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ON EXPANSION OF ESTIMATES FOR MEAN VALUE FUNCTIONS OF HOMOGENEOUS RANDOM FIELDS ON COMPACT HOMOGENEOUS SPACES

By

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§ 1. Summary.

We consider the uniformly minimum variance unbiased linear estimators $\hat{m}(t)$ of mean value functions $m(t)$ of homogeneous random fields on compact homogeneous spaces and their expansions by spherical functions.

§ 2. Expansions of estimates for mean value functions.

Let $T = \{t\}$ be an arbitrary compact homogeneous space, that is, a compact space which admits a transitive transformation group $G = \{g\}$. We denote by $K = \{k\}$ a stationary subgroup of G , that is, a subgroup which leaves invariant a point $t_0 \in T$.

Let $\{X(t), t \in T\}$ be a real-valued homogeneous random field on T having the mean value function

$$m(t) = E\{X(t)\}, \quad t \in T \quad (1)$$

and satisfying the conditions:

$$(C.1) \quad E\{|X(t)|^2\} < \infty, \quad \text{for all } t \in T.$$

(C.2) The covariance functions

$$R(t, s) = E\{(X(t) - m(t))(X(s) - m(s))\}$$

is a continuous positive definite function on $T \times T$.

(C.3) For all $g \in G$,

$$R(t, s) = R(gt, gs), \quad t, s \in T.$$

We denote by $L_2(X)$ a Hilbert space consisting of all random variables which may be represented either as a finite linear combinations $U = \sum_{j=1}^n c_j X(t_j)$, for some integer n , points $t_1, t_2, \dots, t_n$ in T and scalars $c_1, c_2, \dots, c_n$, or as a limit in quadratic mean of such finite linear combinations under the scalar product defined by $(U, W) = E\{U \cdot W\}$.

We denote by $H(R)$ a reproducing kernel Hilbert space generated by the kernel

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$R(s, t)$, $s, t \in T$. $H(R)$ is actually a Hilbert space consisting of functions on T satisfying the conditions:

(R.1) For each $t \in T$, $R(t, \cdot) \in H(R)$.

(R.2) For any $f \in H(R)$,

$$(f, R(t, \cdot))_R = f(t), \quad t \in T,$$

where by $(\cdot, \cdot)_R$ we denote the scalar product in $H(R)$.

Suppose that the mean value function $m(\cdot)$ is known to be in a closed subspace Σ of $H(R)$ but the actual values of $m(t)$, $t \in T$, are not known.

The problem to find "the uniformly minimum variance unbiased linear estimator $\hat{m}(t) \in L_2(X)$ for $m(t)$ at t " in the sense that for each $t \in T$ the followings hold:

$$(E.1) \quad E_m\{\hat{m}(t)\} = m(t) \quad \text{for all } m \in \Sigma,$$

$$(E.2) \quad \text{For any unbiased estimate } \hat{\hat{m}}(t),$$

$$\text{Var}(\hat{m}(t)) \leq \text{Var}(\hat{\hat{m}}(t)), \quad \text{uniformly in } m \in \Sigma.$$

is completely solved (E. Parzen [2]), where by $E_m\{\cdot\}$ we denote the expectation is calculated by the probability with its mean value function $m(\cdot)$.

This solution $\hat{m}(t)$ is given by

$$\hat{m}(t) = \phi(\text{Proj}(R(t, \cdot) | \Sigma)), \quad t \in T$$

where ϕ is a congruence between $H(R)$ and $L_2(X)$, (E. Parzen [2], [3]), such that, for $f, g \in H(R)$,

$$(M.1) \quad \phi(R(t, \cdot)) = X(t), \quad t \in T,$$

$$(M.2) \quad E_m\{\phi(f)\} = (f, m)_R, \quad \text{for all } m \in \Sigma,$$

$$(M.3) \quad \text{Cov}(\phi(f), \phi(g)) = (f, g)_R.$$

We consider here the case where the subspace Σ is invariant under K , that is, $\Sigma = \{f \in H(R) | f(t) = f(kt), k \in K\}$.

