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ON A WALD'S EQUATION AND AVERAGE SAMPLE NUMBER IN A SEQUENTIAL SIGNAL DETECTION

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§ 1. Summary.

It is shown that in detecting sequentially a deterministic signal $\phi(t)$ in white noise $\eta(t)$ a similar identity (iii) in theorem 2.1, to the Wald's holds concerning a stopping time τ determined by making use of a likelihood ratio. It is also shown that τ has finite moments of any order under quite weak conditions over the signal. The exact A. S. N. $E\{\tau\}$ in a constant signal case has been obtained and given by (2, 8).

It is also considered a detection problem of a constant signal $\phi(t) \equiv \alpha$ in a coloured noise based on a sub-optimal statistic which become optimal when the noise were white. Similar properties of a stopping time τ to those in the white noise case have been obtained in theorem 3.1.

§ 2. Detection of a deterministic signal in a white noise.

We consider the following detection problem of a signal $\phi(t)$ in the white noise $\eta(t)$;

$$\begin{aligned} H_0; \quad x(t) &= W(t) \\ H_1; \quad x(t) &= m(t) + W(t), \end{aligned} \tag{2.1}$$

where $m(t) = \int_0^t \phi(s) ds$ is the integrated signal and $\{W(t), 0 \leq t < \infty\}$ is the Wiener process which is considered to be the integrated form of the white noise $\eta(t)$.

By H_0 we mean that there is no signal in the (integrated) observation $x(t)$ whose distribution is induced from the Wiener measure P_0 and by H_1 the observation $x(t)$ is the sum of the signal $m(t)$ and the noise $W(t)$ whose distribution is induced by P_1 , i.e. a shift of P_0 by $m(\cdot)$.

In order for the detection problem (2.1) to be non-singular, we assume that $\phi(\cdot)$ is square integrable on each finite interval $[0, t]$, $0 \leq t < \infty$.

Let us put

$$V(t) = \int_0^t |\phi(s)|^2 ds < \infty. \tag{2.2}$$

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Let \mathfrak{B}_t , $0 \leq t < \infty$, be the σ -field generated by the observation $\{x(s), 0 \leq s \leq t\}$ and P_{it} , $i=0, 1$, restrictions of P_i , $i=0, 1$, to \mathfrak{B}_t respectively.

P_{0t} and P_{1t} are equivalent for each t , and the logarithm of the likelihood ratio $L(x; t)$ of P_{1t} with respect to P_{0t} is given (See [5], [6]) by

$$L(x; t) = \int_0^t \phi(s) dx(s) - \frac{1}{2} V(t). \quad (2.3)$$

The statistic $L(x; t)$ is optimal in the sense that it will give the most powerful critical region in this detection problem for testing H_0 vs. H_1 based on $\{x(s), 0 \leq s \leq t\}$, (See [3]).

At first we set error probabilities to be equal to the prescribed value γ , ($0 < \gamma \leq 1/2$), that is,

$$P(\text{to accept } H_1 | H_0) = P(\text{to accept } H_0 | H_1) = \gamma. \quad (2.4)$$

We define a stopping time τ by

$$\tau = \inf \{t > 0; L(x; t) \leq -\lambda_0 \text{ or } L(x; t) \geq \lambda_1\}, \quad (2.5)$$

where λ_0 and λ_1 are positive constants such that our following decision rule satisfies (2.4).

Our decision rule based on the observations $\{x(s), 0 \leq s \leq t\}$ will be formulated as follows; When $L(x; \tau) = \lambda_1$, (or $-\lambda_0$), we stop sampling at $t = \tau$ and decide H_1 , (or H_0), to be true, while as long as $-\lambda_0 < L(x; s) < \lambda_1$, $0 \leq s \leq t$, we continue sampling.

Since each distribution of $L(x; t)$ under H_i , $i=0, 1$, is symmetric to the other, the thresholds $-\lambda_0$ and λ_1 must be, under the condition (2.4), symmetric, that is, $\lambda_0 = \lambda_1$.

Let F_t be the σ -field generated by $\{W(s), 0 \leq s \leq t\}$ and let us put

$$y(t) = \int_0^t \phi(s) dw(s), \quad 0 \leq t < \infty. \quad (2.6)$$

Then we have

LEMMA 2.1. $\{y(t), F_t, 0 \leq t < \infty\}$ is a Gaussian Martingale with the mean-value zero, its covariance function $R_y(t, s) = V(\min(t, s))$ and its realizations are continuous with probability one.

