ON EXPANSION OF ESTIMATES FOR MEAN VALUE FUNCTIONS OF HOMOGENEOUS RANDOM FIELDS ON COMPACT HOMOGENEOUS SPACES

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ON EXPANSION OF ESTIMATES FOR MEAN VALUE FUNCTIONS OF HOMOGENEOUS RANDOM FIELDS ON COMPACT HOMOGENEOUS SPACES

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§ 1. Summary.

We consider the uniformly minimum variance unbiased linear estimators $\hat{m}(t)$ of mean value functions $m(t)$ of homogeneous random fields on compact homogeneous spaces and their expansions by spherical functions.

§ 2. Expansions of estimates for mean value functions.

Let $T=\{t\}$ be an arbitrary compact homogeneous space, that is, a compact space which admits a transitive transformation group $G=\{g\}$. We denote by $K=\{k\}$ a stationary subgroup of G, that is, a subgroup which leaves invariant a point $t_0 \in T$.

Let $\{X(t),\ t \in T\}$ be a real-valued homogeneous random field on T having the mean value function

$$m(t)=E\{X(t)\}, \qquad t \in T \tag{1}$$

and satisfying the conditions:

(C.1) $$E\{|X(t)|^2\}<\infty, \qquad \text{for all } t \in T.$$

(C.2) The covariance functions

$$R(t, s)=E\{(X(t)-m(t))(X(s)-m(s))\}$$

is a continuous positive definite function on $T \times T$.

(C.3) For all $g \in G$,

$$R(t, s)=R(gt, gs), \qquad t, s \in T.$$

We denote by $L_2(X)$ a Hilbert space consisting of all random variables which may be represented either as a finite linear combinations $U=\sum_{j=1}^{n} c_j X(t_j)$, for some integer n, points $t_1, t_2, \cdots, t_n$ in T and scalars $c_1, c_2, \cdots, c_n$, or as a limit in quadratic mean of such finite linear combinations under the scalar product defined by $(U, W) =E\{U \cdot W\}$.

We denote by $H(R)$ a reproducing kernel Hilbert space generated by the kernel

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$R(s, t)$, $s, t \in T$. $H(R)$ is actually a Hilbert space consisting of functions on T satisfying the conditions:

(R.1) For each $t \in T$, $R(t, \cdot) \in H(R)$.

(R.2) For any $f \in H(R)$,

$$(f, R(t, \cdot))_R = f(t), \qquad t \in T,$$

where by $(\cdot, \cdot)_R$ we denote the scalar product in $H(R)$.

Suppose that the mean value function $m(\cdot)$ is known to be in a closed subspace Σ of $H(R)$ but the actual values of $m(t)$, $t \in T$, are not known.

The problem to find "*the uniformly minimum variance unbiased linear estimator* $\hat{m}(t) \in L_2(X)$ *for* $m(t)$ *at* t" in the sense that for each $t \in T$ the followings hold:

(E.1) $$E_m\{\hat{m}(t)\} = m(t) \qquad \text{for all } m \in \Sigma,$$

(E.2) For any unbiased estimate $\hat{\hat{m}}(t)$,

$$Var(\hat{m}(t)) \leqq Var(\hat{\hat{m}}(t)), \qquad \text{uniformly in } m \in \Sigma.$$

is completely solved (E. Parzen [2]), where by $E_m\{\cdot\}$ we denote the expectation is calculated by the probability with its mean value function $m(\cdot)$.

This solution $\hat{m}(t)$ is given by

$$\hat{m}(t) = \psi(Proj(R(t, \cdot) | \Sigma)), \qquad t \in T$$

where ψ is a congruence between $H(R)$ and $L_2(X)$, (E. Parzen [2], [3]), such that, for f, $g \in H(R)$,

(M.1) $$\psi(R(t, \cdot)) = X(t), \qquad t \in T,$$

(M.2) $$E_m\{\psi(f)\} = (f, m)_R, \qquad \text{for all } m \in \Sigma,$$

(M.3) $$Cov(\psi(f), \psi(g)) = (f, g)_R.$$

We consider here the case where the subspace Σ is invariant under K, that is, $\Sigma = \{f \in H(R) | f(t) = f(kt), k \in K\}$.

