GENERALIZED CONVEXITIES OF CONTINUOUS FUNCTIONS AND THEIR APPLICATION TO MATHEMATICAL PROGRAMMING

Tagawa, Shojiro
International Institute for Advanced Study of Social Information Science

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GENERALIZED CONVEXITIES OF CONTINUOUS FUNCTIONS AND THEIR APPLICATIONS TO MATHEMATICAL PROGRAMMING

By

Shôjirô TAGAWA*

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1. Introduction.

For real valued functions defined on a finite dimensional Euclidean space, the notion of quasi-convexity was first introduced by K. J. Arrow and A. C. Enthoven [1], while the notion of pseudo-convexity by O. L. Mangasarian [3] for differentiable functions. In this paper, we shall generalize these notions to the case where the functions are defined on a linear topological space and differentiable in the sense of Neustadt [4]. We then show that some important results parallel to those obtained in [1] and [3] are valid for the functions in our concern. We shall apply these notions so as to obtain the sufficient conditions for the minimizing problem in the framework of general mathematical programming enunciated in a previous paper of the present author [5]. Our present problem treated in this paper can be considered to belong to quasi-convex programming in a generalized sense.

In Section 2 we shall investigate the differentiability in the sense of Neustadt in a linear topological space. In Section 3 we shall describe the necessary and sufficient conditions for a continuous convex function on an open convex subset of a linear topological space. In Section 4 we introduce the notions of pseudo-convexity and quasi-convexity, and investigate the interrelations among convex, pseudo-convex and quasi-convex functions when these functions are all continuous. In Section 5 we shall discuss the quasi-convex programming regarding the nonlinear programming in a Banach space stated in [5]. Some examples are given in Section 6. Example 1 given in Section 6 shows that the pseudo-convexity in our sense is truely more general than that due to Mangasarian [3]. In example 2 we remark that the norm in a real Hilbert space yields an example of the differentiable functions in the sense of Neustadt.

2. Differentiability in the sense of Neustadt.

In this section, we shall investigate some properties of differentiable functions in the sense of Neustadt.

* International Institute for Advanced Study of Social Information Science.

DEFINITION 2.1 (cf. [4]). Let X and Y be real linear topological spaces and f a mapping of X into Y. Then, f is called to be *differentiable* at $\bar{x}\in X$ *in the sense of Neustadt* if to each $x\in X$ there corresponds a vector $f_{\bar{x}}(x)\in Y$ such that

$$\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon} \xrightarrow[\substack{\varepsilon \to 0+ \\ y \to x}]{} f_{\bar{x}}(x).$$

The mapping f is called to be *differentiable* on an open domain D of X *in the sense of Neustadt* if f is differentiable at $\bar{x}$ in the sense of Neustadt for every $\bar{x}\in D$.

Note that the differential $f_{\bar{x}}(x)$ in the sense of Neustadt is necessarily positively homogeneous, i. e.,

$$f_{\bar{x}}(\lambda x)=\lambda f_{\bar{x}}(x) \qquad \text{for all } \lambda\geqq 0.$$

We now presents the properties of the differential $f_{\bar{x}}$ in the sense of Neustadt.

LEMMA 2.1. *Given linear topological spaces X, Y and Z, a mapping f of X into Y, and a mapping g of Y into Z. If f is differentiable at $\bar{x}\in X$ in the sense of Neustadt, and if g is differentiable at $\bar{y}=f(\bar{x})$ in the sense of Neustadt, then the mapping $g\circ f$ is differentiable at $\bar{x}$ in the sense of Neustadt and*

$$(g\circ f)_{\bar{x}}(x)=g_{\bar{y}}(f_{\bar{x}}(x)) \qquad \textit{for all } x\in X.$$

PROOF.

$$\begin{aligned}&\frac{(g\circ f)(\bar{x}+\varepsilon y)-(g\circ f)(\bar{x})}{\varepsilon}\\ &=\frac{g\left(\bar{y}+\varepsilon\dfrac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon}\right)-g(\bar{y})}{\varepsilon} \xrightarrow[\substack{\varepsilon \to 0+ \\ y \to x}]{} g_{\bar{y}}(f_{\bar{x}}(x))\end{aligned}$$

for every $x\in X$.

