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SOME STATISTICAL PROPERTIES OF ESTIMATORS OF DENSITY AND DISTRIBUTION FUNCTIONS

By

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1. Summary and introduction.

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a population with an unknown probability density function $f(x)$. The estimator of the form $f(x) = \frac{1}{n}\sum_{j=1}^{n} w_n(x-X_j)$ of the unknown density $f(x)$ based on this sample, where $w_n(y) \geqq 0$ on R^1 and $\int_{-\infty}^{\infty} w_n(y)dy = 1$, is shown to be not unbiased for any probability density function. For the class of all continuous probability density functions, the estimator $f_n(x)$ is asymptotically unbiased if and only if the sequence of functions $\left\{\int_{-\infty}^{x} w_n(y)dy\right\}$ converges to the unit distribution function except for the origin. Furthermore for this class the consistency and the asymptotic normality of the estimator $f_n(x)$ is discussed. In case $f(x)$ is symmetric around zero, we propose an estimator $\hat{f}_n(x) = \frac{1}{2}\{f_n(x)+f_n(-x)\}$. Then the variance of the proposed estimator $\hat{f}_n(x)$ is asymptotically half of the variance of the estimator $f_n(x)$ at a non-zero point x of continuity. We also consider the integration of our estimator $F_n(x)$ as an estimator of the distribution function, which is compared with the empirical distribution function $F_n^*(x)$. We propose an estimator $\hat{F}_n(x) = \frac{1}{2}\{F_n(x)+1-F_n(-x)\}$ and the corrected empirical distribution function $\hat{F}_n^*(x) = \frac{1}{2}\{F_n^*(x)+1-F_n^*(-x-0)\}$ for all (absolutely) continuous symmetric distribution functions. The mean square error of $\hat{F}_n^*(x)$ is smaller than half of the mean square error of $F_n^*(x)$ for x with $F(x) \neq 0$ or 1, and the estimator $\hat{F}_n(x)$ is asymptotically at least as good as $F_n(x)$.

The density estimator of the form:

$$f_n(x) = \frac{1}{n}\sum_{j=1}^{n} w_n(x-X_j) \tag{1.1}$$

was introduced by Rosenblatt [8], and several authors have discussed the statistical properties of the estimator $f_n(x)$. Concerning the unbiasedness of density estimators, the following result is obtained by Rosenblatt [8]: let a function $S(y: x_1, \cdots, x_n) \geqq 0$ be Borel measurable in $(y, x_1, \cdots, x_n)$ and symmetric in $(x_1, \cdots, x_n)$. Then there are

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no estimators $S(y: X_1, \cdots, X_n)$ unbiased for all continuous probability density functions, $f(y)$. Therefore it is obvious that there are no unbiased estimators $f_n(x)$ given by (1.1) with $w_n(y) \geqq 0$ on R^1 for all probability density functions. Now there arises a question if there exists a subset of the class of all continuous densities, where density functions have unbiased positive estimators $f_n(x)$. We shall answer the question in section 2.

Next we prepare ourselves with the brief review on the asymptotic unbiasedness, asymptotic variance, consistency and asymptotic normality of the density estimator $f_n(x)$ given by (1.1), which were treated by Parzen [7], Leadbetter [3], Murthy [6] and Craswell [1].

In Parzen [7] it is shown that if a sequence of real positive numbers $\{h_n\}$ converges to zero and a measurable function $K(y)$ satisfies

$$\sup_{-\infty<y<\infty} |K(y)| < \infty, \tag{1.2}$$

$$\int_{-\infty}^{\infty} |K(y)|\, dy < \infty, \tag{1.3}$$

$$\lim_{|y|\to\infty} yK(y) = 0, \tag{1.4}$$

$$\int_{-\infty}^{\infty} K(y)dy = 1, \tag{1.5}$$

then the density estimator $f_n(x)$ given by (1.1) with $w_n(y) = \frac{1}{h_n} K\left(\frac{y}{h_n}\right)$ satisfies

$$\lim_{n\to\infty} Ef_n(x) = f(x) \tag{1.6}$$

and

$$\lim_{n\to\infty} nh_n\, Var[f_n(x)] = f(x)\int_{-\infty}^{\infty} K^2(y)dy \tag{1.7}$$

at all points x of continuity of f. Furthermore, if $nh_n \to \infty$ as $n \to \infty$ then

$$\lim_{n\to\infty} E|f_n(x) - f(x)|^2 = 0 \tag{1.8}$$

and $f_n(x)$ is asymptotically normal at all points x of continuity of f.

If we are intereted in continuous probability density functions, (1.6), (1.7), (1.8) and the asymptotic normality of $f_n(x)$ hold for $K(y)$ satisfying (1.2), (1.3) and (1.5), but not necessarily (1.4).

Next, we shall state the results in general case given by Leadbetter [3], in which $\{w_n\}$ is a δ-function sequence, that is, $\{w_n\}$ satisfies the following conditions:

$$\int_{-\infty}^{\infty} |w_n(x)|\, dx < A \qquad \text{for all } n \text{ and some fixed } A,$$

$$\int_{-\infty}^{\infty} w_n(x)dx = 1 \qquad \text{for all } n,$$

$$w_n(x) \to 0 \text{ uniformly in } |x| > \lambda \text{ for any fixed } \lambda > 0,$$

$$\int_{|x|>\lambda} |w_n(x)|\, dx \to 0 \qquad \text{as } n\to\infty \text{ for any fixed } \lambda > 0.$$

If $\{w_n\}$ is a δ-function sequence, then (1.6) and

$$\lim_{n\to\infty} \frac{n}{\alpha_n} Var[f_n(x)] = f(x)$$

hold at a point x of continuity of f, where $\alpha_n = \int_{-\infty}^{\infty} w_n^2(x)dx < \infty$. Furthermore

$$\lim_{n\to\infty} n \, Cov\,(f_n(x), f_n(y)) = -f(x)f(y)$$

at two distinct points x and y of continuity of f. If $\{w_n\}$ be a δ-function sequence such that $\alpha_n < \infty$, $\alpha_n = 0(n)$ and for some constant K_0, $|w_n(u)| < K_0\alpha_n$ for all n and u, then the distribution of

$$\frac{\sqrt{n}\,[f_n(x) - Ef_n(x)]}{[\alpha_n f(x)]^{1/2}}$$

converges to the standardized normal distribution at a point x of continuity of f with $f(x) \neq 0$.

