A SIMPLE TIGHTNESS CONDITION FOR RANDOM ELEMENTS ON $ C([0,1]^2) $

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A SIMPLE TIGHTNESS CONDITION FOR RANDOM ELEMENTS ON $C([0, 1]^2)$

By

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§1. Based on moments of variations of random elements on $C([0, 1]^2)$, a simple sufficient condition for tightness** is considered.

§2. Let $C_2 \equiv C([0, 1]^2)$ be the set of all real valued continuous functions on $[0, 1]^2$ with the uniform topology.

W. J. Park [2], [3] considered random elements on C_2 to prove the existence of Wiener measure and invariance principle on C_2.

W. J. Park's sufficient condition for tightness (lemma 3 in [3]) is not necessarily easy to apply directly.

The object of this paper is to give a handy sufficient condition for tightness of random elements on C_2.

We consider random elements $\{Z_n(t, s),\ 0 \leqq t,\ s \leqq 1\}\ n \geqq 1$ satisfying:

$$Z_n(t, 0) = Z_n(0, s) = 0\,, \qquad 0 \leqq t\,,\ s \leqq 1\,, \quad w.pr.\ 1\,, \tag{1}$$

$$E\{Z_n(t, s)\} = 0\,, \qquad 0 \leqq t\,,\ s \leqq 1\,. \tag{2}$$

For intervals $A = [t', t]$ and $B = [s', s]$ in $I = [0, 1]$, we denote a variation of $Z_n(\cdot, \cdot)$ on $A \times B$ by $Z_n(A, B) \equiv Z_n(t, s) - Z_n(t', s) - Z_n(t, s') + Z_n(t', s')$ and for a function $G(\cdot)$ on I, we put $G(A) \equiv G(t) - G(t')$.

THEOREM. *The sequence of random elements* $\{Z_n(t, s),\ 0 \leqq t,\ s \leqq 1\}\ n \geqq 1$ *on* C_2 *is tight if for some real* $\gamma > 0$ *and* $\alpha > 1$,

$$E\{|Z_n(A, B)|^\gamma\} \leqq K_0\{G_1(A) \cdot G_2(B)\}^\alpha\,, \tag{3}$$

holds for any intervals A and B in I, where K_0 is a finite constant independent of n, A and B, and $G_i(\cdot)$, $i = 1, 2$ are continuous monotone non-decreasing functions on I.

PROOF. For a function $x \in C_2$, we write

$$\omega(x\,;\delta) \equiv \sup_{\substack{|t-t'| \leqq \delta \\ |s-s'| \leqq \delta}} |x(t, s) - x(t', s')|\,.$$

As was shown in lemma 2 of [3], the necessary and sufficient conditions for $\{Z_n(\cdot, \cdot)\}\, n \geqq 1$ to be tight are followings:

(4) For each positive η, there exists an a such that

$$P(|Z_n(0, 0)| > a) \leqq \eta\,, \qquad n \geqq 1\,.$$

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** See [1], pp. 37 and 40.

(5) For each positive ε and η, there exist a δ, with $0<\delta<1$, and an n_0 such that

$$P(\omega(Z_n\,;\,\delta)\geqq\varepsilon)\leqq\eta\,,\qquad n\geqq n_0\,.$$

Since $Z_n(0,0)=0$ w. pr. 1, we only need to prove that the condition (3) implies the condition (5).

At first, by dividing I into $1+[\delta^{-1}]$ subintervals of length δ, except the last one, where $[\delta^{-1}]$ is the largest integer not greater than δ^{-1}, we see that

$$\{\omega(Z_n:\delta)\geqq\varepsilon\}\subset\Big\{\sup_{j_1,j_2<\delta^{-1}}\ \sup_{\substack{j_1\delta\leqq t\leqq j_1\delta+\delta\\ j_2\delta\leqq s\leqq j_2\delta+\delta}}|Z_n(t,s)-Z_n(j_1\delta,j_2\delta)|\geqq\frac{\varepsilon}{4}\Big\}\,.$$

It is easily seen that for each (j_1,j_2),

$$\begin{aligned}\sup_{\substack{j_1\delta\leqq t\leqq j_1\delta+\delta\\ j_2\delta\leqq s\leqq j_2\delta+\delta}}|Z_n(t,s)-Z_n(j_1\delta,j_2\delta)|&\leqq\sup_{\substack{j_1\delta\leqq t\leqq j_1\delta+\delta\\ j_2\delta\leqq s\leqq j_2\delta+\delta}}|Z_n([j_1\delta,t],[j_2\delta,s])|\\&+\sup_{j_1\delta\leqq t\leqq j_1\delta+\delta}\Big|\sum_{l=1}^{j_2}Z_n([j_1\delta,t],[(l-1)\delta,l\delta])\Big|\\&+\sup_{j_2\delta\leqq s\leqq j_2\delta+\delta}\Big|\sum_{l=1}^{j_1}Z_n([(l-1)\delta,l\delta],[j_2\delta,s])\Big|.\end{aligned}$$