LEMMA 2.2. *Let f be a real-valued function defined over a real line R^1. If f is differentiable at $\bar{x}\in R^1$ in the sense of Neustadt, then f has right and left derivatives at $\bar{x}$. Conversely, if f has right and left derivatives at $\bar{x}\in R^1$, then f is differentiable at $\bar{x}$ in the sense of Neustadt.*

PROOF. First of all, assume that f is differentiable at $\bar{x}\in R^1$ in the sense of Neustadt, i. e.,

$$\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon} \xrightarrow[\substack{\varepsilon \to 0+ \\ y \to x}]{} f_{\bar{x}}(x) \qquad \text{for all } x\in R^1.$$

Setting $x=1$, we have

$$\frac{f(\bar{x}+\varepsilon)-f(\bar{x})}{\varepsilon} \xrightarrow[\varepsilon \to 0+]{} f_{\bar{x}}(1),$$

which means that f has right derivative at $\bar{x}$.

In the same fashion, we obtain

$$\frac{f(\bar{x}-\varepsilon)-f(\bar{x})}{\varepsilon} \xrightarrow[\varepsilon \to 0+]{} -f_{\bar{x}}(-1),$$

and hence f has left derivative at $\bar{x}$.

Conversely, suppose that f has right and left derivatives. Then,

$$\frac{f(\bar{x}+\varepsilon)-f(\bar{x})}{\varepsilon} \xrightarrow[\varepsilon \longrightarrow 0+]{} f'_+(\bar{x}),$$

$$\frac{f(\bar{x}-\varepsilon)-f(\bar{x})}{-\varepsilon} \xrightarrow[\varepsilon \longrightarrow 0+]{} f'_-(\bar{x}).$$

For each $x>0$, we have

$$\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon}$$

$$=\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon y} y \xrightarrow[\substack{\varepsilon \longrightarrow 0+ \\ y \longrightarrow x}]{} f'_+(\bar{x})x$$

since every y sufficiently close to x is positive. This implies that $f_{\bar{x}}(x)$ exists and $f_{\bar{x}}(x)=f'_+(\bar{x})x$.

In the same way, for each $x<0$, we have

$$\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon}$$

$$=\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon y} y \xrightarrow[\substack{\varepsilon \longrightarrow 0+ \\ y \longrightarrow x}]{} f'_-(\bar{x})x,$$

so that $f_{\bar{x}}(x)$ exists and

$$f_{\bar{x}}(x)=f'_-(\bar{x})x.$$

Now, consider the case where $x=0$. It is true that there is a real number δ_0, $0<\delta_0<1$, such that

$$\left|\frac{f(\bar{x}+\varepsilon)-f(\bar{x})}{\varepsilon}-f'_+(\bar{x})\right|<1 \qquad \text{whenever } 0<\varepsilon<\delta_0,$$

and

$$\left|\frac{f(\bar{x}+\varepsilon)-f(\bar{x})}{\varepsilon}-f'_-(\bar{x})\right|<1 \qquad \text{whenever } -\delta_0<\varepsilon<0.$$

Setting $M=\max\{|f'_+(\bar{x})|, |f'_-(\bar{x})|\}$, we have

$$\left|\frac{f(\bar{x}+\varepsilon)-f(\bar{x})}{\varepsilon}\right|<M+1 \qquad \text{whenever } 0<|\varepsilon|<\delta_0.$$

For an arbitrary positive number ξ, let

$$\delta=\min\left\{\frac{\xi}{M+1}, \delta_0\right\}.$$

It is then valid that

$$\left|\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon}\right|=\left|\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon y}\right||y|<(M+1)\delta \leqq \xi$$

$$\text{whenever } 0<\varepsilon<\delta,\ 0<|y|<\delta,$$

which implies that

$$\frac{f(\bar{x}+\varepsilon y)-f(\bar{x})}{\varepsilon} \xrightarrow[\substack{\varepsilon \longrightarrow 0+ \\ y \longrightarrow 0}]{} 0=f_{\bar{x}}(0) .$$

This completes the proof of Lemma 2.2.