Murthy [6] discussed the properties (1.6), (1.7), (1.8) and the asymptotic normality of the same one as the density estimator in Parzen [7], in case a random sample $X_1, \cdots, X_n$ is obtained from a distribution with no singular part. He proved that if $h_n \to 0$ and $nh_n \to \infty$ as $n \to \infty$ and the measurable function $K(y)$ satisfies $K(y) \geqq 0$ on R^1, $K(-y) = K(y)$, (1.4) and (1.5), then the density estimator $f_n(x)$ given by (1.1) with $w_n(y) = \frac{1}{h_n} K\left(\frac{y}{h_n}\right)$ have the properties (1.6), (1.7), (1.8) and the asymptotic normality at a point of continuity of the distribution function and the derivative of the absolutely continuous part.

At last, we state the results given by Craswell [1] in case the sample space is R^1. Suppose w_n is real-valued, non-negative, symmetric and integrable function on R^1 such that

(1.9) $$\int_{-\infty}^{\infty} w_n(y)dy = 1,$$

(1.10) for any $\varepsilon > 0$

(a) $$\lim_{n\to\infty} \left\{\int_{-\infty}^{-\varepsilon} w_n(y)dy + \int_{\varepsilon}^{\infty} w_n(y)dy\right\} = 0$$

(b) $w_n(y) \to 0$ uniformly for almost all $y \in (-\infty, -\varepsilon) \cup (\infty, \varepsilon)$.

Then the density estimator $f_n(x)$ is asymptotically unbiased at a point of continuity of f. In addition, if $\{C_n\}$ is a sequence of positive constants, converging to zero, for which $\{C_n w_n^2(y)\}$ satisfies (1.9), (1.10a) and (1.10b), then

$$\lim_{n\to\infty} nC_n \, Var[f_n(x)] = f(x)$$

at a point x of continuity of f and $\{f_n(s), f_n(t)\}$ is jointly asymptotically normal and independent when s and t are two distinct continuous points of f with $f(s)+f(t) \neq 0$.

In section 3, we shall treat the necessary and sufficient condition that the density estimator $f_n(x)$ given by (1.1) is asymptotically unbiased for the class of all continuous density functions.

In section 4, we shall make use of a sequence similar to $\{C_n\}$ in Craswell [1] and $\{\alpha_n\}$ in Leadbetter [3], determined by $\{w_n\}$ itself and discuss the asymptotic properties of $Var[f_n(x)]$ and the consistency of $f_n(x)$ for continuous density functions.

In section 5, we shall give the limit distribution of $f_n(x)$ for continuous density functions, which is derived from the result in section 3.

In section 6, we shall treat the case where the density function is symmetric around the origin. Keeping in mind the fact that the covariance of density estimators $f_n(u)$ and $f_n(v)$ is asymptotically zero at two distinct points u, v of continuity of f, which is given by Leadbetter [3], we propose the estimator

$$\hat{f}_n(x)=\frac{1}{2}\{f_n(x)+f_n(-x)\}$$

for symmetric density functions. We discuss the asymptotic unbiasedness, consistency and asymptotic normality of $\hat{f}_n(x)$, and the variance of $\hat{f}_n(x)$ is shown to be asymptotically half of the variance of $f_n(x)$.

In section 7, we shall compare the estimator of the distribution function obtained by integrating the density estimator $f_n(x)$ with the empirical distribution function. We propose the estimator obtained by integrating $\hat{f}_n(x)$ and the corrected empirical distribution function for all (absolutely) continuous symmetric distribution functions and four estimators of distribution functions are compared.

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2. Unbiasedness.

THEOREM 1. *Let $X_1, X_2, \cdots, X_n$ be independently, identically distributed radom variables with a probability density function $f(x)$. Let $\mathcal{W}$ be the class of measurable functions w satisfying*

$$w(y)\geqq 0 \qquad on\ R^1 \tag{2.1}$$

and

$$\int_{-\infty}^{\infty} w(y)dy=1\,. \tag{2.2}$$

Then there does not exist any function $w_n\in\mathcal{W}$ such that the estimator $f_n(x)$, given by (1.1), of the density function $f(x)$ is unbiased.

PROOF. We suppose conversely that there were the function $w_n\in\mathcal{W}$ such that

$$Ef_n(x)=f(x) \qquad \text{on } R^1\,,$$

i. e.,

$$\int_{-\infty}^{\infty} w_n(x-y)f(y)dy=f(x) \qquad \text{on } R^1\,. \tag{2.3}$$

By using Fourier transform of the function w_n, Φ_{w_n}, and the characteristic function, φ, we can reduce (2.3) to

$$\Phi_{w_n}(u)\cdot\varphi(u)=\varphi(u) \qquad \text{on } R^1\,. \tag{2.4}$$

Since the characteristic function $\varphi(u)$ is equal to 1 and continuous at $u=0$, there exists a $\varepsilon>0$ such that

$$\varphi(u) \neq 0 \quad \text{on } (-\varepsilon, \varepsilon). \tag{2.5}$$

From (2.4) and (2.5) we have

$$\Phi_{w_n}(u)=1 \quad \text{on } (-\varepsilon, \varepsilon).$$

Therefore by Proposition a on p. 202 in Loeve [5] we have

$$\Phi_{w_n}(u)=1 \quad \text{on } R^1. \tag{2.6}$$

There does not exist the function w_n satisfying (2.1), (2.2) and (2.6), which contradicts to the assumption. Thus the theorem is proved.

