A SIMPLE TIGHTNESS CONDITION FOR RANDOM ELEMENTS ON $C([0, 1]^2)$

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A SIMPLE TIGHTNESS CONDITION FOR RANDOM ELEMENTS ON $C([0, 1]^2)$

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§ 1. Based on moments of variations of random elements on $C([0, 1]^2)$, a simple sufficient condition for tightness** is considered.

§ 2. Let $C_2 \equiv C([0, 1]^2)$ be the set of all real valued continuous functions on $[0, 1]^2$ with the uniform topology.

W. J. Park [2], [3] considered random elements on C_2 to prove the existence of Wiener measure and invariance principle on C_2 .

W. J. Park's sufficient condition for tightness (lemma 3 in [3]) is not necessarily easy to apply directly.

The object of this paper is to give a handy sufficient condition for tightness of random elements on C_2 .

We consider random elements $\{Z_n(t, s), 0 \leq t, s \leq 1\}$ $n \geq 1$ satisfying:

$$(1) \quad Z_n(t, 0) = Z_n(0, s) = 0, \quad 0 \leq t, s \leq 1, \quad w. pr. 1,$$

$$(2) \quad E\{Z_n(t, s)\} = 0, \quad 0 \leq t, s \leq 1.$$

For intervals $A = [t', t]$ and $B = [s', s]$ in $I = [0, 1]$, we denote a variation of $Z_n(\cdot, \cdot)$ on $A \times B$ by $Z_n(A, B) \equiv Z_n(t, s) - Z_n(t', s) - Z_n(t, s') + Z_n(t', s')$ and for a function $G(\cdot)$ on I , we put $G(A) \equiv G(t) - G(t')$.

THEOREM. The sequence of random elements $\{Z_n(t, s), 0 \leq t, s \leq 1\}$ $n \geq 1$ on C_2 is tight if for some real $\gamma > 0$ and $\alpha > 1$,

$$(3) \quad E\{|Z_n(A, B)|^\gamma\} \leq K_0 \{G_1(A) \cdot G_2(B)\}^\alpha,$$

holds for any intervals A and B in I , where K_0 is a finite constant independent of n , A and B , and $G_i(\cdot)$, $i=1, 2$ are continuous monotone non-decreasing functions on I .

PROOF. For a function $x \in C_2$, we write

$$\omega(x; \delta) \equiv \sup_{\substack{|t-t'| \leq \delta \\ |s-s'| \leq \delta}} |x(t, s) - x(t', s')|.$$

As was shown in lemma 2 of [3], the necessary and sufficient conditions for $\{Z_n(\cdot, \cdot)\}$ $n \geq 1$ to be tight are followings:

(4) For each positive η , there exists an a such that

$$P(|Z_n(0, 0)| > a) \leq \eta, \quad n \geq 1.$$

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** See [1], pp. 37 and 40.

(5) For each positive ε and η , there exist a δ , with $0 < \delta < 1$, and an n_0 such that

$$P(\omega(Z_n; \delta) \geq \varepsilon) \leq \eta, \quad n \geq n_0.$$

Since $Z_n(0, 0) = 0$ w. pr. 1, we only need to prove that the condition (3) implies the condition (5).

At first, by dividing I into $1 + [\delta^{-1}]$ subintervals of length δ , except the last one, where $[\delta^{-1}]$ is the largest integer not greater than δ^{-1} , we see that

$$\{\omega(Z_n; \delta) \geq \varepsilon\} \subset \left\{ \sup_{j_1, j_2 < \delta^{-1}} \sup_{\substack{j_1 \delta \leq t \leq j_1 \delta + \delta \\ j_2 \delta \leq s \leq j_2 \delta + \delta}} |Z_n(t, s) - Z_n(j_1 \delta, j_2 \delta)| \geq \frac{\varepsilon}{4} \right\}.$$