3. Convexity.

We shall present the property of the convex functions.

DEFINITION 3.1. Let X be a linear topological space, A a convex subset of X, and f a real-valued function defined over X. Then, f is called to be convex on A if for every x_1 and x_2 in A

$$f(\lambda x_1+(1-\lambda)x_2) \leqq \lambda f(x_1)+(1-\lambda)f(x_2)$$

for every λ, $0 \leqq \lambda \leqq 1$.

PROPOSITION 3.1. *The function f is continuous convex on an open convex subset A of X if and only if*

(a) *f is differentiable on A in the sense of Neustadt,*

(b) *$f(x_2)-f(x_1) \geqq f_{x_1}(x_2-x_1)$ for all $x_1, x_2 \in A$.*

PROOF. "only if" part: This is an immediate consequence of Proposition 4.1 in [5].

"if" part: First of all, note that f is continuous at $\bar{x} \in X$ if f is differentiable at $\bar{x}$ in the sense of Neustadt (see Lemma 3.1 in [5]).

To show the contrary, assume that $f(x)$ is not convex on A. That is, there are vectors x_1, x_2 $(\neq x_1) \in A$ and a real number α, $0<\alpha<1$, such that

$$f(\alpha x_1+(1-\alpha)x_2) > \alpha f(x_1)+(1-\alpha)f(x_2) .$$

Define the function h: $[0, 1] \rightarrow R^1$ as follows:

$$h(\lambda)=f(\lambda x_2+(1-\lambda)x_1)+\lambda(f(x_1)-f(x_2)) \quad \text{for all } \lambda \in [0, 1] . \tag{3.1}$$

It is then clear that h is differentiable on $[0, 1]$ in the sense of Neustadt and

$$h(0)=h(1)=f(x_1) .$$

Furthermore, we have

$$h(1-\alpha) > f(x_1)=h(0)=h(1) .$$

Since h is continuous on $[0, 1]$, there exists a real number $\bar{\lambda}$, $0<\bar{\lambda}<1$, such that

$$h(\bar{\lambda})=\max_{\lambda \in [0,1]} h(\lambda) > h(0)=h(1) . \tag{3.2}$$

It is then valid that

$$h(\bar{\lambda})=f(\bar{x})+\bar{\lambda}(f(x_1)-f(x_2)) ,$$

where

$$\bar{x}=\bar{\lambda}x_2+(1-\bar{\lambda})x_1 .$$

Since h is differentiable on $(\bar{\lambda}, 1]$ in the sense of Neustadt, it follows from Lemma 2.2 that h has left derivative and

$$h_{\lambda}(\delta)=h'_{-}(\lambda)\delta \qquad \text{for all } \delta<0 , \tag{3.3}$$

whenever $\bar{\lambda} < \lambda \leqq 1$.

Now, we shall show that

$$h_{\delta_1}(\delta_2-\delta_1) \leqq h(\delta_2)-h(\delta_1) \qquad \text{whenever } 0<\delta_1<1,\ 0\leqq \delta_2 \leqq 1. \tag{3.4}$$