We note that the estimator given by (1.1) with $w_n \in \mathcal{W}$ is also a probability density function. If we allow that the function w_n to take negative values, then we can give an artificial example of the unbiased density estimator $f_n(x)$ given by (1.1). Such the estimator may not be non-negative. Consider the density function

$$f(x)=\frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2$$

and the function

$$w_n(x)=\frac{1}{2\pi a_n}\left((1+a_n)^2\left\{\frac{\sin((1+a_n)x/2)}{(1+a_n)x/2}\right\}^2-\left(\frac{\sin(x/2)}{x/2}\right)^2\right),$$

where $\{a_n\}$ is a sequence of positive numbers diverging to $+\infty$. Then we have

$$\int_{-\infty}^{\infty} w_n(x)dx=1,$$

$$\lim_{n\to\infty}\int_{-\infty}^{x} w_n(x)dx=\begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$$

$$\varphi(u)=\begin{cases} 1-|u| & \text{on } [-1, 1] \\ 0 & \text{on } (-\infty, -1)\cup(1, \infty) \end{cases}$$

$$\Phi_{w_n}(u)=\begin{cases} 1 & \text{on } [-1, 1] \\ 0 & \text{on } (-\infty, -1-a_n)\cup(1+a_n, \infty), \end{cases}$$

where φ is the characteristic function of f and Φ_{w_n} is Fourier transform of w_n. Thus we have

$$\Phi_{w_n}(u)\varphi(u)=\varphi(u) \quad \text{on } R^1,$$

and consequently

$$\int_{-\infty}^{\infty} w_n(x-y)f(y)dy=f(x) \quad \text{on } R^1.$$

Therefore the density estimator $f_n(x)$ given by (1.1) with the above $w_n(x)$ is unbiased.

3. Asymptotic unbiasedness.

By the similar method to the proof of Theorem 1, we can show the following

LEMMA 1. *For arbitrary probability density function $f(x)$, the distribution function G_0 satisfying*

$$\int_{-\infty}^{\infty} f(x-y)dG_0(y)=f(x) \qquad \text{on } R^1$$

is the unit distribution function.

Hereafter we shall denote the unit distribution function by G_0, i.e.,

$$G_0(x)=\begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x\geqq 0 \end{cases}$$

and the distribution function induced by the corresponding function $w_n \in \mathcal{W}$ by W_n, i.e.,

$$W_n(x)=\int_{-\infty}^{x} w_n(t)dt \qquad \text{for all } w_n \in \mathcal{W}\,, \tag{3.1}$$

where $\mathcal{W}$ is defined in Theorem 1. It should be noted that the function W_n is a distribution function. Now, we can give the necessary and sufficient condition for the density estimator $f_n(x)$ to be asymptotically unbiased for the class of continuous probability density functions.

THEOREM 2. *If the sequence of functions $\{W_n\}$ converges to the unit distribution function G_0 except for the origin, then the density estimator $f_n(x)$ given by* (1.1) *is asymptotically unbiased for all continuous density functions, $f(x)$. Conversely if the estimator $f_n(x)$ is asymptotically unbiased for all continuous probability density functions, then the sequence $\{W_n\}$ converges to the unit distribution function G_0 except for the origin.*

PROOF. At first, we shall note that

$$\begin{aligned} Ef_n(x) &= \int_{-\infty}^{\infty} w_n(x-y)f(y)dy \\ &= \int_{-\infty}^{\infty} f(x-y)w_n(y)dy \\ &= \int_{-\infty}^{\infty} f(x-y)dW_n(y)\,. \end{aligned} \tag{3.2}$$

If the sequence of the distribution functions $\{W_n\}$ converges to the unit distribution function G_0, then by the continuity of $f(x)$ and Theorem 11.2 in Ito [2], we have

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-y)dW_n(y) &= \int_{-\infty}^{\infty} f(x-y)dG_0(y) \\ &= f(x)\,. \end{aligned}$$

Conversely we assume that

(3.3) $$\lim_{n\to\infty} Ef_n(x)=f(x) \quad \text{on } R^1$$

for all continuous density functions, f. From (3.2), we can reduce (3.3) to

(3.4) $$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(y)w_n(x-y)dy=f(x) \quad \text{on } R^1$$

for all continuous density functions, f. Let C_0 be the class of all continuous functions equal to zero except for on some bounded set on R^1. Then, for any $h\in C_0$, $h^+(y)=max(h(y), 0)$ and $h^-(y)=max(-h(y), 0)$ are also contained in C_0 and $h(y)=h^+(y)-h^-(y)$. Since for any $\tau>0$ both

$$\frac{h^+(y-\tau)}{\int_{-\infty}^{\infty} h^+(t)dt} \quad \text{and} \quad \frac{h^-(y-\tau)}{\int_{-\infty}^{\infty} h^-(t)dt}$$

are continuous density functions in y, inserting these in (3.4), taking off constant terms and taking the difference, we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h(y-\tau)W_n(x-y)dy=h(x-\tau) \quad \text{for any } h\in C_0.$$

By the transformation after putting τ equal to x, we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h(-y)w_n(y)dy=h(0) \quad \text{for any } h\in C_0.$$

Consequently

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h(y)w_n(y)dy=h(0) \quad \text{for any } h\in C_0,$$

i. e.,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h(y)dW_n(y)=\int_{-\infty}^{\infty} h(y)dG_0(y) \quad \text{for any } h\in C_0,$$

which implies that the sequence of distribution functions $\{W_n\}$ converges to the unit distribution function G_0 at all points of continuity of G_0, that is, except for the origin (See, for example, Theorem 11.2 in Ito [2]). Thus the theorem is proved.

4. Variance.

In section 3, we showed that the density estimator $f_n(x)$ given by (1.1) is asymptotically unbiased if and only if the sequence of functions $\{W_n\}$ converges to the unite distribution function, in other words, the sequence of functions $\{W_n\}$ converges to Dirac δ-function δ in distribution. Whereas the above convergence does not imply that $\lim_{n\to\infty} w_n(x)=\delta(x)$ on R^1. We shall consider the convergence of $\sup_{-\infty<x<\infty} w_n(x)$ in the following

LEMMA 2. *Let $w_n\in\mathcal{W}$ be continuous on $[-c, c]$ for all n and some positive constant c and*

$$M_n = \sup_{-\infty<x<\infty} w_n(x) < \infty \qquad \textit{for all } n. \tag{4.1}$$

Let the corresponding sequence of distribution functions $\{W_n\}$ *converge to the unit distribution function* G_0 *except for the origin. Then we have*

$$\lim_{n\to\infty} M_n = \infty.$$

PROOF. Conversely, we suppose that there exist a constant $K>0$ and a sequence of positive integers $\{n_k\}$ such that

$$M_{n_k} < K \qquad \text{for all } n_k. \tag{4.2}$$

For the above constant $c>0$, let us put

$$G^*_{n_k}(x) = \int_{-c}^{x} w_{n_k}(t)dt \qquad \text{for all } n_k \text{ on } [-c, c]. \tag{4.3}$$