It is easily seen that for each (j_1, j_2) ,

$$\begin{aligned} \sup_{\substack{j_1 \delta \leq t \leq j_1 \delta + \delta \\ j_2 \delta \leq s \leq j_2 \delta + \delta}} |Z_n(t, s) - Z_n(j_1 \delta, j_2 \delta)| &\leq \sup_{\substack{j_1 \delta \leq t \leq j_1 \delta + \delta \\ j_2 \delta \leq s \leq j_2 \delta + \delta}} |Z_n([j_1 \delta, t], [j_2 \delta, s])| \\ &\quad + \sup_{j_1 \delta \leq t \leq j_1 \delta + \delta} \left| \sum_{l=1}^{j_2} Z_n([j_1 \delta, t], [(l-1)\delta, l\delta]) \right| \\ &\quad + \sup_{j_2 \delta \leq s \leq j_2 \delta + \delta} \left| \sum_{l=1}^{j_1} Z_n([(l-1)\delta, l\delta], [j_2 \delta, s]) \right|. \end{aligned}$$

Hence, we have

$$\begin{aligned} (6) \quad P(\omega(Z_n; \delta) \geq \varepsilon) &\leq P\left(\bigcup_{j_1 < \delta^{-1}} \bigcup_{j_2 < \delta^{-1}} \sup_{\substack{j_1 \delta \leq t \leq j_1 \delta + \delta \\ j_2 \delta \leq s \leq j_2 \delta + \delta}} |Z_n([j_1 \delta, t], [j_2 \delta, s])| \geq \frac{\varepsilon}{12} \right) \\ &\quad + P\left(\bigcup_{j_1 < \delta^{-1}} \sup_{j_2 < \delta^{-1}} \sup_{j_1 \delta \leq t \leq j_1 \delta + \delta} \left| \sum_{l=1}^{j_2} Z_n([j_1 \delta, t], [(l-1)\delta, l\delta]) \right| \geq \frac{\varepsilon}{12} \right) \\ &\quad + P\left(\bigcup_{j_2 < \delta^{-1}} \sup_{j_1 < \delta^{-1}} \sup_{j_2 \delta \leq s \leq j_2 \delta + \delta} \left| \sum_{l=1}^{j_1} Z_n([(l-1)\delta, l\delta], [j_2 \delta, s]) \right| \geq \frac{\varepsilon}{12} \right). \end{aligned}$$

By dividing further the intervals $[j_1 \delta, j_1 \delta + \delta]$ and $[j_2 \delta, j_2 \delta + \delta]$ into m -intervals of length δ/m , we have

$$\begin{aligned} (7) \quad P\left(\bigcup_{\substack{j_1 < \delta^{-1} \\ j_2 < \delta^{-1}}} \sup_{\substack{j_1 \delta \leq t \leq j_1 \delta + \delta \\ j_2 \delta \leq s \leq j_2 \delta + \delta}} |Z_n([j_1 \delta, t], [j_2 \delta, s])| \geq \frac{\varepsilon}{12} \right) \\ \leq \sum_{j_1 < \delta^{-1}} \sum_{j_2 < \delta^{-1}} \lim_{m \rightarrow \infty} P\left(\sup_{1 \leq i_1, i_2 \leq m} \left| \sum_{l=1}^{i_1} \sum_{p=1}^{i_2} \xi_{lp}^{(n)} \right| \geq \frac{\varepsilon}{12} \right) \end{aligned}$$

where we put

$$\xi_{lp}^{(n)} = Z_n\left(\left[j_1 \delta + \frac{l-1}{m} \delta, j_1 \delta + \frac{l}{m} \delta\right], \left[j_2 \delta + \frac{p-1}{m} \delta, j_2 \delta + \frac{p}{m} \delta\right]\right),$$

and also we have

$$\begin{aligned} (8) \quad P\left(\bigcup_{j_1 < \delta^{-1}} \sup_{j_2 < \delta^{-1}} \sup_{j_1 \delta \leq t \leq j_1 \delta + \delta} \left| \sum_{l=1}^{j_2} Z_n([j_1 \delta, t], [(l-1)\delta, l\delta]) \right| \geq \frac{\varepsilon}{12} \right) \\ \leq \sum_{j_1 < \delta^{-1}} \lim_{m \rightarrow \infty} P\left(\sup_{j_2 < \delta^{-1}} \sup_{1 \leq i_1 \leq m} \left| \sum_{l=p}^{i_1} \sum_{p=1}^{j_2} \hat{\xi}_{lp}^{(n)} \right| \geq \frac{\varepsilon}{12} \right) \end{aligned}$$

where

$$\hat{\xi}_{lp}^{(n)} = Z_n \left(\left[j_1 \delta + \frac{l-1}{m} \delta, j_1 \delta + \frac{l}{m} \delta \right], [(p-1)\delta, p\delta] \right).$$

The third term of the R. H. S. of (6) is quite analogous to (8).