It follows from (3.1) that

$$h_{\delta_1}(\delta_2-\delta_1) = \lim_{\substack{\varepsilon\to 0+ \\ \mu\to\delta_2-\delta_1}} \Big[\frac{f(x(\delta_1)+\varepsilon\mu(x_2-x_1))-f(x(\delta_1))}{\varepsilon}+\mu(f(x_1)-f(x_2))\Big]$$

$$= f_{x(\delta_1)}((\delta_2-\delta_1)(x_2-x_1))+(\delta_2-\delta_1)(f(x_1)-f(x_2)),$$

where

$$x(\delta_1)=\delta_1 x_2+(1-\delta_1)x_1 .$$

Moreover, we have

$$(\delta_2-\delta_1)(x_2-x_1)=x(\delta_2)-x(\delta_1),$$

where $x(\delta_2)=\delta_2 x_2+(1-\delta_2)x_1$. Therefore, it follows from the condition (b) that

$$h_{\delta_1}(\delta_2-\delta_1) \leqq h(\delta_2)-h(\delta_1).$$

It is true, by virtue of (3.2), that

$$h_{\bar{\lambda}}(\delta) \leqq 0 \qquad \text{for all real numbers } \delta .$$

If we suppose that

$$h_{\bar{\lambda}}(1-\bar{\lambda})=0,$$

then it is immediate, by (3.4), that

$$0=h_{\bar{\lambda}}(1-\bar{\lambda}) \leqq h(1)-h(\bar{\lambda}),$$

which contradicts to (3.2). Hence, we have

$$h_{\bar{\lambda}}(1-\bar{\lambda})=\lim_{\varepsilon\to 0+}\frac{h(\bar{\lambda}+\varepsilon(1-\bar{\lambda}))-h(\bar{\lambda})}{\varepsilon}<0$$

It is then valid that there is a real number ε_0, $0<\varepsilon_0<1$, such that

$$h(\bar{\lambda}+\varepsilon(1-\bar{\lambda}))-h(\bar{\lambda})<0 \qquad \text{for all } \varepsilon\in(0, \varepsilon_0].$$

This can be rewritten in the form: There is a real number $\hat{\lambda}$, $\bar{\lambda}<\hat{\lambda}<1$, such that

$$h(\lambda)<h(\bar{\lambda}) \qquad \text{for all } \lambda\in(\bar{\lambda}, \hat{\lambda}].$$

Furthermore, the continuity of h allows us to take $\hat{\lambda}$, $\bar{\lambda}<\hat{\lambda}<1$, as the number satisfying

$$h(0)=h(1)<h(\lambda)<h(\bar{\lambda}) \qquad \text{whenever } \bar{\lambda}<\lambda\leqq\hat{\lambda}. \tag{3.5}$$

Recall that h has left derivative on $(\bar{\lambda}, \hat{\lambda}]$. Then, there exists a real number $\lambda_0 \in (\bar{\lambda}, \hat{\lambda}]$ satisfying

$$h'_{-}(\lambda_0)<0.$$

For if we assume that

$$h'_{-}(\lambda)\geqq 0 \qquad \text{for all } \lambda\in(\bar{\lambda}, \hat{\lambda}],$$

then it follows from Proposition 2 in Bourbaki [2; Ch. 1, § 2, N°2] that

$$h(\hat{\lambda})\geqq h(\bar{\lambda}),$$

which contradicts to (3.5).

It is then valid, by virtue of (3.4) and Lemma 2.2, that

$$0 < h'_{-}(\lambda_0)(-\lambda_0) = h_{\lambda_0}(0-\lambda_0) \leqq h(0) - h(\lambda_0),$$

which contradicts to (3.5).

Consequently, we can conclude that f is convex on A. This completes the proof of Proposition 3.1.

4. Pseudo-convexity and quasi-convexity.

We shall introduce the notions of pseudo-convexity and quasi-convexity for the functions defined on a linear topological space.

DEFINITION 4.1. Let X be a linear topological space, A a subset of X, and f a real-valued function defined on X which is differentiable on A in the sense of Neustadt. Then, f is called to be *pseudo-convex* on A if for every x_1 and x_2 in A,

$$f_{x_1}(x_2 - x_1) \geqq 0 \qquad \text{implies } f(x_2) \geqq f(x_1).$$

Now, we shall investigate the properties of pseudo-convex functions which are almost parallel to those described in [3]. Let X be a linear topological space and A a convex subset of X. Let f be a real-valued function defined over X.