Hence, we have

$$\begin{aligned}(6)\qquad &P(\omega(Z_n:\delta)\geqq\varepsilon)\\&\leqq P\Big(\bigcup_{j_1<\delta^{-1}}\bigcup_{j_2<\delta^{-1}}\sup_{\substack{j_1\delta\leqq t\leqq j_1\delta+\delta\\ j_2\delta\leqq s\leqq j_2\delta+\delta}}|Z_n([j_1\delta,t],[j_2\delta,s])|\geqq\frac{\varepsilon}{12}\Big)\\&+P\Big(\bigcup_{j_1<\delta^{-1}}\sup_{j_2<\delta^{-1}}\sup_{j_1\delta\leqq t\leqq j_1\delta+\delta}\Big|\sum_{l=1}^{j_2}Z_n([j_1\delta,t],[(l-1)\delta,l\delta])\Big|\geqq\frac{\varepsilon}{12}\Big)\\&+P\Big(\bigcup_{j_2<\delta^{-1}}\sup_{j_1<\delta^{-1}}\sup_{j_2\delta\leqq s\leqq j_2\delta+\delta}\Big|\sum_{l=1}^{j_1}Z_n([(l-1)\delta,l\delta],[j_2\delta,s])\Big|\geqq\frac{\varepsilon}{12}\Big)\,.\end{aligned}$$

By dividing further the intervals $[j_1\delta,j_1\delta+\delta]$ and $[j_2\delta,j_2\delta+\delta]$ into m-intervals of length δ/m, we have

$$\begin{aligned}(7)\qquad &P\Big(\bigcup_{\substack{j_1<\delta^{-1}\\ j_2<\delta^{-1}}}\sup_{\substack{j_1\delta\leqq t\leqq j_1\delta+\delta\\ j_2\delta\leqq s\leqq j_2\delta+\delta}}|Z_n([j_1\delta,t],[j_2\delta,s])|\geqq\frac{\varepsilon}{12}\Big)\\&\leqq\sum_{j_1<\delta^{-1}}\sum_{j_2<\delta^{-1}}\lim_{m\to\infty}P\Big(\sup_{1\leqq i_1,i_2\leqq m}\Big|\sum_{l=1}^{i_1}\sum_{p=1}^{i_2}\xi_{lp}^{(n)}\Big|\geqq\frac{\varepsilon}{12}\Big)\end{aligned}$$

where we put

$$\xi_{lp}^{(n)}=Z_n\Big(\Big[j_1\delta+\frac{l-1}{m}\delta,\ j_1\delta+\frac{l}{m}\delta\Big],\Big[j_2\delta+\frac{p-1}{m}\delta,\ j_2\delta+\frac{p}{m}\delta\Big]\Big)\,,$$

and also we have

$$\begin{aligned}(8)\qquad &P\Big(\bigcup_{j_1<\delta^{-1}}\sup_{j_2<\delta^{-1}}\sup_{j_1\delta\leqq t\leqq j_1\delta+\delta}\Big|\sum_{l=1}^{j_2}Z_n([j_1\delta,t],[(l-1)\delta,l\delta])\Big|\geqq\frac{\varepsilon}{12}\Big)\\&\leqq\sum_{j_1<\delta^{-1}}\lim_{m\to\infty}P\Big(\sup_{j_2<\delta^{-1}}\sup_{1\leqq i_1\leqq m}\Big|\sum_{l=p}^{i_1}\sum_{p=1}^{j_2}\hat{\xi}_{lp}^{(n)}\Big|\leqq\frac{\varepsilon}{12}\Big)\end{aligned}$$

where

$$\hat{\xi}_{lp}^{(n)} = Z_n\Big(\Big[j_1\delta + \frac{l-1}{m}\delta,\ j_1\delta + \frac{l}{m}\delta\Big], [(p-1)\delta, p\delta]\Big).$$

The third term of the R. H. S. of (6) is quite analogous to (8).