PROOF. Clear.

From the symmetricity of the distribution of $L(x; t)$, we may and do proceed our discussion under the assumption that H_0 is always true.

We have the following evaluation of the tail probability of τ :

LEMMA 2.2. For sufficiently large t ,

$$P(\tau > t) \leq \frac{2}{\pi} \frac{\sqrt{V(t)}}{(V(t) - 2\lambda_0)} \exp\left\{-\frac{V(t) - 2\lambda_0}{8V(t)}\right\}. \quad (2.7)$$

PROOF. Since $[\tau > t] \subset [|\gamma(t) - \frac{1}{2} V(t)| < \lambda_0]$, we have from lemma 2.1,

$$P(\tau > t) \leq \int_{-\lambda_0 + \frac{1}{2} V(t)}^{\infty} \frac{1}{\sqrt{2\pi V(t)}} \exp\left[-\frac{y^2}{2V(t)}\right] dy.$$

For a large t such that $V(t) > 2\lambda_0$, the inequality (2.7) easily follows. Q. E. D.

LEMMA 2.3. *If there is a positive constant $\alpha > 0$, whatever small it is, such that the signal power $V(t)$ diverges to infinity with the same order as $O(t^\alpha)$ or faster, then for all positive $\beta > 0$, $E\{\tau^\beta\} < \infty$.*

PROOF. Let $F(t)$ be the c.d.f. of τ . From the assumption that $V(t) = O(t^\alpha)$, we can find positive numbers T_0 and A^* such that $\sqrt{V(t)} - 2\lambda_0 / \sqrt{V(t)} \geq A^* t^{\alpha/2}$, for all $t \geq T_0$. It is enough for us to show that $\int_{T_0}^{\infty} t^\beta dF(t) < \infty$, for all $\beta > 0$. Indeed, it is easily seen that the integral is dominated by a convergent series $K_0 \sum_{\nu=1}^{\infty} (1+\nu)^\beta e^{-K_1 \nu^\alpha} < \infty$, where K_0 and K_1 are suitably chosen positive constants. Q. E. D.

Let us put

$$U(t) = y(t)^2 - V(t), \quad 0 \leq t < \infty,$$

and for each λ , $-\infty < \lambda < \infty$,

$$Z(t, \lambda) = \exp \left\{ \lambda y(t) - \frac{\lambda^2}{2} V(t) \right\}, \quad 0 \leq t < \infty.$$

Then, we have

LEMMA 2.4. $\{U(t), F_t, 0 \leq t < \infty\}$ is a martingale with the mean value zero and $\{Z(t, \lambda), F_t, 0 \leq t < \infty\}$ is also a martingale with the mean value 1 for each real λ .

PROOF. It is clear that $E\{U(t)\} = 0$. Let us put $\xi(s, t) = \int_s^t \phi(u) dW(u)$. Then

$$U(t+h) = U(t) + 2y(t)\xi(t, t+h) + \{\xi(t, t+h)\}^2 - V(t+h) + V(t).$$

Thus, we have

$$E\{U(t+h) | F_t\} = U(t), \quad \text{a. s.}$$

On the other hand, $E\{e^{\lambda y(t)}\} = \exp \left\{ -\frac{\lambda^2}{2} V(t) \right\}$, and hence $E\{Z(t, \lambda)\} \equiv 1$, for each real λ . Since it is written as follows:

$$Z(t+h, \lambda) = Z(t, \lambda) \exp \left\{ \lambda \xi(t, t+h) - \frac{\lambda^2}{2} [V(t+h) - V(t)] \right\},$$

we have

$$E\{Z(t+h, \lambda) | F_t\} = Z(t, \lambda), \quad \text{a. s.}$$

This shows that $\{Z(t, \lambda), F_t, 0 \leq t < \infty\}$ is a martingale for each real λ . Q. E. D.

By noticing that τ is the Brownian stopping time, that is $\{\tau > t\} \in F_t$ for each t , we have

- THEOREM 2.1. (i) $E\{y(\tau)\} = 0$,
(ii) $E\{|y(\tau)|^2\} = E\{V(\tau)\}$, and
(iii) $E\left\{\exp \left\{ \lambda y(\tau) - \frac{\lambda^2}{2} V(\tau) \right\}\right\} = 1$, for each real λ .