PROPERTY 1. *If $f(x)$ is continuous convex on A, then $f(x)$ is pseudo-convex on A, but not conversely.*

PROOF. This is an immediate consequence of Proposition 3.1.

DEFINITION 4.2 (cf. [1]). The function f is called to be *quasi-convex* on A if for every real number λ the set

$$\{x \in A \mid f(x) \leqq \lambda\}$$

is convex.

If the real-valued function f defined over X is quasi-convex on A and if f is differentiable on A in the sense of Neustadt, then

$$f(x) \leqq f(\bar{x}) \qquad \text{implies } f_{\bar{x}}(x - \bar{x}) \leqq 0.$$

DEFINITION 4.3 (cf. [3]). Let be given a linear topological space X, a convex set A in X and a real-valued function f defined over X. The function f is called to be *strictly quasi-convex* on A if for every x_1 and x_2 in A, $x_1 \neq x_2$,

$$f(x_1) < f(x_2) \qquad \text{implies } f(\lambda x_1 + (1-\lambda)x_2) < f(x_2)$$

for every λ, $0 < \lambda < 1$.

PROPERTY 2. *If $f(x)$ is pseudo-convex on A, then $f(x)$ is strictly quasi-convex on A, but not conversely.*

PROOF. We shall prove by contradiction. That is, assume that $f(x)$ is not strictly quasi-convex on A, then it follows from the Definition 4.3 that there are vectors x_1 and x_2 ($\neq x_1$) in A and a real number λ_1, $0 < \lambda_1 < 1$, such that

$$f(x_1) < f(x_2)$$

$$f(\lambda_1 x_1 + (1-\lambda_1)x_2) \geqq f(x_2). \tag{4.1}$$

Define the function h from the interval $[0, 1]$ into a real line as follows:

$$h(\lambda)=f(\lambda x_1+(1-\lambda)x_2) \qquad \text{for all } \lambda\in[0, 1]. \tag{4.2}$$

Since the function h is continuous, there is (by (4.1)) a real number $\bar{\lambda}$ such that

$$0<\bar{\lambda}<1,$$

$$h(\bar{\lambda})=\max_{\lambda\in[0,1]} h(\lambda)\geqq f(x_2)>f(x_1). \tag{4.3}$$

It is then true, on the basis of Lemma 2.1, that

$$\begin{aligned} 0\geqq h_{\bar{\lambda}}(\delta) &= \lim_{\substack{\varepsilon\to0+\\ \mu\to\delta}} \frac{h(\bar{\lambda}+\varepsilon\mu)-h(\bar{\lambda})}{\varepsilon} \\ &= f_{\bar{x}}(\delta(x_1-x_2)) \end{aligned}$$

for every real number δ, where $\bar{x}=\bar{\lambda}x_1+(1-\bar{\lambda})x_2$. Hence we have

$$f_{\bar{x}}(x_2-\bar{x})=f_{\bar{x}}(\bar{\lambda}(x_2-x_1))\leqq 0. \tag{4.4}$$

If we assume that

$$f_{\bar{x}}(x_2-\bar{x})=0,$$

then for every $\mu\in[0, 1]$ we obtain

$$f_{\bar{x}}(\mu x_2+(1-\mu)\bar{x}-\bar{x})=\mu f_{\bar{x}}(x_2-\bar{x})=0.$$

Since f is pseudo-convex, it is immediate that

$$f(\mu x_2+(1-\mu)\bar{x})\geqq f(\bar{x}) \qquad \text{for all } \mu\in[0, 1].$$

It then follows from (4.2) and (4.3) that

$$f(\mu x_2+(1-\mu)\bar{x})=f(\bar{x}) \qquad \text{for all } \mu\in[0, 1],$$

which implies that

$$f_{x_2}(x_1-x_2)=\lim_{\varepsilon\to0+}\frac{f(x_2+\varepsilon(x_1-x_2))-f(x_2)}{\varepsilon}=0.$$

Therefore, it is valid, on the basis of pseudo-convexity of f, that

$$f(x_1)\geqq f(x_2),$$

which contradicts to (4.3).