The function $G^*_{n_k}(x)$ is continuous on $[-c, c]$ for all n_k and

$$\lim_{n_k\to\infty} G^*_{n_k}(x) = G_0(x) \qquad \text{for } x \neq 0 \text{ on } [-c, c]. \tag{4.4}$$

By the first mean value theorem and the continuity of w_n we have

$$|G^*_{n_k}(x) - G^*_{n_k}(y)| = |x-y| w_{n_k}(\xi) \qquad \text{for all } n_k, \tag{4.5}$$

where ξ is some value between x and $y \in [-c, c]$. From (4.3) and (4.5), we have

$$|G^*_{n_k}(x) - G^*_{n_k}(y)| \leqq |x-y| K$$

for all n_k and $x, y \in [-c, c]$, which implies that $\{G^*_{n_k}(x)\}$ are equicontinuous on $[-c, c]$. On the other hand, it is obvious that $|G^*_{n_k}(x)| \leqq 1$ for all n_k and $x \in [-c, c]$, that is, $\{G^*_{n_k}(x)\}$ is uniformly bounded. Therefore by Ascoli-Arzela's theorem the convergence of (4.4) holds uniformly. On the other hand we have

$$\sup_{-c\leqq x\leqq c} |G^*_{n_k}(x) - G_0(x)| = \max\{G^*_{n_k}(0), 1 - G^*_{n_k}(0)\}$$

which does not converge to zero. Thus the lemma is proved.

The result in Lemma 2.1 of Leadbetter [3] can be generalized to the case where the sequence of functions $\{W_n\}$, induced by $\{w_n\}$, converges to the unit distribution except for the origin, because the property that

$$\lim_{n\to\infty} \int_{-\lambda}^{\lambda} w_n(y)dy = 1 \qquad \text{for any } \lambda > 0$$

holds also for our sequence $\{w_n\}$. Note that $\{w_n\}$ contains the δ-function sequences which he considered, in non-negative case.

LEMMA 3. *If the sequence of functions* $\{W_n\}$ *converges to the unit distribution function except for the origin and*

$$\alpha_n = \int_{-\infty}^{\infty} w_n^2(x)dx < \infty \qquad \textit{for all } n, \tag{4.6}$$

then we have

$$\lim_{n\to\infty} \alpha_n = \infty .$$

PROOF. Although the proof is identical with that of Leadbetter [3], we dare to present the proof as the present author is not able to locate any literature, widely available, which gives the proof.

Since the sequence of distribution functions $\{W_n\}$ converges to the unit distribution function, we have easily

$$\lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} w_n(x)dx = 1 \qquad \text{for any } \varepsilon > 0 .$$

By Scharz's inequality we have

$$0 \leqq \int_{-\varepsilon}^{\varepsilon} w_n(x)dx \leqq (2\varepsilon)^{1/2}\Big(\int_{-\varepsilon}^{\varepsilon} w_n^2(x)dx\Big)^{1/2}$$

Hence it follows that

$$\begin{aligned}(2\varepsilon)^{1/2} \liminf_{n\to\infty} &\Big(\int_{-\infty}^{\infty} w_n^2(x)dx\Big)^{1/2} \\ &\geqq \liminf_{n\to\infty} (2\varepsilon)^{1/2}\Big(\int_{-\varepsilon}^{\varepsilon} w_n^2(x)dx\Big)^{1/2} \\ &\geqq \liminf_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} w_n(x)dx = 1 .\end{aligned}$$

Thus for any $\varepsilon > 0$ we have

$$\liminf_{n\to\infty} \alpha_n \geqq \frac{1}{2\varepsilon} ,$$

which yields the conclusion.

Obviously for M_n and α_n given by (4.1) and (4.6) respectively

$$0 \leqq \frac{\alpha_n}{M_n} \leqq 1$$

and the limit also lies, if exists, between 0 and 1.

THEOREM 3. *Let $w_n \in \mathcal{W}$ be continuous on $[-c, c]$ for all n and some constant $c > 0$. Suppose that the corresponding sequence of functions $\{W_n\}$ converges to the unit distribution G_0 except for the origin and the limit in the left hand side of* (4.7) *exists*

$$\lim_{n\to\infty} \frac{\alpha_n}{M_n} = \beta . \tag{4.7}$$

Then for the density estimator $f_n(x)$ of continuous density functions $f(x)$, given by (1.1), *we have*

$$\lim_{n\to\infty} \frac{n}{M_n} Var[f_n(x)] = \beta f(x) \qquad on\ R^1 . \tag{4.8}$$

PROOF. In case of $\beta = 0$, (4.8) is obvious. We consider the case of $\beta \neq 0$. We have

$$(4.9) \qquad \frac{n}{\alpha_n} Var[f_n(x)] = \frac{1}{\alpha_n}\int_{-\infty}^{\infty} f(x-y)w_n^2(y)dy - \frac{1}{\alpha_n}\left\{\int_{-\infty}^{\infty} f(x-y)w_n(y)dy\right\}^2.$$

The second term of the right hand side of (4.9) tends to zero as n tends to ∞ by Theorem 2 and Lemma 3. On the other hand we can reduce the first term of the right hand side of (4.9) to

$$\int_{-\infty}^{\infty} f(x-y)dH_n(y),$$

where $H_n(y) = \frac{1}{\alpha_n}\int_{-\infty}^{y} w_n^2(t)dt$. By (4.1), (4.6), (4.7) and the convergence of $\{W_n\}$ it is shown that $\{H_n\}$ is a sequence of distribution functions and converges to the unit distribution function G_0 except for the origin. Therefore, from the continuity of $f(x)$, we have

$$(4.10) \qquad \lim_{n\to\infty}\int_{-\infty}^{\infty} f(x-y)dH_n(y) = \int_{-\infty}^{\infty} f(x-y)dG_0(y) = f(x) \quad \text{on } R^1.$$

Consequently we have

$$\lim_{n\to\infty}\frac{n}{M_n} Var[f_n(x)] = \lim_{n\to\infty}\frac{\alpha_n}{M_n}\frac{n}{\alpha_n}\int_{-\infty}^{\infty} f(x-y)w_n^2(y)dy = \beta f(x) \quad \text{on } R^1.$$

Thus the theorem is proved.