PROOF. Let us define a sequence of stopping times τ_n by

$$\tau_n = \min(n, \tau), \quad n = 1, 2, \dots$$

Let $\check{y}_n(t)$, $n = 1, 2, \dots$ be a sequence of stopped processes of $y(t)$ by τ_n , that is, $\check{y}_n(t) = y(t)$ for $t < \tau_n$; $= y(\tau_n)$ for $t \geq \tau_n$ and $\check{\mathfrak{B}}_t^{(n)}$ the σ -field generated by τ_n , that is,

the totality of measurable sets A whose intersection with $\{\min(t, \tau_n) \leq s\}$ belongs to F_s for each s , $0 \leq s < \infty$.

Since τ_n is bounded a.s. for each n , it is seen that $\{\check{y}_n(t), \check{\mathfrak{B}}_t^{(n)}, 0 \leq t < \infty\}$ is a martingale and $E\{\check{y}_n(t)\} = \sup_{t'} E\{y(t')\} = 0$ for all t , $0 \leq t < \infty$. (See [1]).

Hence, for all $t > n$, ($n = 1, 2, \dots$),

$$\begin{aligned} E\{\check{y}_n(t)\} &= E\{y(\tau_n)\} = \int_{[\tau \leq n]} y(\tau) dP + \int_{[\tau > n]} y(n) dP \\ &= E\{y(t)\} = 0. \end{aligned}$$

Since for $\tau > n$, $|y(n)| \leq \lambda_0 + \frac{1}{2} V(n)$, we have

$$\begin{aligned} \left| \int_{[\tau > n]} y(n) dP \right| &\leq \left(\lambda_0 + \frac{1}{2} V(n) \right) P(\tau > n) \\ &\leq \text{Const.} \times \frac{\sqrt{V(n)}(V(n) + 2\lambda_0)}{V(n) - 2\lambda_0} \exp \left\{ -\frac{(V(n) - 2\lambda_0)^2}{8V(n)} \right\} \\ &\longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, we have

$$E\{y(\tau)\} = \lim_{n \rightarrow \infty} E\{y(\tau_n)\} = 0.$$

Similarly, we write

$$\begin{aligned} \check{U}_n(t) &= U(t) \quad \text{for } t < \tau_n \\ &= U(\tau_n) \quad \text{for } t \geq \tau_n \\ \check{Z}_n(t, \lambda) &= Z(t, \lambda) \quad \text{for } t < \tau_n \\ &= Z(\tau_n, \lambda) \quad \text{for } t \geq \tau_n. \end{aligned}$$

We have then new martingale processes $\{\check{U}_n(t), \check{\mathfrak{B}}_t^{(n)}, 0 \leq t < \infty\}$ and $\{\check{Z}_n(t, \lambda), \check{\mathfrak{B}}_t^{(n)}, 0 \leq t < \infty\}$. Therefore we have

$$\begin{aligned} E\{U_n(t)\} &= \int_{[\tau \leq n]} U(\tau) dP + \int_{[\tau > n]} U(n) dP \\ &= \int_{[\tau \leq n]} \{y^2(\tau) - V(\tau)\} dP + \int_{[\tau > n]} \{y(n)^2 - V(n)\} dP \\ &= 0. \end{aligned}$$

Since,

$$\begin{aligned} \left| \int_{[\tau > n]} \{y(n)^2 - V(n)\} dP \right| &\leq \left\{ V(n) + \frac{1}{4} (2\lambda_0 + V(n))^2 \right\} \cdot P(\tau > n) \\ &\longrightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} \int_{[\tau \leq n]} [y^2(\tau) - V(\tau)] dP = E\{(y(\tau))^2\} - E\{V(\tau)\} = 0.$$

Similarly we have

$$1 = E\{\tilde{Z}_n(t, \lambda)\} = \int_{[\tau \leq n]} e^{\lambda y(\tau) - \frac{\lambda^2}{2} V(\tau)} dP \\ + \int_{[\tau > n]} e^{\lambda y(n) - \frac{\lambda^2}{2} V(n)} dP.$$