Consequently, it follows from (4.4) that

$$f_{\bar{x}}(x_2-\bar{x})<0. \tag{4.5}$$

It is easy to verify (by (4.5)) that there is a real number ε_0, $0<\varepsilon_0<1$, such that

$$f(\bar{x}+\varepsilon(x_2-\bar{x}))<f(\bar{x}) \qquad \text{whenever } 0<\varepsilon<\varepsilon_0.$$

Since f is continuous, there exists a real number ε_1, $0<\varepsilon_1<\varepsilon_0$, such that

$$f(x_1)<f(\bar{x}+\varepsilon(x_2-\bar{x}))<f(\bar{x}) \qquad \text{whenever } 0<\varepsilon<\varepsilon_1,$$

or equivalently, there exists (see (4.2)) a real number λ_2, $0<\lambda_2<\bar{\lambda}$, such that

$$h(1)<h(\lambda)<h(\bar{\lambda}) \qquad \text{whenever } \lambda_2\leqq\lambda<\bar{\lambda}. \tag{4.6}$$

Since $h(\lambda)$ is differentiable on $(0, 1)$ in the sense of Neustadt, it follows from Lemma 2.2 that h has right derivative and that

$$h'_+(\lambda)\mu = h_\lambda(\mu) \qquad \text{for all } \mu > 0 .$$

We now show that there is a real number λ_3, $\lambda_2 \leqq \lambda_3 < \bar{\lambda}$, such that

$$h'_+(\lambda_3) > 0 .$$

To show the contrary, assume that

$$h'_+(\lambda) \leqq 0 \qquad \text{for all } \lambda \in [\lambda_2, \bar{\lambda}) .$$

It then follows from Proposition 2 in [2; Ch. 1, § 2, N°2] that

$$h(\lambda_2) \geqq h(\bar{\lambda}) ,$$

which contradicts to (4.6).

Therefore, it is clear that

$$f_{x_3}(\delta(x_1 - x_2)) = h_{\lambda_3}(\delta) = h'_+(\lambda_3)\delta > 0 \qquad \text{for all } \delta > 0 ,$$

where $x_3 = \lambda_3 x_1 + (1-\lambda_3)x_2$. Since $x_1 - x_3 = (1-\lambda_3)(x_1 - x_2)$, we have

$$f_{x_3}(x_1 - x_3) = f_{x_3}((1-\lambda_3)(x_1 - x_2)) > 0 ,$$

which together with the pseudo-convexity of f imply that

$$h(1) = f(x_1) \geqq f(x_3) = h(\lambda_3) .$$

This contradicts to (4.6).

Consequently, we can conclude that (4.5) does not hold, and hence f is strictly quasi-convex on A. This completes the proof of Property 2.

PROPERTY 3 (cf. [3]). *If f is continuous, strictly quasi-convex on A, then f is quasi-convex on A.*

PROPERTY 4 (cf. [3]). *If f is strictly quasi-convex on A, then every local minimum is a global minimum.*

PROPERTY 5. *Let $f(x)$ be pseudo-convex on A (probably not convex only in this case). If*

$$f_{\bar{x}}(x - \bar{x}) \geqq 0 \qquad \textit{for all } x \in A ,$$

then $\bar{x}$ is a global minimum over A.

It follows from Properties 1-3 that there is a hierarchy among differentiable functions in the sense of Neustadt. More precisely, if we let F_1, F_2, F_3, and F_4 represent the sets of all differentiable functions in the sense of Neustadt defined on a convex set A in a linear topological space that are convex, pseudoconvex, strictly quasi-convex, and quasi-convex, respectively, then

$$F_1 \subset F_2 \subset F_3 \subset F_4 .$$

5. Quasi-convex programming.

In this section, we shall present the programming in which the conditions described in [5] are sufficient for optimality.