In the above theorem with $\beta \neq 0$, (4.9) is reduced to

$$(4.11) \qquad \lim_{n\to\infty}\frac{n}{\alpha_n} Var[f_n(x)] = f(x),$$

which is the same representation as in Theorem 2.5 of Leadbetter [3]. Now we shall put $w_n(x) = \frac{1}{h_n}K\left(\frac{x}{h_n}\right)$, where $\{h_n\}$ is a sequence of positive constants converging to zero and $K(x)$ is a continuous, positive, bounded function and $\int_{-\infty}^{\infty} K(x)dx = 1$. Then we have

$$M_n = \frac{1}{h_n}\sup_{-\infty<x<\infty} K(x),$$

$$\alpha_n = \frac{1}{h_n}\int_{-\infty}^{\infty} K^2(x)dx$$

and

$$\frac{\alpha_n}{M_n} = \frac{\int_{-\infty}^{\infty} K^2(x)dx}{\sup_{-\infty<x<\infty} K(x)}.$$

In Theorem 3 if we asssume furthermore $\lim_{n\to\infty}(M_n/n) = 0$, then we have

$$(4.12) \qquad \lim_{n\to\infty} Var[f_n(x)] = 0 \quad \text{on } R^1.$$

and the combination of Theorem 2 and (4.12) yields

$$\lim_{n\to\infty} E[|f_n(x)-f(x)|^2]=0 \qquad \text{on } R^1 .$$

Thus we have the following

COROLLARY. *If we assume*

$$\lim_{n\to\infty} \frac{M_n}{n}=0 \tag{4.13}$$

in addition to the assumption in Theorem 3, *then we have*

$$\lim_{n\to\infty} Var[f(x)]=0 \qquad \text{on } R^1$$

and

$$\lim_{n\to\infty} E[|f_n(x)-f(x)|^2]=0 \qquad \text{on } R^1 . \tag{4.14}$$

Now by (4.14) we have

$$f_n(x) \xrightarrow{p} f(x) \qquad \text{on } R^1 ,$$

and therefore the estimator $f_n(x)$ is consistent at all point x. We assume in Theorem 3 that $w_n \in \mathcal{W}$ is continuous on $[-c, c]$ for all n and some positive constant c and it is natural that we use the continuous function $w_n \in \mathcal{W}$ in order to estimate a unknown continuous density.

5. Asymptotic normality.

We shall show the asymptotic normality of the density estimator $f_n(x)$ for continuous density functions.

THEOREM 4. *Under the same assumptions as in Theorem* 3 *and its corollary except for* $\beta=0$, *the density estimator* $f_n(x)$ *is asymptotically normal.*

PROOF. By putting

$$V_{nj}=w_n(x-X_j) \qquad \text{for } j=1, 2, \cdots, n$$

our estimator can be expressed as

$$f_n(x)=\frac{1}{n}\sum_{j=1}^{n} V_{nj} ,$$

where $V_{n1}, V_{n2}, \cdots, V_{nn}$ are statistically independent and identically distributed as $V_n=w_n(x-X_n)$. By (4.10) we have

$$\frac{1}{\alpha_n} Var[V_n]=\frac{n}{\alpha_n} Var[f_n(x)] \longrightarrow f(x) \qquad \text{as } n\to\infty . \tag{5.1}$$

On the other hand,

$$\begin{aligned} E|V_n|^3 &=\int_{-\infty}^{\infty} w_n^3(x-y)f(y)dy=\int_{-\infty}^{\infty} w_n^3(y)f(x-y)dy \\ &= M_n\int_{-\infty}^{\infty} w_n^2(y)f(x-y)dy . \end{aligned} \tag{5.2}$$

By applying (4.10) on (5.2), we have

(5.3) $$\lim_{n\to\infty} \frac{E|V_n|^3}{\alpha_n M_n} \leqq f(x).$$

From (4.7), (4.13), (5.1) and (5.3), we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} E|V_{nj}|^3}{\{\sum_{j=1}^{n} Var[V_{nj}]\}^{3/2}} = \lim \frac{E|V_n|^3}{n^{1/2}\{Var[V_n]\}^{3/2}}$$

$$= \lim_{n\to\infty} \left(\frac{M_n}{n}\right)^{1/2} \left(\frac{M_n}{\alpha_n}\right)^{1/2} \frac{E|V_n|^3}{\alpha_n M_n} \Big/ \left(\frac{1}{\alpha_n} Var[V_n]\right)^{3/2}$$

$$= 0.$$

The condition of the basic lemma on p. 277 in Loeve [5] is satisfied and therefore the distribution of

$$\frac{f_n(x)-Ef_n(x)}{\sqrt{Var[f_n(x)]}} = \frac{\sum_{j=1}^{n}(V_{nj}-EV_{nj})}{\sqrt{\sum_{j=1}^{n} Var[V_{nj}]}}$$

converges to the normal distribution $N(0, 1)$. Thus the theorem is proved.

6. Estimation of symmetric density functions.

In this section, we suppose that $\{w_n\}$ is a sequence of measurable functions satisfying

(6.1) $w_n(y) = w_n(-y)$ for all $y \in R^1$,

(6.2) $w_n(y) \geqq 0$ on R^1,

(6.3) $\int_{-\infty}^{\infty} w_n(y)dy = 1$,

(6.4) $\lim_{n\to\infty} \int_{-\delta}^{\delta} w_n(y)dy = 1$ for any $\delta > 0$,

(6.5) $w_n(y) \to 0$ uniformly as $n \to \infty$ on $(-\infty, -\delta) \cup (\delta, \infty)$ for any $\delta > 0$.

Following the definition given in Craswell [1] and Leadbetter [3], we call the sequence of these functions $\{w_n\}$ as a δ-function sequence. It can be easily seen that the sequence of functions $\{W_n\}$, induced by the δ-function sequence with (3.1), converges to the unit distribution function G_0 except for the origin. We state the asymptotic unbiasedness, asymptotic covariance and asymptotic variance of the estimators $f_n(x)$, given by (1.1) with the δ-function sequence $\{w_n\}$, in the following Lemmas 4 and 5, which are found in Craswell [1] and Leadbetter [3].