Since, for each real λ ,

$$\left| \int_{[\tau > n]} \exp \left\{ \lambda y(n) - \frac{\lambda^2}{2} V(n) \right\} dP \right| \\ \leq \text{Const.} \times \frac{\sqrt{V(n)}}{|V(n) - 2\lambda_0|} \times \exp \left\{ -\frac{1}{8} \left[(1 - 2\lambda)^2 V(n) - 4\lambda_0 + \frac{4\lambda_0^2}{V(n)} \right] \right\} \\ \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

it follows immediately that for each real λ

$$1 = \lim_{n \rightarrow \infty} \int_{[\tau \leq n]} \exp \left\{ \lambda y(\tau) - \frac{\lambda^2}{2} V(\tau) \right\} dP \\ = E \left\{ \exp \left\{ \lambda y(\tau) - \frac{\lambda^2}{2} V(\tau) \right\} \right\}. \quad \text{Q. E. D.}$$

EXAMPLE 2.1. (c.f. [2], [7]). From theorem 2.1, it is easily obtain the A.S. N.'s $E\{\tau|H_0\}$ and $E\{\tau|H_1\}$ of our detection problem when the signal $\phi(s)$ is constant $\alpha > 0$ which is in the white noise. From (2, 4), it is well known that λ_0 is given by

$$\lambda_0 = \log \left(\frac{1-\gamma}{\gamma} \right).$$

Let $E_1 = \{W(\tau) = \frac{\lambda_0}{\alpha} + \frac{\alpha}{2}\}$ and E_2 be the complementary event of E_1 . Ther., since τ is define by

$$\tau = \inf \left\{ t > 0; \left| W(t) - \frac{\alpha}{2} t \right| \geq \lambda_0 / \alpha \right\},$$

we have

$$E\{y(\tau)\} = \gamma E \left\{ \lambda_0 + \frac{\alpha^2}{2} \cdot \tau | E_1 \right\} + (1-\gamma) E \left\{ -\lambda_0 + \frac{\alpha^2}{2} \cdot \tau | E_2 \right\} \\ = \frac{\alpha^2}{2} E\{\tau\} - (1-2\gamma)\lambda_0 = 0,$$

that is,

$$E\{\tau\} = E\{\tau|H_0\} = E\{\tau|H_1\} \\ = \frac{2}{\alpha^2} (1-2\gamma) \log \left(\frac{1-\gamma}{\gamma} \right). \quad (2.8)$$

EXAMPLE 2.2. Let $\Delta > 0$ be a suitably chosen small interval and let us put

$$\phi_j(s) = 1 \quad \text{for } (j-1)\Delta \leq s < j\Delta, \quad j = 1, 2, \dots, \\ = 0 \quad \text{otherwise.}$$

Let us assume that $\phi(s)$ is a pulsed signal and is approximately expressed by

$$\phi(s) = \sum_{j=1}^{\infty} h \cdot \epsilon_j \cdot \phi_j(s), \quad 0 \leq s < \infty,$$

where $h > 0$ is a given constant and $\varepsilon_j = 0$ or 1 , $j = 1, 2, 3, \dots$.

We also assume that a relative frequency of the occurrence of pulses $(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n)/n$ is close to a constant a , ($0 < a \leq 1$), except for the first several n .

Then, we have approximately

$$V(t) \doteq \alpha^2 a t, \quad \text{and} \quad y(t) = h \sum_{j=1}^n \varepsilon_j z_j + h \varepsilon_{n+1} [W(t) - W(n\Delta)], \quad (2.9)$$

where $z_j = W(j\Delta) - W((j-1)\Delta)$ and n is the largest integer not greater than t/Δ .

It is clear that $\{y(t), 0 \leq t < \infty\}$ is a Gaussian process with the mean value zero and its covariance function $R(s, t)$ is given approximately by

$$R(s, t) \doteq ah^2 \min(s, t).$$

Hence, this detection problem of the pulsed signal $\phi(s)$ in the white noise is nearly equivalent to the problem to detect a constant signal $\phi_0(s) \equiv h \cdot \sqrt{a}$ in the white noise as shown in Example 2.1.

Thus, we have the A.S.N. which are given approximately by

$$E\{\tau | H_0\} = E\{\tau | H_1\} \doteq \frac{2(1-2\gamma)}{ah^2} \cdot \log\left(\frac{1-\gamma}{\gamma}\right).$$

§ 3. Detection of a constant signal in a non-white Gaussian noise.

Similary in § 2, let $\{W(s), F_s, 0 \leq s < \infty\}$ be the Wiener process.