By noting that from (3), we have

$$E\Big\{\Big|\sum_{l=1}^{i_1}\sum_{p=1}^{i_2}\xi_{lp}^{(n)}\Big|^{\gamma}\Big\} \leqq K_0\Big\{\sum_{l=1}^{i_1}\sum_{p=1}^{i_2} G_{lp}\Big\}^{\alpha},$$

where

$$G_{lp} = G_1\Big(\Big[j_1\delta + \frac{l-1}{m}\delta,\ j_1\delta + \frac{l}{m}\delta\Big] \times G_2\Big(\Big[j_2\delta + \frac{p-1}{m}\delta,\ j_2\delta + \frac{p}{m}\delta\Big]\Big) \geqq 0,$$

and also

$$E\Big\{\Big|\sum_{l=1}^{i_1}\sum_{p=1}^{j_2}\hat{\xi}_{lp}^{(n)}\Big|^{\gamma}\Big\} \leqq K_0\Big\{\sum_{l=1}^{i_1}\sum_{p=1}^{j_2}\hat{G}_{lp}\Big\}^{\alpha},$$

where

$$\hat{G}_{lp} = G_1\Big(\Big[j_1\delta + \frac{l-1}{m}\delta,\ j_1\delta + \frac{l}{m}\delta\Big]\Big) \times G_2([(p-1)\delta, p\delta]) \geqq 0,$$

we can evaluate the probabilities (7) and (8) by making use of theorem 12.2 in [1].

Thus, we have

$$\begin{aligned}
(9)\qquad & \sum_{j_1<\delta^{-1}}\sum_{j_2<\delta^{-1}}\lim_{m\to\infty} P\Big(\sup_{1\leqq i_1, i_2\leqq m}\Big|\sum_{l=1}^{i_1}\sum_{p=1}^{i_2}\xi_{lp}^{(n)}\Big| \geqq \frac{\varepsilon}{12}\Big) \\
& \leqq \sum_{j_1<\delta^{-1}}\sum_{j_2<\delta^{-1}} K_0^*\Big\{\sum_{l=1}^{m}\sum_{p=1}^{m} G_{lp}\Big\}^{\alpha} \\
& = K_0^* \sum_{j_1<\delta^{-1}}\sum_{j_2<\delta^{-1}} G_1([j_1\delta, j_1\delta+\delta] \times G_2([j_2\delta, j_2\delta+\delta])\}^{\alpha} \\
& \leqq K_0^{**}\cdot \max_{j_1, j_2<\delta^{-1}} \{G_1([j_1\delta, j_1\delta+\delta]) \times G_2([j_2\delta, j_2\delta+\delta])\}^{\alpha-1}.
\end{aligned}$$

where K_0^* and K_0^{**} are finite constants independent of n.

Similarly, we have

$$\begin{aligned}
(10)\qquad & \sum_{j_1<\delta^{-1}}\lim_{m\to\infty} P\Big(\sup_{j_2<\delta^{-1}}\sup_{1\leqq i_1\leqq m}\Big|\sum_{l=1}^{i_1}\sum_{p=1}^{j_2}\hat{\xi}_{lp}^{(n)}\Big| \geqq \frac{\varepsilon}{12}\Big) \\
& \leqq K_1^* \sum_{j_1<\delta^{-1}}\Big\{\sum_{l=1}^{m}\sum_{p=1}^{1+[\delta^{-1}]}\hat{G}_{lp}\Big\}^{\alpha} \\
& \leqq K_1^{**} \max_{j_1<\delta^{-1}} G_1\{[j_1\delta, j_1\delta+\delta])\}^{\alpha-1},
\end{aligned}$$

where K_1^* and K_1^{**} are finite constants independent of n.

By choosing sufficiently small δ, $0<\delta<1$, we can make $\max_{j_i<\delta^{-1}}\{G_i([j_i\delta, j_i\delta+\delta])\}^{\alpha-1}$ $i=1, 2$, arbitrarily small, that is, for each $\varepsilon>0$ and $\eta>0$, we can find δ, $0<\delta<1$ such that

$$P(\omega(Z_n;\ \delta) \geqq \varepsilon) \leqq \eta \qquad n \geqq 1.$$

Thus, we have proved the condition (5). Q. E. D.

References

[1] P. BILLINGSLEY, *Convergence of probability measures,* Wiley, (1968), New York.
[2] W. J. PARK, *A multi-parameter Gaussian process,* Ann. Math. Statist., 41 (1970), 1582-1595.
[3] W. J. PARK, *Weak convergence of Probability measures on the function Space* $C([0, 1]^2)$. J. Multi. Analysis, 1 (1971), 433-444.