In this case, we shall see that the estimator $\hat{m}(t)$ can be expressed quite in a simple form by making use of zonal spherical functions.

Let $\{T^{(\lambda)}(g) = [T_{ij}^{(\lambda)}(g)], g \in G, \lambda = 1, 2, \dots, 1 \leq i, j \leq d_\lambda\}$ be the complete system of unitary continuous non-equivalent representations of the group G and let us choose in the space of these representations a basis such that these representations decompose into irreducible representations of the subgroup K . Suppose, for instance, that the representation $T^{(\lambda)}$ of the group G contains r_λ times the identity representation of K . Suppose that in our basis $l_1, l_2, \dots, l_{d_\lambda}$, these identity representations correspond to the first r_λ basis vectors so that $T^{(\lambda)}(k)l_j = l_j$ for $k \in K$ and $j = 1, 2, \dots, r_\lambda$.

In this case the functions of t ,

$$\Phi_{ij}^{(\lambda)}(t) = T_{ij}^{(\lambda)}(g); \quad i = 1, 2, \dots, d_\lambda, j = 1, 2, \dots, r_\lambda, \lambda = 1, 2, 3, \dots,$$

will be called spherical functions over T , while the functions

$$\Phi_{ij}^{(\lambda)}(t) = T_{ij}^{(\lambda)}(g); \quad i, j = 1, 2, \dots, r_\lambda, \lambda = 1, 2, 3, \dots$$

will be called zonal spherical functions over T . (The matrix elements $T_{ij}^{(\lambda)}(g)$ are constant over all left cosets of G modulo K .)

Under these notations, we have the following theorem:

THEOREM: *Let the mean function $m(\cdot)$ be invariant under the subgroup K of G , that is,*

$$\Sigma = \{f \in H(R) \mid f(t) = f(kt), k \in K\}.$$

Then the uniformly minimum variance unbiased linear estimator $\hat{m}(t)$ for $m(t)$ at t has the following expansion:

$$\hat{m}(t) = \sum_{\lambda=1}^{\infty} \sum_{i=1}^{\tau_{\lambda}} \sum_{j=1}^{\tau_{\lambda}} X_{ij}^{(\lambda)} \cdot Q_{ij}^{(\lambda)}(t), \quad t \in T$$

where

$$Q_{ij}^{(\lambda)}(t) = \Phi_{ij}^{(\lambda)}(t) / \|\Phi_{ij}^{(\lambda)}\|$$

$$X_{ij}^{(\lambda)} = \int_T X(s) Q_{ij}^{(\lambda)}(s) ds$$

$$\|\Phi_{ij}^{(\lambda)}\|^2 = \int_T |\Phi_{ij}^{(\lambda)}(s)|^2 ds.$$

PROOF. Let $r(\hat{g}) = R(\hat{g}t_0, t_0)$. Then, since for any $g \in G$ $R(t, s) = R(gt, gs)$, we see that for each $k \in K$ $r(\hat{g}) = r(k^{-1}\hat{g}k)$, that is, $r(\cdot)$ is a class function of K . Hence, from the Yaglom's result (Yoglem [4], Theorem 5, p. 604) the covariance function $R(t, s)$ can be represented in the form;

$$R(t, s) = \sum_{\lambda=1}^{\infty} \sum_{i=1}^{\tau_{\lambda}} \sum_{j=1}^{\tau_{\lambda}} f_{ij}^{(\lambda)} \cdot \sum_{l=1}^{d_{\lambda}} \Phi_{il}^{(\lambda)}(t) \Phi_{lj}^{(\lambda)}(s).$$