Let $\{\xi(s), 0 \leq s < \infty\}$ be a Gaussian noise process defined by

$$\xi(t) = \int_0^t e^{-\beta(t-u)} dW(u), \quad 0 \leq t < \infty, \dots \quad (3.1)$$

where $\beta \geq 0$ is a non-negative constant.

We shall consider the following detection problem;

$$\begin{aligned} H_0: \quad x(t) &= \xi(t), \\ H_1: \quad x(t) &= \alpha t + \xi(t), \end{aligned} \quad (3.2)$$

where $\alpha > 0$, is a constant signal to be detected.

Let P_{it} , $i = 0, 1$, be the distribution of the observation $\{x(s), 0 \leq s \leq t\}$ under H_i , $i = 0, 1$, respectively.

Then, the detection problem (3.2) is nonsingular and the logarithm of the likelihood ratio of P_{1t} with respect to P_{0t} is given by

$$\begin{aligned} L(x; t) &= \log \frac{dP_{1t}}{dP_{0t}}(x) \\ &= L_0(x; t) + \frac{\beta}{2} \cdot \{2m(t)x(t) - (m(t))^2\} \\ &\quad + \frac{\beta^2}{2} \left\{ 2 \int_0^t m(s)x(s)ds - \int_0^t (m(s))^2 ds \right\}, \end{aligned} \quad (3.3)$$

where $m(t) = \alpha t$, and

$$L_0(x; t) = \alpha x(t) - \frac{\alpha^2}{2} t. \quad (3.4)$$

Suppose that we adopt $L_0(x; t)$ as a statistic for the problem (3.2) instead of the log-likelihood ratio $L(x; t)$.

$L_0(x; t)$ is actually not optimal for the problem (3.2) (See [3]) but it has several sub-optimal properties because it become optimal when $\beta = 0$, that is, the noise $\xi(t)$ were white, and $L_0(x; t)$ does not contain the parameter β which is intrinsic in the noise.

We set error probabilities to the equal to the prescribed value γ , that is,

$$P(\text{to accept } H_1 | H_0) = P(\text{to accept } H_0 | H_1) = \gamma, \quad (3.5)$$

and consider only such decision rules that (3.5) holds.

We define a stopping τ^* by

$$\tau^* = \inf \{t > 0; L_0(x; t) \leq -\lambda_0 \text{ or } L_0(x; t) \geq \lambda_1\} \quad (3.6)$$

where λ_0, λ_1 are positive constants such that our following decision rule satisfies (3.5).

Our decision rule based on the observation $\{x(s), 0 \leq s \leq t\}$ will be formulated as follows; When $L_0(x; \tau^*) = \lambda_1$, (or $-\lambda_0$), we stop sampling at $t = \tau^*$ and decide H_1 , (or H_0) to be true, while as long as $-\lambda_0 < L_0(x, s) < \lambda_1$, $0 \leq s \leq t$, we continue sampling.

Since each distribution of $L_0(x; t)$ under H_i , $i = 0, 1$, is symmetric to the other, the constants λ_0 and λ_1 must be equal under the condition (3.5).

Let us put

$$V(t) = E\{\xi(t)^2\} = (1 - e^{-2\beta t})/2\beta.$$

Let us consider a continuous function $f(t)$, $0 \leq t < \infty$, such that

- (i) $f(0) = -\lambda^* < 0$,
- (ii) $f(t) = O(t^\alpha) \nearrow \infty$ as $t \rightarrow \infty$,

for some positive constant $\alpha > 0$.

We shall now define a stopping time τ as follows;

$$\tau = \inf \{t > 0; \xi(t) \leq f(t) \text{ or } \xi(t) \geq 2\lambda^* + f(t)\}. \quad (3.7)$$

The stopping time τ^* defined by (3.6) is a special case of τ by (3.7). Indeed, τ^* corresponds to the case where $f(t) = \alpha t/2 - \lambda^*$ and $\lambda^* = \lambda_0/\alpha$.

For a large t , it is easily seen that

$$\begin{aligned} P(\tau \geq t) &\leq P(f(t) \leq \xi(t) \leq 2\lambda^* + f(t)) \\ &\leq P(\xi(t) \geq f(t)) \\ &\leq \frac{1}{2\sqrt{\pi\beta}} \times \frac{1}{|f(t)|} \times \exp\{-\beta \times [f(t)]^2\}. \end{aligned}$$

Thus, we have

LEMMA 3.1. For all real $k \geq 0$,

$$E\{\tau^k\} < \infty.$$

Proof of lemma 3.1 is analogous to that of lemma 2.3.