In this case, we shall see that the estimator $\hat{m}(t)$ can be expressed quite in a simple form by making use of zonal spherical functions.

Let $\{T^{(\lambda)}(g) = [T^{(\lambda)}_{ij}(g)], g \in G, \lambda = 1, 2, \cdots, 1 \leqq i, j \leqq d_\lambda\}$ be the complete system of unitary continuous non-equivalent representations of the group G and let us choose in the space of these representations a basis such that these representations decompose into irreducible representations of the subgroup K. Suppose, for instance, that the representation $T^{(\lambda)}$ of the group G contains r_λ times the identity representation of K. Suppose that in our basis $l_1, l_2, \cdots, l_{d_\lambda}$, these identity representations correspond to the first r_λ basis vectors so that $T^{(\lambda)}(k)l_j = l_j$ for $k \in K$ and $j = 1, 2, \cdots, r_\lambda$.

In this case the functions of t,

$$\Phi^{(\lambda)}_{ij}(t) = T^{(\lambda)}_{ij}(g); \qquad i = 1, 2, \cdots, d_\lambda,\ j = 1, 2, \cdots, r_\lambda,\ \lambda = 1, 2, 3, \cdots,$$

will be called spherical functions over T, while the functions

$$\Phi^{(\lambda)}_{ij}(t) = T^{(\lambda)}_{ij}(g); \qquad i, j = 1, 2, \cdots, r_\lambda,\ \lambda = 1, 2, 3, \cdots$$

will be called zonal spherical functions over T. (The matrix elements $T_{ij}^{(\lambda)}(g)$ are constant over all left cosets of G modulo K.)

Under these notations, we have the following theorem:

THEOREM: *Let the mean function* $m(\cdot)$ *be invariant under the subgroup* K *of* G, *that is,*

$$\Sigma = \{f \in H(R) \mid f(t) = f(kt),\ k \in K\}.$$

Then the uniformly minimum variance unbiased linear estimator $\hat{m}(t)$ *for* $m(t)$ *at* t *has the following expansion:*

$$\hat{m}(t) = \sum_{\lambda=1}^{\infty} \sum_{i=1}^{r_\lambda} \sum_{j=1}^{r_\lambda} X_{ij}^{(\lambda)} \cdot Q_{ij}^{(\lambda)}(t), \qquad t \in T$$

where

$$Q_{ij}^{(\lambda)}(t) = \Phi_{ij}^{(\lambda)}(t) / \|\Phi_{ij}^{(\lambda)}\|$$

$$X_{ij}^{(\lambda)} = \int_T X(s) Q_{ij}^{(\lambda)}(s) ds$$

$$\|\Phi_{ij}^{(\lambda)}\|^2 = \int_T |\Phi_{ij}^{(\lambda)}(s)|^2 ds.$$

PROOF. Let $r(\hat{g}) = R(\hat{g}t_0, t_0)$. Then, since for any $g \in G$ $R(t, s) = R(gt, gs)$, we see that for each $k \in K$ $r(\hat{g}) = r(k^{-1}\hat{g}k)$, that is, $r(\cdot)$ is a class function of K. Hence, from the Yaglom's result (Yoglem [4], Theorem 5, p. 604) the covariance function $R(t, s)$ can be represented in the form;

$$R(t, s) = \sum_{\lambda=1}^{\infty} \sum_{i=1}^{r_\lambda} \sum_{j=1}^{r_\lambda} f_{ij}^{(\lambda)} \cdot \sum_{l=1}^{d_\lambda} \Phi_{il}^{(\lambda)}(t) \Phi_{ij}^{(\lambda)}(s).$$

