sen [9], it follows that $n^{\frac{1}{2}}$ $(Z_{1i} - (\theta_{[1]} - \theta_{[i]}),\ i = 2, \cdots, c)$ is asymptotically normally distributed with mean zero and covariance matrix $\|\sigma_{ij}\|$ where

$$\sigma_{ij} = (A/B)^2(1+\delta_{ij}), \qquad i, j = 1, \cdots, c-1.$$

Hence it holds asymptotically that

$$P^* = (c-s)\binom{c-t}{s-t} Q_{c-1}(\underbrace{\Delta^{(n)}(B/A)(n/2)^{\frac{1}{2}}, \cdots, \Delta^{(n)}(B/A)(n/2)^{\frac{1}{2}}}_{t}, \underbrace{0, \cdots, 0}_{c-t-1}). \tag{2.18}$$

The result (2.14) follows from (2.16) and (2.18). It has been shown by Sen [9] that $1/4 \leqq \lambda(G) \leqq 7/24$ holds.

3. Comparison with the mean procedure M.

For the problem, we may define the normal theory procedure as in Bechhofer [2], say the mean procedure M. Let $\bar{X}_i = n^{-1}\sum_{\alpha=1}^{n} X_{i\alpha}$ and denote the ordered values of $\bar{X}$'s by $\bar{X}_{[1]} < \cdots < \bar{X}_{[c]}$.

The procedure M: Select the s treatments associated with $\bar{X}_{[c-s+1]}, \cdots, \bar{X}_{[c]}$.

Then the following lemma is analogous to the result of Puri and Puri [5] and hence its proof is omitted.

LEMMA 3.1. *For fixed P^*, let n be determined by*

$$\inf_{D(\Delta^{(n)})} P[CS \mid M, \underset{\sim}{\theta}^{(n)}] = P^*. \tag{3.1}$$

Then as $n \to \infty$,

$$\Delta^{(n)} = \Delta\sigma(G)n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}) \tag{3.2}$$

where $\sigma^2(G)$ is the variance of G and Δ is defined by (2.16).

The asymptotic relative efficiency of the two procedures is defined as the limiting ratio of the sample sizes to attain the same minimum probability of correct selection subject to the same condition (2.13) in both the case. The following is directly obtained from Theorem 2.1 and Lemma 3.1.

COROLLARY 3.1. *The asymptotic relative efficiency of the procedure R with respect to the mean procedure M is given by the followig*

$$e_{R,M} = e(G)c[2+(c-2)\{24\lambda(G)-6\}]^{-1} \tag{3.3}$$

where

$$e(G) = 12\sigma^2(G)\left[\int_{-\infty}^{\infty} g^2(x)dx\right]^2.$$

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