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Nagai, Takeaki
Faculty of Engineering Science, Osaka University

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ON A WALD'S EQUATION AND AVERAGE SAMPLE NUMBER IN A SEQUENTIAL SIGNAL DETECTION

By

Takeaki NAGAI*

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§ 1. Summary.

It is shown that in detecting sequentially a deterministic signal $\phi(t)$ in white noise $\eta(t)$ a similar identity (iii) in theorem 2.1, to the Wald's holds concerning a stopping time τ determined by making use of a likelihood ratio. It is also shown that τ has finite moments of any order under quite weak conditions over the signal. The exact A. S. N. $E\{\tau\}$ in a constant signal case has been obtained and given by (2, 8).

It is also considered a detection problem of a constant signal $\phi(t)\equiv\alpha$ in a coloured noise based on a sub-optimal statistic which become optimal when the noise were white. Similar properties of a stopping time τ to those in the white noise case have been obtained in theorem 3.1.

§ 2. Detection of a deterministic signal in a white noise.

We consider the following detection problem of a signal $\phi(t)$ in the white noise $\eta(t)$;

$$H_0;\quad x(t)=W(t)$$
$$H_1;\quad x(t)=m(t)+W(t), \tag{2.1}$$

where $m(t)=\int_0^t \phi(s)ds$ is the integrated signal and $\{W(t),\ 0\leqq t<\infty\}$ is the Wiener process which is considered to be the integrated form of the white noise $\eta(t)$.

By H_0 we mean that there is no signal in the (integrated) observation $x(t)$ whose distribution is induced from the Wiener measure P_0 and by H_1 the observation $x(t)$ is the sum of the signal $m(t)$ and the noise $W(t)$ whose distribution is induced by P_1, i. e. a shift of P_0 by $m(\cdot)$.

In order for the detection problem (2.1) to be non-singular, we assume that $\phi(\cdot)$ is square integrable on each finite interval $[0, t]$, $0\leqq t<\infty$.

Let us put

$$V(t)=\int_0^t |\phi(s)|^2 ds<\infty\,. \tag{2.2}$$

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* Faculty of Engineering Science, Osaka University, Toyonaka, Osaka.

Let $\mathfrak{B}_t$, $0 \leqq t < \infty$, be the σ-field generated by the observation $\{x(s),\ 0 \leqq s \leqq t\}$ and P_{it}, $i=0, 1$, restrictions of P_i, $i=0, 1$, to $\mathfrak{B}_t$ respectively.

P_{0t} and P_{1t} are equivalent for each t, and the logarithm of the likelihood ratio $L(x; t)$ of P_{1t} with respect to P_{0t} is given (See [5], [6]) by

$$L(x; t)=\int_0^t \phi(s)dx(s)-\frac{1}{2} V(t). \tag{2.3}$$

The statistic $L(x; t)$ is optimal in the sense that it will give the most powerful critical region in this detection problem for testing H_0 *vs.* H_1 based on $\{x(s),\ 0 \leqq s \leqq t\}$, (See [3]).

At first we set error probabilities to be equal to the prescribed value γ, $(0 < \gamma \leqq 1/2)$, that is,

$$P(\textit{to accept } H_1 | H_0) = P(\textit{to accept } H_0 | H_1) = \gamma. \tag{2.4}$$

We define a stopping time τ by

$$\tau = \inf\{t > 0;\ L(x; t) \leqq -\lambda_0 \text{ or } L(x; t) \geqq \lambda_1\}, \tag{2.5}$$

where λ_0 and λ_1 are positive constants such that our following decision rule satisfies (2.4).

Our decision rule based on the observations $\{x(s),\ 0 \leqq s \leqq t\}$ will be formulated as follows; When $L(x; \tau)=\lambda_1$, (or $-\lambda_0$), we stop sampling at $t=\tau$ and decide H_1, (or H_0), to be true, while as long as $-\lambda_0 < L(x; s) < \lambda_1$, $0 \leqq s \leqq t$, we continue sampling.

Since each distribution of $L(x; t)$ under H_i, $i=0, 1$, is symmetric to the other, the thresholds $-\lambda_0$ and λ_1 must be, under the condition (2.4), symmetric, that is, $\lambda_0 = \lambda_1$.