DEFINITION 5.1 (cf. [5]). Let X be a linear topological space and A a subset of

X. By the local closed convex cone of A at $\bar{x}$ ($\in A$) we mean the set

$$\mathrm{P}(A, \bar{x}) = \bigcap_{N \in \mathfrak{N}(\bar{x})} \mathrm{CC}(A \cap N, \bar{x}),$$

where $\mathfrak{N}(\bar{x})$ is the class of all neighborhoods of $\bar{x}$ and $\mathrm{CC}(A \cap N, \bar{x})$ is the intersection of all closed convex cones containing the set $A - \bar{x} = \{a - \bar{x} \mid a \in A\}$.

DEFINITION 5.2 (cf. [6]). By the local closed cone of A at $\bar{x}$ we mean the set

$$\mathrm{LC}(A, \bar{x}) = \bigcap_{N \in \mathfrak{N}(\bar{x})} \mathrm{C}(A \cap N, \bar{x}),$$

where $C(A \cap N, \bar{x})$ is the intersection of all closed cones containing the set $A - \bar{x}$.

DEFINITION 5.3 (cf. [5]). The set A in X is called a *pseudo-cone* with vertex at $\bar{x}$ ($\in A$) if

$$x - \bar{x} \in LC(A, \bar{x}) \qquad \text{for all } x \in A.$$

Before we discuss the quasi-convex programming, we shall generalize the notion of quasi-convexity.

DEFINITION 5.4. Given linear topological spaces X and Y, a convex set A in X, a closed convex cone* B in Y, and a mapping g of X into Y. Then, g is called to be *B-quasi-convex* on A if for each vector $y \in Y$ the set

$$\{x \in A \mid g(x) \in y + B\}$$

is convex.

LEMMA 5.1. *Let X be a linear topological space, R^n an n-Euclidean space, A a convex set in X and $g = (g_1, \cdots, g_n)$ a mapping of X into R^n. If all g_i's are quasi-convex on A, then g is B-quasi-convex on A, where B is given by*

$$B = \{y = (y_1, \cdots, y_n) \in R^n \mid y_i \leqq 0,\ i = 1, \cdots, n\}.$$

PROOF. For any vector $y = (y_1, \cdots, y_n) \in R^n$ the sets

$$\{x \in A \mid g_i(x) \leqq y_i\} \qquad \text{for } i = 1, \cdots, n$$

are all convex since g_i's are quasi-convex. Then, the set

$$\{x \in A \mid g(x) \in y + B\} = \{x \in A \mid g_i(x) \leqq y_i,\ i = 1, \cdots, n\}$$
$$= \bigcap_{i=1}^{n} \{x \in A \mid g_i(x) \leqq y_i\}$$

is convex. Hence g is B-quasi-convex on A.

This lemma shows that the B-quasi-convexity is a natural extension of the quasi-convexity.

THEOREM 5.1. *Let X and Y be real Banach spaces, A a convex subset of X, and B a closed convex cone in Y. Let a real-valued function f defined over X be pseudo-convex on A and let a differentiable mapping g of X into Y in the sense of Neustadt be B-quasi-convex on A. If there exist a vector $\bar{x} \in A$, a real number $\eta \neq 0$, and a linear continuous functional $\bar{y}^* \in Y^*$ such that*

$$\eta \geqq 0, \tag{5.1}$$

* A subset B of a linear space is called a cone if $\alpha B \subset B$ for all $\alpha \geqq 0$.