LEMMA 4. *Let $\{w_n\}$ be a δ-function sequence. Then the density estimators $f_n(x)$ and $f_n(y)$ satisfy*

(6.6) $$\lim_{n\to\infty} Ef_n(x) = f(x)$$

at a point x *of continuity of density* f *and*

(6.7) $$\lim_{n\to\infty} n\, Cov\,(f_n(x), f_n(y)) = -f(x)f(y)$$

at two distinct points x, y *of continuity of* f.

The above lemma implies that

(6.8) $$\lim_{n\to\infty} Cov\,(f_n(x), f_n(y)) = 0$$

at two distinct points x, y of continuity of f.

LEMMA 5. *Let* $\{w_n\}$ *be a* δ*-function sequence with* $\alpha_n = \int_{-\infty}^{\infty} w_n^3(y)dy < \infty$ *for all* n. *Then for the density estimator* $f_n(x)$ *we have*

(6.9) $$\lim_{n\to\infty} \frac{n}{\alpha_n} Var\,[f_n(x)] = f(x)$$

at a point x *of continuity of* f.

We give the asymptotic normality of $f_n(x)$ in the following lemma, which is essentially identical with Theorem 2.7 in Leadbetter [3] except for the condition of continuity of w_n.

LEMMA 6. *Let* $\{w_n\}$ *be a* δ*-function sequence and continuous on* $[-c, c]$ *for all* n *and some positive constant* c. *Suppose that* $M_n = 0(n)$ *and* $M_n = 0(\alpha_n)$, *where* $M_n = \sup_{-\infty<y<\infty} w_n(y) < \infty$ *and* $\alpha_n = \int_{-\infty}^{\infty} w_n^2(y)dy$ *for all* n. *Then the density estimator* $f_n(x)$ *is asymptotically normal at all points* x *of continuity of* f.

PROOF. We dare to present the different proof from the one given by Leadbetter [3].

From the assumption $M_n < \infty$ it is obvious that $\alpha_n < \infty$. Hence, by Lemma 2.2 in Leadbetter [3], $\{w_n^2(y)/\alpha_n\}$ is also a δ-function sequence. Consequently we can prove the lemma by the same method as that of Theorem 4.

Motivated by the asymptotic property of the covariance of the density estimator $f_n(x)$, (6.8), let us propose the density estimator

(6.10) $$\hat{f}_n(x) = \frac{1}{2}\{f_n(x) + f_n(-x)\}$$

of density function $f(x)$ which is symmetric with respect to the axis of ordinates. The density estimators $\hat{f}_n(x)$ are asymptotically at least better than the density estimators $f_n(x)$ given by (1.1).

THEOREM 5. *Let the density function* $f(x)$ *be symmetric with respect to the axis of ordinates and* $\{w_n\}$ *be a* δ*-function sequence with* $\alpha_n < \infty$ *for all* n. *Then for density estimators* $f_n(x)$ *and* $\hat{f}_n(x)$, *given by* (1.1) *and* (6.10) *respectively, we have*

(6.11) $$Var\,[\hat{f}_n(x)] \leqq Var\,[f_n(x)] \qquad \textit{for all } n \textit{ and } x,$$

(6.12) $$\lim_{n\to\infty} \frac{n}{\alpha_n} Var[\hat{f}_n(x)] = \frac{1}{2} f(x)$$

at all points x of continuity of f and

(6.13) $$\lim_{n\to\infty} \frac{E|\hat{f}_n(x)-f(x)|^2}{E|f_n(x)-f(x)|^2} \leq 1$$

at all non-zero points x of continuity of f.

PROOF. (6.11) is obvious. (6.12) is proved as follows: we have

(6.14) $$\frac{n}{\alpha_n} Var[\hat{f}_n(x)] = \frac{1}{4}\Big\{\frac{n}{\alpha_n} Var[f_n(x)] + \frac{n}{\alpha_n} Var[f_n(-x)] + 2\frac{1}{\alpha_n} Cov(f_n(x), f_n(-x))\Big\}.$$

By applying (6.7) and (6.9) on the right hand side of (6.14), we have (6.12). (6.13) is easily obtained by (6.9) and (6.12). Thus the theorem is proved.

At all points x of continuity of the symmetric density function f, it follows that the density estimator $\hat{f}_n(x)$ given by (6.10) is asymptotically unbiased under the condition of Lemma 4, and consistent under the condition of Lemma 4, 5 and, in addition, $\alpha_n/n \to 0$ as $n\to\infty$. The asymptotic normality of $\hat{f}_n(x)$ holds under the condition of Lemma 6.

In case the density function $f(x)$ is symmetric with respect to the axis of ordinates and continuous on R^1, it follows that the density estimator $\hat{f}_n(x)$ is asymptotically unbiased at all points x under the condition of Theorem 2 and consistent at all points x under the condition of Corollary of Theorem 3. Furthermore we have the asymptotic normality of $\hat{f}_n(x)$ under the condition of Theorem 4.

In practice we may not be sure that the underlying distribution is symmetric. The use of the estimator $\hat{f}_n(x)$ may be still recommended over the use of the ordinary estimator $f_n(x)$, after testing the symmetry of the distribution by the method such as the permutation test, etc. Because the bias of the estimator $f_n(x)$ is identical with the one of the estimator $\hat{f}_n(x)$ for all n and x, which converges to zero and the variance of the estimator $\hat{f}_n(x)$ is asymptotically half of the one of the estimator $f_n(x)$.

7. Estimation of distribution functions.

We shall consider the estimation of the absolutely continuous distribution function $F(x)$ with the unknown density function $f(x)$. We can easily obtain the estimator of the distribution function $F(x)$ by integrating the density estimator $f_n(x)$,

$$F_n(x) = \int_{-\infty}^{x} f_n(t)dt$$

which may be denoted in another form by W_n

(7.1)
$$F_n(x)=\frac{1}{n}\sum_{j=1}^{n}W_n(x-X_j),$$

where W_n is given by (3.1) for $w_n\in\mathcal{W}$. The estimator $F_n(x)$ is absolutely continuous. We have

(7.2)
$$EF_n(x)=\int_{-\infty}^{\infty}W_n(x-t)dF(t)$$

and

(7.3)
$$n\ Var\,[F_n(x)]=\int_{-\infty}^{\infty}W_n^2(x-t)dF(t)-\left\{\int_{-\infty}^{\infty}W_n(x-t)dF(t)\right\}^2.$$