Let F_t be the σ-field generated by $\{W(s), 0 \leqq s \leqq t\}$ and let us put

$$y(t)=\int_0^t \phi(s)dw(s), \qquad 0 \leqq t < \infty. \tag{2.6}$$

Then we have

LEMMA 2.1. *$\{y(t), F_t, 0 \leqq t < \infty\}$ is a Gaussian Martingale with the mean-value zero, its cavariance function $R_y(t, s)=V(min(t, s))$ and its realizations are continuous with probability one.*

PROOF. Clear.

From the symmetricity of the distribution of $L(x; t)$, we may and do proceed our discussion under the assumption that H_0 is always true.

We have the following evaluation of the tail probability of τ:

LEMMA 2.2. *For sufficiently large t,*

$$P(\tau > t) \leqq \frac{2}{\pi} \frac{\sqrt{V(t)}}{(V(t)-2\lambda_0)} exp\left\{-\frac{V(t)-2\lambda_0}{8V(t)}\right\}. \tag{2.7}$$

PROOF. Since $[\tau > t] \subset \left[\,|y(t)-\frac{1}{2}V(t)| < \lambda_0\right]$, we have from lemma 2.1,

$$P(\tau > t) \leqq \int_{-\lambda_0+\frac{1}{2}V(t)}^{\infty} \frac{1}{\sqrt{2\pi V(t)}} exp\left[-\frac{y^2}{2V(t)}\right] dy.$$

For a large t such that $V(t) > 2\lambda_0$, the inequality (2.7) easily follows. Q. E. D.

LEMMA 2.3. *If there is a positive constant $\alpha > 0$, whatever small it is, such that the signal power $V(t)$ diverges to infinity with the same order as $O(t^\alpha)$ or faster, then for all positive $\beta > 0$, $E\{\tau^\beta\} < \infty$.*

PROOF. Let $F(t)$ be the c. d. f. of τ. From the assumption that $V(t) = O(t^\alpha)$, we can find positive numbers T_0 and A^* such that $\sqrt{V(t)} - 2\lambda_0/\sqrt{V(t)} \geqq A^* t^{\alpha/2}$, for all $t \geqq T_0$. It is enough for us to show that $\int_{T_0}^{\infty} t^\beta dF(t) < \infty$, for all $\beta > 0$. Indeed, it is easily seen that the integral is dominated by a convergent series $K_0 \sum_{\nu=1}^{\infty} (1+\nu)^\beta e^{-K_1 \nu^\alpha} < \infty$, where K_0 and K_1 are suitably chosen positive constants. Q. E. D.

Let us put

$$U(t) = y(t)^2 - V(t), \qquad 0 \leqq t < \infty,$$

and for each λ, $-\infty < \lambda < \infty$,

$$Z(t, \lambda) = exp\left\{\lambda y(t) - \frac{\lambda^2}{2} V(t)\right\}, \qquad 0 \leqq t < \infty.$$

Then, we have

LEMMA 2.4. *$\{U(t), F_t, 0 \leqq t < \infty\}$ is a martingale with the mean value zero and $\{Z(t, \lambda), F_t, 0 \leqq t < \infty\}$ is also a martingale with the mean value 1 for each real λ.*

PROOF. It is clear that $E\{U(t)\} = 0$. Let us put $\xi(s, t) = \int_s^t \phi(u) dW(u)$. Then

$$U(t+h) = U(t) + 2y(t)\xi(t, t+h) + \{\xi(t, t+h)\}^2 - V(t+h) + V(t).$$

Thus, we have

$$E\{U(t+h) | F_t\} = U(t), \qquad \text{a. s.}$$

On the other hand, $E\{e^{\lambda \cdot y(t)}\} = exp\left\{\frac{\lambda^2}{2} V(t)\right\}$, and hence $E\{Z(t, \lambda)\} \equiv 1$, for each real λ. Since it is written as follows:

$$Z(t+h, \lambda) = Z(t, \lambda)\, exp\left\{\lambda \xi(t, t+h) - \frac{\lambda^2}{2} [V(t+h) - V(t)]\right\},$$

we have

$$E\{Z(t+h, \lambda) | F_t\} = Z(t, \lambda), \qquad \text{a. s.}$$

This shows that $\{Z(t, \lambda), F_t, 0 \leqq t < \infty\}$ is a martingale for each real λ. Q. E. D.