By noting that from (3), we have

$$E \left\{ \left| \sum_{l=1}^{i_1} \sum_{p=1}^{i_2} \hat{\xi}_{lp}^{(n)} \right|^r \right\} \leq K_0 \left\{ \sum_{l=1}^{i_1} \sum_{p=1}^{i_2} G_{lp} \right\}^\alpha,$$

where

$$G_{lp} = G_1 \left(\left[j_1 \delta + \frac{l-1}{m} \delta, j_1 \delta + \frac{l}{m} \delta \right] \times G_2 \left(\left[j_2 \delta + \frac{p-1}{m} \delta, j_2 \delta + \frac{p}{m} \delta \right] \right) \geq 0,$$

and also

$$E \left\{ \left| \sum_{l=1}^{i_1} \sum_{p=1}^{j_2} \hat{\xi}_{lp}^{(n)} \right|^r \right\} \leq K_0 \left\{ \sum_{l=1}^{i_1} \sum_{p=1}^{j_2} \hat{G}_{lp} \right\}^\alpha,$$

where

$$\hat{G}_{lp} = G_1 \left(\left[j_1 \delta + \frac{l-1}{m} \delta, j_1 \delta + \frac{l}{m} \delta \right] \right) \times G_2([(p-1)\delta, p\delta]) \geq 0,$$

we can evaluate the probabilities (7) and (8) by making use of theorem 12.2 in [1].

Thus, we have

$$\begin{aligned} (9) \quad & \sum_{j_1 < \delta^{-1}} \sum_{j_2 < \delta^{-1}} \lim_{m \rightarrow \infty} P \left(\sup_{1 \leq i_1, i_2 \leq m} \left| \sum_{l=1}^{i_1} \sum_{p=1}^{i_2} \hat{\xi}_{lp}^{(n)} \right| \geq \frac{\varepsilon}{12} \right) \\ & \leq \sum_{j_1 < \delta^{-1}} \sum_{j_2 < \delta^{-1}} K_0^* \left\{ \sum_{l=1}^m \sum_{p=1}^m G_{lp} \right\}^\alpha \\ & = K_0^* \sum_{j_1 < \delta^{-1}} \sum_{j_2 < \delta^{-1}} G_1([j_1 \delta, j_1 \delta + \delta]) \times G_2([j_2 \delta, j_2 \delta + \delta])^\alpha \\ & \leq K_0^{**} \cdot \max_{j_1, j_2 < \delta^{-1}} \{G_1([j_1 \delta, j_1 \delta + \delta]) \times G_2([j_2 \delta, j_2 \delta + \delta])\}^{\alpha-1}. \end{aligned}$$

where K_0^* and K_0^{**} are finite constants independent of n .

Similarly, we have

$$\begin{aligned} (10) \quad & \sum_{j_1 < \delta^{-1}} \lim_{m \rightarrow \infty} P \left(\sup_{j_2 < \delta^{-1}} \sup_{1 \leq i_1 \leq m} \left| \sum_{l=1}^{i_1} \sum_{p=1}^{j_2} \hat{\xi}_{lp}^{(n)} \right| \geq \frac{\varepsilon}{12} \right) \\ & \leq K_1^* \sum_{j_1 < \delta^{-1}} \left\{ \sum_{l=1}^m \sum_{p=1}^{1+[\delta^{-1}]} \hat{G}_{lp} \right\}^\alpha \\ & \leq K_1^{**} \max_{j_1 < \delta^{-1}} G_1([j_1 \delta, j_1 \delta + \delta])^{\alpha-1}, \end{aligned}$$

where K_1^* and K_1^{**} are finite constants independent of n .

By choosing sufficiently small δ , $0 < \delta < 1$, we can make $\max_{j_i < \delta^{-1}} \{G_i([j_i \delta, j_i \delta + \delta])\}^{\alpha-1}$ $i=1, 2$, arbitrarily small, that is, for each $\varepsilon > 0$ and $\eta > 0$, we can find δ , $0 < \delta < 1$ such that

$$P(\omega(Z_n; \delta) \geq \varepsilon) \leq \eta \quad n \geq 1.$$

Thus, we have proved the condition (5).

Q. E. D.

References

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