Hence, we have

$$\begin{aligned} R_1(t_1, \cdot) &= Proj(R(t, \cdot) | \Sigma) \\ &= \sum_{\lambda=1}^{\infty} \sum_{i=1}^{\tau_{\lambda}} \sum_{j=1}^{\tau_{\lambda}} f_{ij}^{(\lambda)} \sum_{l=1}^{\tau_{\lambda}} \Phi_{il}^{(\lambda)}(t) \cdot \Phi_{lj}^{(\lambda)}(\cdot). \end{aligned}$$

Since $\int_T \int_T R(t, s) \Phi_{ij}^{(\lambda)}(t) \Phi_{ij}^{(\lambda)}(s) dt ds = f_{ij}^{(\lambda)} \cdot \|\Phi_{ij}^{(\lambda)}\|^2 < \infty$,

$$U_{ij}^{(\lambda)} = \int_T X(s) \Phi_{ij}^{(\lambda)}(s) ds$$

is well-defined and therefore there exists a unique function $p_{ij}^{(\lambda)} \in H(R)$ such that $\phi(p_{ij}^{(\lambda)}) = U_{ij}^{(\lambda)}$.

The function $p_{ij}^{(\lambda)}$ can be easily determined and given by

$$p_{ij}^{(\lambda)}(t) = \|\Phi_{ij}^{(\lambda)}\|^2 \cdot \sum_{l=1}^{\tau_{\lambda}} f_{jl}^{(\lambda)} \Phi_{il}^{(\lambda)}(t).$$

Thus, we have

$$U_{ij}^{(\lambda)} = \|\Phi_{ij}^{(\lambda)}\|^2 \cdot \phi\left(\sum_{l=1}^{\tau_{\lambda}} f_{jl}^{(\lambda)} \cdot \Phi_{il}^{(\lambda)}(\cdot)\right).$$

Let us write

$$X_{ij}^{(\lambda)} = U_{ij}^{(\lambda)} / \|\Phi_{ij}^{(\lambda)}\| = \int_T X(s) Q_{ij}^{(\lambda)}(s) ds$$

where

$$Q_{ij}^{(\lambda)}(t) = \Phi_{ij}^{(\lambda)}(t) / \|\Phi_{ij}^{(\lambda)}\|.$$

Then, we have the following expansion

$$\begin{aligned} \hat{m}(t) = \phi(R_1(t, \cdot)) &= \sum_{\lambda=1}^{\infty} \sum_{i=1}^{r_{\lambda}} \sum_{j=1}^{r_{\lambda}} \Phi_{ii}^{(\lambda)}(t) \phi\left(\sum_{j=1}^{r_{\lambda}} f_{ij}^{(\lambda)} \Phi_{ij}^{(\lambda)}(\cdot)\right) \\ &= \sum_{\lambda=1}^{\infty} \sum_{i=1}^{r_{\lambda}} \sum_{i=1}^{r_{\lambda}} Q_{ii}^{(\lambda)}(t) \cdot X_{ii}^{(\lambda)}. \end{aligned} \quad \text{Q. E. D.}$$

EXAMPLE (NAGAI [1]). Let $T = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$ and $x = \sin \theta \cos \phi$, $y = \sin \theta \sin \phi$ and $z = \cos \theta$; $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. Let $\{X(t), t \in T\}$ be a homogeneous random field on T and its mean value function $m(\theta, \phi)$ be invariant under the rotation group $K = SO(2)$. Then the uniformly minimum variance unbiased linear estimator $\hat{m}(\theta, \phi)$ at $(\theta, \phi) \in T$ can be expressed as follows;

$$\hat{m}(\theta, \phi) = \sum_{v=0}^{\infty} X_v \cdot \phi_v(\theta), \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi,$$

where

$$\begin{aligned} \phi_v(\theta) &= \sqrt{2v+1} P_v(\cos \theta), \\ X_v &= \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} X(\theta, \phi) \phi_v(\theta) \sin \theta d\phi d\theta \end{aligned}$$

and

$$\{P_v(z), \quad |z| \leq 1, \quad v = 0, 1, 2, \dots\}$$

are Legendre's polynomials.

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