Now, we obtain

THEOREM 3.1. Let τ be defined by (3.7). Then we have

$$(i) \quad E\{\xi(\tau)\} = 0,$$

$$(ii) \quad E\{\xi(\tau)^2\} = E\{V(\tau)\}, \quad \text{and}$$

$$(iii) \quad \text{for each real } \lambda,$$

$$E\left\{\exp\left[\lambda\xi(\tau) - \frac{\lambda^2}{2} V(\tau)\right]\right\} = 1. \quad (3.8)$$

PROOF. It is clear that τ is a stopping time with respect to F_t , $t \geq 0$. The stochastic process $\{\xi(t), 0 \leq t < \infty\}$ is the unique non-anticipated solution of a stochastic differential equation;

$$d\xi(t) = -\beta\xi(t)dt + dW(t), \quad (3.9)$$

with $\xi(0) = 0$. Hence, it enjoys the strong Markov property with respect to a Brownian stopping time, for example, say, τ . (See [4]).

For any random variable g and any measurable set A , we will write

$$E_A\{g\} = E\{I_A \cdot g\},$$

where I_A is the indicator function of A .

Let \mathfrak{B}_τ be the σ -field generated by τ , that is, the totality of sets whose intersections with $[\tau > t]$ belong to F_t for every t , $0 \leq t < \infty$. Then, we have from the strong Markov property,

$$\begin{aligned} E_{[\tau \leq t]} \{\xi(t)\} &= E\{E\{I_{[\tau \leq t]} \cdot \xi(t) | \mathfrak{B}_\tau\}\} \\ &= E\{I_{[\tau \leq t]} \cdot E\{\xi(t) | \tau, \xi(\tau)\}\} \\ &= E_{[\tau \leq t]} \{\xi(\tau)\}. \end{aligned}$$

Since for $\tau > t$, $f(t) < \xi(t) < 2\lambda^* + f(t)$, we have

$$\begin{aligned} |E_{[\tau > t]} \{\xi(t)\}| &\leq [2\lambda^* + f(t)] \cdot P(\tau > t) \\ &\leq \frac{1}{2\sqrt{\pi\beta}} \times \frac{|2\lambda^* + f(t)|}{|f(t)|} \times \exp\{-\beta|f(t)|^2\} \\ &\longrightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus, it is seen that

$$\begin{aligned} E\{\xi(t)\} &= \lim_{t \rightarrow \infty} E_{[\tau \leq t]} \{\xi(\tau)\} + \lim_{t \rightarrow \infty} E_{[\tau > t]} \{\xi(t)\} \\ &= E\{\xi(\tau)\} = 0. \end{aligned}$$

We have shown that (i) holds.

Let us write

$$U(t) = \xi(t)^2 - V(t).$$

Then, $U(t)$ is a functional of the Markov process $\{\xi(s), 0 \leq s < \infty\}$ and hence we have

$$\begin{aligned} E_{[\tau \leq t]} \{U(t)\} &= E\{E\{I_{[\tau \leq t]} \cdot U(t) | \mathfrak{B}_\tau\}\} \\ &= E\{I_{[\tau \leq t]} \cdot E\{U(t) | \tau, \xi(\tau)\}\} \\ &= E_{[\tau \leq t]} \{U(\tau)\} = E_{[\tau \leq t]} \{\xi(\tau)^2 - V(\tau)\}. \end{aligned}$$

Since, for $\tau > t$, $|\xi(t)| \leq 2\lambda^* + f(t)$, it follows that

$$\begin{aligned} |E_{[\tau > t]} \{U(t)\}| &\leq [V(t) + (2\lambda^* + f(t))^2] \cdot P(\tau > t) \\ &\leq \frac{1}{2\sqrt{\pi}\beta} \cdot \frac{[V(t) + (2\lambda^* + f(t))^2]}{|f(t)|} \cdot e^{-\beta|f(t)|^2} \\ &\longrightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus, we have

$$\begin{aligned} E\{U(t)\} &= \lim_{t \rightarrow \infty} E_{[\tau \leq t]} \{U(\tau)\} + \lim_{t \rightarrow \infty} E_{[\tau > t]} \{U(t)\} \\ &= E\{\xi(\tau)^2 - V(\tau)\} = 0. \end{aligned}$$

We have shown that $E\{\xi(\tau)^2\} = E\{V(\tau)\}$.