Hence, we have

$$\begin{aligned} R_1(t_1, \cdot) &= Proj\,(R(t, \cdot) \mid \Sigma) \\ &= \sum_{\lambda=1}^{\infty} \sum_{i=1}^{r_\lambda} \sum_{j=1}^{r_\lambda} f_{ij}^{(\lambda)} \sum_{l=1}^{r_\lambda} \Phi_{il}^{(\lambda)}(t) \cdot \Phi_{ij}^{(\lambda)}(\cdot). \end{aligned}$$

Since $\int_T \int_T R(t, s) \Phi_{ij}^{(\lambda)}(t) \Phi_{ij}^{(\lambda)}(s) dt ds = f_{ij}^{(\lambda)} \cdot \|\Phi_{ij}^{(\lambda)}\|^4 < \infty$,

$$U_{ij}^{(\lambda)} = \int_T X(s) \Phi_{ij}^{(\lambda)}(s) ds$$

is well-defined and therefore there exists a unique function $p_{ij}^{(\lambda)} \in H(R)$ such that $\psi(p_{ij}^{(\lambda)}) = U_{ij}^{(\lambda)}$.

The function $p_{ij}^{(\lambda)}$ can be easily determined and given by

$$p_{ij}^{(\lambda)}(t) = \|\Phi_{ij}^{(\lambda)}\|^2 \cdot \sum_{l=1}^{r_\lambda} f_{jl}^{(\lambda)} \Phi_{il}^{(\lambda)}(t).$$

Thus, we have

$$U_{ij}^{(\lambda)} = \|\Phi_{ij}^{(\lambda)}\|^2 \cdot \psi\Big(\sum_{l=1}^{r_\lambda} f_{jl}^{(\lambda)} \cdot \Phi_{il}^{(\lambda)}(\cdot)\Big).$$

Let us write

$$X_{ij}^{(\lambda)} = U_{ij}^{(\lambda)} / \|\Phi_{ij}^{(\lambda)}\| = \int_T X(s) Q_{ij}^{(\lambda)}(s) ds$$

where

$$Q_{ij}^{(\lambda)}(t) = \Phi_{ij}^{(\lambda)}(t)/\|\Phi_{ij}^{(\lambda)}\| .$$

Then, we have the following expansion

$$\hat{m}(t) = \psi(R_1(t, \cdot)) = \sum_{\lambda=1}^{\infty} \sum_{i=1}^{r_\lambda} \sum_{j=1}^{r_\lambda} \Phi_{li}^{(\lambda)}(t)\psi(\sum_{j=1}^{r_\lambda} f_{ij}^{(\lambda)} \Phi_{ij}^{(\lambda)}(\cdot))$$

$$= \sum_{\lambda=1}^{\infty} \sum_{l=1}^{r_\lambda} \sum_{i=1}^{r_\lambda} Q_{li}^{(\lambda)}(t) \cdot X_{li}^{(\lambda)} . \qquad \text{Q. E. D.}$$

EXAMPLE (NAGAI [1]). Let $T = \{(x, y, z) | x^2+y^2+z^2=1\}$ and $x = \sin\theta \cos\psi$, $y = \sin\theta \sin\psi$ and $z = \cos\theta$; $0 \leqq \theta \leqq \pi$, $0 \leqq \psi < 2\pi$. Let $\{X(t), t \in T\}$ be a homogeneous random field on T and its mean value function $m(\theta, \psi)$ be invariant under the rotation group $K = SO(2)$. Then the uniformly minimum variance unbiased linear estimator $\hat{m}(\theta, \psi)$ at $(\theta, \psi) \in T$ can be expressed as follows;

$$\hat{m}(\theta, \psi) = \sum_{v=0}^{\infty} X_v \cdot \psi_v(\theta) , \qquad 0 \leqq \theta \leqq \pi, \ 0 \leqq \psi \leqq 2\pi ,$$

where

$$\psi_v(\theta) = \sqrt{2v+1}\, P_v(\cos\theta) ,$$

$$X_v = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} X(\theta, \psi)\psi_v(\theta) \sin\theta d\psi\, d\theta$$

and

$$\{P_v(z), \quad |z| \leqq 1, \quad v = 0, 1, 2, \cdots\}$$

are Legendre's polynomials.

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