If $W_n\to G_0$ as $n\to\infty$, then $W_n^2\to G_0$ as $n\to\infty$. Therefore we have by the continuity of $F(x)$

(7.4)
$$\lim_{n\to\infty}EF_n(x)=\int_{-\infty}^{\infty}G_0(x-t)dF(t)$$
$$=F(x)\qquad\text{at all points } x$$

and

(7.5)
$$\lim_{n\to\infty}n\ Var\,[F_n(x)]=\int_{-\infty}^{\infty}G_0(x-t)dF(t)-\left\{\int_{-\infty}^{\infty}G_0(x-t)dF(t)\right\}^2$$
$$=F(x)[1-F(x)]\qquad\text{at all points } x.$$

Thus the condition $W_n\to G_0$ as $n\to\infty$ is sufficient for the asymptotic unbiaseness of $F_n(x)$. It is interesting to note that this condition is necessary and sufficient for the asymptotic unbiasedness of the estimator $f_n(x)$ of a continuous density, and is satisfied if $\{w_n\}$ is a δ-function sequence, which is the sufficient condition for $f_n(x)$ to be asymptotically unbiased at all continuous points x of a density.

Now let us denote the empirical distribution function by $F_n^*(x)$, which can be expressed by G_0,

(7.6)
$$F_n^*(x)=\frac{1}{n}\sum_{j=1}^{n}G_0(x-X_j).$$

For the empirical distribution function $F_n^*(n)$, we have

(7.7)
$$EF_n^*(x)=F(x)\qquad\text{for all } n \text{ and } x$$

and

(7.8)
$$n\ Var\,[F_n^*(x)]=F(x)[1-F(x)]\qquad\text{for all } n \text{ and } x,$$

which is given by Rosenblatt [8]. From (7.5) and (7.8) we have

(7.9)
$$\lim_{n\to\infty}\frac{E|F_n^*(x)-F(x)|^2}{E|F_n(x)-F(x)|^2}\leqq 1\qquad\text{for all } x \text{ with } F(x)\neq 0 \text{ or } 1.$$

Thus we have the following

THEOREM 6. *If the distribution function $F(x)$ is absolutely continuous and the sequence of distribution functions $\{W_n\}$ satisfies that $\{W_n(y)-G_0(y)\}\to 0$ as $n\to\infty$ for $y\neq 0$. Then for estimators $F_n(x)$ and $F_n^*(x)$ of $F(x)$ we have*

(7.10)
$$\lim_{n\to\infty}EF_n(x)=F(x),$$

(7.11)
$$\lim_{n\to\infty} n\, Var\,[F_n(x)] = F(x)[1-F(x)]$$

for all x and

$$\lim_{n\to\infty} \frac{E|F_n^*(x)-F(x)|^2}{E|F_n(x)-F(x)|^2} \leq 1$$

for all x with $F(x)\neq 0$ *or* 1.

(7.10) and (7.11) are a generalization of the result given by Leadbetter, which is stated on p. 25 in [5], where he considered the case of δ-function sequence, whereas we discussed on the more general class of functions w_n, in nonnegative case.

On the other hand for arbitrarily fixed x the empirical distribution function $F_n^*(x)$ is the uniformly minimum variance unbiased estimator of $F(x)$ for $F\in\mathscr{F}$, where $\mathscr{F}$ is the family of all absolutely continuous distribution functions or the family of all continuous distribution functions. Because the order statistic is sufficient and complete for $\mathscr{F}$ (See, for example, p. 40-42 and p. 133 in Lehmann [4]), and $F_n^*(x)$ is symmetric in $X_1, \cdots, X_n$ and satisfies (7.7) for all $F\in\mathscr{F}$.

The empirical distribution function $F_n^*(x)$ may be said to be preferable as it is the uniformly minimum variance unbiased estimator. Also the estimator $F_n(x)$ is not the uniformly minimum variance unbiased estimator but it may be still said to to preferable because of its absolutely continuity and asymptotic properties.

Concerning the covariance of estimators $F_n(x)$ and $F_n(y)$ we have the following result.

LEMMA 7. *If the sequence of distribution functions* $\{W_n\}$ *converges to the unit distribution function except for the origin, then we have*

(7.12)
$$\lim_{n\to\infty} n\, Cov\,(F_n(x), F_n(y)) = F(min\,(x, y)) - F(x)F(y) \qquad \text{for } x\neq y.$$

PROOF. We have

$$n\, Cov\,(F_n(x), F_n(y)) = \int_{-\infty}^{\infty} W_n(x-t)W_n(y-t)dF(t)$$
$$-\int_{-\infty}^{\infty} W_n(x-t)dF(t)\int_{-\infty}^{\infty} W_n(y-t)dF(t)$$

(7.13)
$$\to \int_{-\infty}^{\infty} G_0(x-t)G_0(y-t)dF(t) - F(x)F(y) \qquad \text{as } n\to\infty.$$

Application of the equality $G_0(x-t)G_0(y-t) = G_0(min\,(x, y)-t)$ on (7.13) implies (7.12).

For the empirical distribution functions $F_n^*(x)$ and $F_n^*(y)$, it hold that

(7.14)
$$n\, Cov\,(F_n(x), F_n(y)) = F(min\,(x, y)) - F(x)F(y) \qquad \text{for } x\neq y,$$

which is given by Rosenblatt [8].

Next we consider the estimation of the symmetric and absolutely continuous distribution function $F(x)$ with the derivative $f(x)$, in which case for all x

$$F(x) = 1-F(-x) \quad \text{and} \quad f(x) = f(-x).$$

Integrating the estimator $\hat{f}_n(x)$ of $f(x)$, given by (6.10), we have an estimator of the distributian function

$$\hat{F}_n(x)=\int_{-\infty}^{x}\hat{f}_n(x)dx$$

$$=\frac{1}{2}\left\{\int_{-\infty}^{x}f_n(x)dx+\int_{-\infty}^{x}f_n(-x)dx\right\},$$

which can be expressed by the estimator $F_n(x)$ given by (7.1),

$$\hat{F}_n(x)=\frac{1}{2}\{F_n(x)+1-F_n(-x)\}. \tag{7.15}$$

Our estimator may be understood as the arithmetic mean of the estimators of quantities $F(x)$ and $1-F(-x)$. Motivated by the above consideration we propose to revise the empirical distribution function $F_n^*(x)$ as follows,

$$\hat{F}_n^*(x)=\frac{1}{2}\{F_n^*(x)+1-F_n^*(-x-0)\}. \tag{7.16}$$

The estimator $\hat{F}_n^*(x)$ may be called as the corrected empirical distribution function for symmetric distribution functions. Both estimators $\hat{F}_n(x)$ and $\hat{F}_n^*(x)$ are symmetric, that is,

$$\hat{F}_n(x)=1-\hat{F}_n(-x-0) \quad \text{and} \quad \hat{F}_n^*(x)=1-\hat{F}_n^*(-x-0).$$