Let us put for each real λ ,

$$Z(t, \lambda) = \exp\left[\lambda\xi(t) - \frac{\lambda^2}{2} V(t)\right], \quad 0 \leq t < \infty.$$

Then, it is clear that $Z(t, \lambda)$ is F_t -measurable and $E\{Z(t, \lambda)\} \equiv 1$.

Thus, we have

$$\begin{aligned} E_{[\tau \leq t]} \{Z(t, \lambda)\} &= E\{E\{I_{[\tau \leq t]} Z(t, \lambda) | \mathfrak{B}_\tau\}\} \\ &= E\{I_{[\tau \leq t]} E\{Z(t, \lambda) | \tau, \xi(\tau)\}\} \\ &= E_{[\tau \leq t]} \{Z(\tau, \lambda)\}. \end{aligned}$$

Now, we shall evaluate $E_{[\tau > t]} \left\{ \exp\left[\lambda\xi(t) - \frac{\lambda^2}{2} V(t)\right] \right\}$.

Since, for $\tau > t$, $f(t) < \xi(t) < 2\lambda^* + f(t)$, we have for each non-negative real λ ,

$$\begin{aligned} E_{[\tau > t]} \{Z(t, \lambda)\} &\leq \exp\left\{\lambda[2\lambda^* + f(t)] - \frac{\lambda^2}{2} V(t)\right\} \cdot P(\tau > t) \\ &\leq \frac{1}{2|f(t)|\sqrt{\pi}\beta} \exp\left\{2\lambda\lambda^* - \beta|f(t)|^2\left(1 - \frac{\lambda}{\beta f(t)}\right)\right\} \\ &\longrightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned}$$

and for each negative real λ ,

$$\begin{aligned} E_{[\tau > t]} \{Z(t, \lambda)\} &\leq \exp\left\{\lambda f(t) - \frac{\lambda^2}{2} V(t)\right\} \\ &\longrightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus, it follows that for each real λ ,

$$\begin{aligned} E\{Z(t, \lambda)\} &= \lim_{t \rightarrow \infty} E_{[\tau \leq t]} \{Z(\tau, \lambda)\} + \lim_{t \rightarrow \infty} E_{[\tau > t]} \{Z(t, \lambda)\} \\ &= E\{Z(\tau, \lambda)\} \equiv 1. \end{aligned}$$

This completes the proof of theorem 3.1.

Q. E. D.

The stopping time τ^* is the special case of τ in (3.7) and hence from theorem 3.1 it is seen that

$$\begin{aligned} E\{\xi(\tau^*)\} &= 0, \\ E\{\xi(\tau^*)^2\} &= E\{V(\tau^*)\}, \end{aligned}$$

and for each real λ ,

$$E\left\{\exp\left[\lambda\xi(\tau^*) - \frac{\lambda^2}{2} V(\tau^*)\right]\right\} \equiv 1.$$

COROLLARY.

$$E\{\tau^*|H_0\} = E\{\tau^*|H_1\} = 2\lambda_0(1-2\gamma)/\alpha^2,$$

where λ_0 is such a constant that the error probabilities satisfy (3.5).

PROOF. Let $E_0 = \{\xi(\tau^*) = \frac{\alpha}{2}\tau^* - \frac{\lambda_0}{\alpha}\}$ and E_1 be the complementary event of E_0 .

Then, by noticing that τ^* is equal to τ when $f(t) = \frac{\alpha}{2}t - \lambda^*$ and $\lambda^* = \lambda_0/\alpha$. and also that $P(E_1|H_0) = \gamma$ and $P(E_0|H_0) = 1 - \gamma$, it follows from theorem 3.1 that

$$\begin{aligned} E\{\xi(\tau^*)\} &= \frac{\alpha}{2} [E\{\tau^*|E_0\} \cdot P(E_0|H_0) + E\{\tau^*|E_1\} \cdot P(E_1|H_0)] \\ &\quad - \frac{\lambda_0}{\alpha} P(E_0|H_0) + \frac{\lambda_0}{\alpha} P(E_1|H_0) \\ &= \frac{\alpha}{2} E\{\tau^*|H_0\} - \frac{\lambda_0}{\alpha} (1-2\gamma) = 0. \end{aligned}$$

Thus, we have proved corollary.

Q. E. D.

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