(5.2) $$\bar{y}^*(y) \leqq 0 \quad \textit{for all } y \in B,$$

(5.3) $$\bar{y}^*(g(\bar{x})) = 0,$$

(5.4) $$g(\bar{x}) \in B,$$

(5.5) $$\eta f_{\bar{x}}(x) + \bar{y}^*(g_{\bar{x}}(x)) \geqq 0 \quad \textit{for all } x \in P(A, \bar{x}),$$

then

$$f(\bar{x}) = \min\{f(x) \mid x \in A,\ g(x) \in B\}.$$

PROOF. Define the set D in X as follows:

$$D = \{x \in A \mid g(x) \in B\}.$$

It is then clear, by virtue of (5.4), that $\bar{x} \in D$. Since g is B-quasi-convex on A, the set D is convex, and hence it is a pseudo-cone with vertex at $\bar{x}$. Since f is pseudo-convex on A, it is pseudo-convex on D. Consequently, it follows, on the basis of Theorem 3.4 in [5], that the conditions (5.1)-(5.5) are sufficient for minimality of $f(\bar{x})$ on D. This completes the proof of Theorem 5.1.

Now, we can state, by virtue of Lemma 5.1, the corollary of Theorem 5.1.

COROLLARY. *Let X be a real Banach space, and A a convex subset of X. Let a real-valued function f defined on X be pseudo-convex on A, and real-valued functions $g_1, \cdots, g_n$ differentiable in the sense of Neustadt and quasi-convex on A. If there exist a vector $\bar{x} \in A$, a real number $\eta \neq 0$, and a vector $\zeta = (\zeta_1, \cdots, \zeta_n) \in R^n$ such that*

(5.6) $$\eta \geqq 0,$$

(5.7) $$\zeta_i \geqq 0 \quad \textit{for } i = 1, \cdots, n,$$

(5.8) $$\zeta_1 g_1(\bar{x}) + \cdots + \zeta_n g_n(\bar{x}) = 0,$$

(5.9) $$g_i(\bar{x}) \leqq 0 \quad \textit{for } i = 1, \cdots, n,$$

(5.10) $$\eta f_{\bar{x}}(x) + \zeta_1 g_{1\bar{x}}(x) + \cdots + \zeta_n g_{n\bar{x}}(x) \geqq 0 \quad \textit{for all } x \in P(A, \bar{x}),$$

then

$$f(\bar{x}) = \min\{f(x) \mid x \in A,\ g_i(x) \leqq 0 \quad \textit{for } i = 1, \cdots, n\}.$$

PROOF. Let Y be an Eucledian space R^n. If we define the mapping g of X into Y and the closed convex cone B in Y as follows:

$$g(x) = (g_1(x), \cdots, g_n(x)) \quad \text{for all } x \in X,$$

$$B = \{y = (y_1, \cdots, y_n) \in R^n \mid y_i \leqq 0 \quad \text{for } i = 1, \cdots, n\},$$

then it is easily verified, on the basis of Theorem 5.1, that the assertion of the corollary holds.

6. Remarks.

Our concept of pseudo-convexity is an extension of the pseudo-convexity described in [3]. For instance, consider the following examples.

EXAMPLE 1. Let $f(x)$ be the function from R^1 into R^1 defined by

$$f(x)=\begin{cases} x^2 & \text{if } x\leqq 0 \\ \sqrt{x+1}-1 & \text{if } x>0. \end{cases}$$

Then, $f(x)$ is not differentiable at 0 in the usual sense, but differentiable at 0 in the sense of Neustadt. Therefore, $f(x)$ is not pseudo-convex in the sence of Mangasarian, but pseudo-convex in our sense.

EXAMPLE 2. As more simple, but important example, we can consider the norm in a real Hilbert space X, i. e., the norm $\|x\|$ is defined by

$$\|x\|=(x\,|\,x)^{1/2} \qquad \text{for all } x\in X,$$

where $(x\,|\,y)$ is an inner product in X. Then, the real valued function $f(x)=\|x\|$ is differentiable on X in the sense of Neustadt, and the differential $f_{\bar{x}}$ is:

$$f_{\bar{x}}(x)=\begin{cases} \|x\| & \text{if } \bar{x}=0, \\ \dfrac{(\bar{x}\,|\,x)}{\|\bar{x}\|} & \text{if } \bar{x}\neq 0. \end{cases}$$

It is clear that the above function $f(x)$ is not differentiable at $\bar{x}=0$ in the usual sense.

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