We discuss the unbiasedness and variances of $\hat{F}_n(x)$ and $\hat{F}_n^*(x)$ in the following

THEOREM 7. *Let the distribution function $F(x)$ be symmetric and absolutely continuous and the sequence of symmetric distribution functions $\{W_n\}$ converge to the unit distribution function except for the origin. Then for estimators $\hat{F}_n(x)$ and $\hat{F}_n^*(x)$, we have*

$$\lim_{n\to\infty} E\hat{F}_n(x)=F(x) \qquad \textit{for all } x, \tag{7.17}$$

$$E\hat{F}_n^*(x)=F(x) \qquad \textit{for all } n \textit{ and } x, \tag{7.18}$$

$$\lim_{n\to\infty} n\, Var[\hat{F}_n(x)]=\frac{1}{2}\{F(x)F(-x)-F^2(-|x|)\} \qquad \textit{for all } x \tag{7.19}$$

and

$$n\, Var[F_n^*(x)]=\frac{1}{2}\{F(x)F(-x)-F^2(-|x|)\} \qquad \textit{for all } n \textit{ and } x. \tag{7.20}$$

PROOF. (7.17) is easily seen by applying (7.4) on (7.15) and (7.19) is from (7.5), (7.12) and (7.15). (7.18) and (7.20) are proved as follows: at first we note that

$$\hat{F}_n^*(x)=\frac{1}{2n}\sum_{j=1}^{n}\{G_0(x-X_j)+G_0(x+X_j)\}$$

and (7.18) is easily seen. Since

$$G_0(x-y)G_0(x+y)=\begin{cases}0 & \text{if } x<0\\ 1 & \text{when } -x\leqq y\leqq x \text{ if } x\geqq 0\end{cases}$$

we have

$$Cov\,(G_0(x-X_j),\,G_0(x+X_j))=\begin{cases}0 & \text{if } x<0\\ F(x)-F(-x)-F^2(x) & \text{if } x\geqq 0\end{cases}$$

for $j=1, 2, \cdots, n$. By applying the above relation and

$$Var[G_0(x-X_j)]=Var[G_0(x+X_j)]=F(x)[1-F(x)] \qquad \text{for } j=1, 2, \cdots, n$$

on

$$Var[\hat{F}_n^*(x)]=\frac{1}{4n^2}\sum_{j=1}^{n}\{Var[G_0(x-X_j)]+Var[G_0(x+X_j)] \\ +2\,Cov[G_0(x-X_j), G_0(x+X_j)]\},$$

we have (7.20).

On the other hand for arbitrarily fixed x the corrected empirical distribution function $\hat{F}_n^*(x)$ is the uniformly minimum variance unbiased estimator of $F(x)$ for $F \in \mathcal{F}'$, where $\mathcal{F}'$ is the family of all absolutely continuous symmetric distribution functions or the family of all continuous symmetric distribution functions. Because $\hat{F}_n^*(x)$ can be reduced to

$$\hat{F}_n^*(x)=\begin{cases} \dfrac{1}{2n}\sum_{j=1}^{n} G_0(x+|X_j|) & \text{if } x \leqq 0 \\ \dfrac{1}{2}+\dfrac{1}{2n}\sum_{j=1}^{n} G_0(x-|X_j|) & \text{if } x>0, \end{cases}$$

which is symmetric in $|X_1|, \cdots, |X_n|$, and it is well known that the statistic $(|X|^{(1)}, \cdots, |X|^{(n)})$, where $|X|^{(1)}, \cdots, |X|^{(n)}$ are values $|X_1|, \cdots, |X_n|$ arranged in order of magnitude, is sufficient and complete for $\mathcal{F}'$.

Now we shall compare the mean square error of the corrected empirical distribution function, which is best in the above sense, with the mean square error of other estimators. It is easily shown that

$$F(x)F(-x)-F^2(-|x|) \leqq F(x)[1-F(x)] \qquad \text{for all } x, \tag{7.21}$$

where the equality holds for x with $F(x)=0$ or 1. Combination of Theorem 6, 7 and (7.21) yields

COROLLARY. *Under the assumptions in Theorem 7, for estimators $F_n(x)$, $\hat{F}_n(x)$, $F_n^*(x)$ and $\hat{F}_n^*(x)$ we have*

$$\frac{E|\hat{F}_n^*(x)-F(x)|^2}{E|F_n^*(x)-F(x)|^2}<\frac{1}{2} \qquad \textit{for all } n \textit{ and } x \textit{ with } F(x)\neq 0 \textit{ or } 1,$$

$$\lim_{n\to\infty}\frac{E|\hat{F}_n(x)-F(x)|^2}{E|F_n(x)-F(x)|^2}\leqq 1 \qquad \textit{for all } x \textit{ with } F(x)\neq 0 \textit{ or } 1$$

and

$$\lim_{n\to\infty}\frac{E|\hat{F}_n^*(x)-F(x)|^2}{E|\hat{F}_n(x)-F(x)|^2}\leqq 1 \qquad \textit{for all } x \textit{ with } F(x)\neq 0, \tfrac{1}{2} \textit{ or } 1. \tag{7.22}$$

At the point $x=0$ both $\hat{F}_n^*(x)$ and $\hat{F}_n(x)$ are equal to $F(x)=1/2$ with probability one. Thus the point $x=0$ is excepted from (7.22). From the above corollary it follows that the corrected empirical distribution function $\hat{F}_n^*(x)$ has less than half of the variance of the empirical distribution function.

In this section up to this point we considered the case where the distribution is absolultely continuous. But the above discussion can be extended to the case of continuous distribution functions. Theorem 6, Lemma 7, Theorem 7 and its Corollary also hold for continuous distribution functions and a sequence of given distribution functions $\{W_n\}$.

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