By noticing that τ is the Brownian stopping time, that is $\{\tau > t\} \in F_t$ for each t, we have

THEOREM 2.1. (i) $E\{y(\tau)\} = 0$,

(ii) $E\{|y(\tau)|^2\} = E\{V(\tau)\}$, *and*

(iii) $E\left\{exp\left\{\lambda \cdot y(\tau) - \frac{\lambda^2}{2} \cdot V(\tau)\right\}\right\} = 1$, *for each real λ.*

PROOF. Let us define a sequence of stopping times τ_n by

$$\tau_n = min(n, \tau), \qquad n = 1, 2, \cdots.$$

Let $\breve{y}_n(t)$, $n = 1, 2, \cdots$ be a sequence of stopped processes of $y(t)$ by τ_n, that is, $\breve{y}_n(t) = y(t)$ for $t < \tau_n$; $= y(\tau_n)$ for $t \geqq \tau_n$ and $\breve{\mathfrak{B}}_t^{(n)}$ the σ-field generated by τ_n, that is, the totality of measurable sets A whose intersection with $\{min(t, \tau_n) \leqq s\}$ belongs to F_s for each s, $0 \leqq s < \infty$.

Since τ_n is bounded a. s. for each n, it is seen that $\{\breve{y}_n(t), \breve{\mathfrak{B}}_t^{(n)}, 0 \leqq t < \infty\}$ is a martingale and $E\{\breve{y}_n(t)\} = \sup_{t'} E\{y(t')\} = 0$ for all t, $0 \leqq t < \infty$. (See [1]).

Hence, for all $t > n$, $(n = 1, 2, \cdots)$,

$$E\{\breve{y}_n(t)\} = E\{y(\tau_n)\} = \int_{[\tau \leqq n]} y(\tau) dP + \int_{[\tau > n]} y(n) dP.$$

$$= E\{y(t)\} = 0.$$

Since for $\tau > n$, $|y(n)| \leqq \lambda_0 + \frac{1}{2} V(n)$, we have

$$\left| \int_{[\tau > n]} y(n) dP \right| \leqq \left(\lambda_0 + \frac{1}{2} V(n) \right) P(\tau > n)$$

$$\leqq Const. \times \frac{\sqrt{V(n)}(V(n) + 2\lambda_0)}{V(n) - 2\lambda_0} exp\left\{ \frac{-(V(n) - 2\lambda_0)^2}{8V(n)} \right\}$$

$$\longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus, we have

$$E\{y(\tau)\} = \lim_{n \to \infty} E\{y(\tau_n)\} = 0.$$

Similarly, we write

$$\breve{U}_n(t) = U(t) \qquad \text{for } t < \tau_n$$

$$= U(\tau_n) \qquad \text{for } t \geqq \tau_n$$

$$\breve{Z}_n(t, \lambda) = Z(t, \lambda) \qquad \text{for } t < \tau_n$$

$$= Z(\tau_n, \lambda) \qquad \text{for } t \geqq \tau_n.$$

We have then new martingale processes $\{\breve{U}_n(t), \breve{\mathfrak{B}}_t^{(n)}, 0 \leqq t < \infty\}$ and $\{\breve{Z}_n(t, \lambda), \breve{\mathfrak{B}}_t^{(n)}, 0 \leqq t < \infty\}$. Therefore we have

$$E\{U_n(t)\} = \int_{[\tau \leqq n]} U(\tau) dP + \int_{[\tau > n]} U(n) dP$$

$$= \int_{[\tau \leqq n]} \{y^2(\tau) - V(\tau)\} dP + \int_{[\tau > n]} \{y(n)^2 - V(n)\} dP$$

$$= 0.$$

Since,

$$\left| \int_{[\tau > n]} \{y(n)^2 - V(n)\} dP \right| \leqq \left\{ V(n) + \frac{1}{4} (2\lambda_0 + V(n))^2 \right\} \cdot P(\tau > n)$$

$$\longrightarrow 0 \quad \text{as } n \to \infty,$$

We have

$$\lim_{n \to \infty} \int_{[\tau \leqq n]} [y^2(\tau) - V(\tau)] dP = E\{(y(\tau))^2\} - E\{V(\tau)\} = 0.$$

Similarly we have

$$1=E\{\breve{Z}_n(t,\lambda)\}=\int_{[\tau\leqq n]} e^{\lambda y(\tau)-\frac{\lambda^2}{2}V(\tau)}dP$$

$$+\int_{[\tau>n]} e^{\lambda y(n)-\frac{\lambda^2}{2}V(n)}dP.$$

Since, for each real λ,

$$\left|\int_{[\tau>n]} exp\left\{\lambda y(n)-\frac{\lambda^2}{2}V(n)\right\}dP\right|$$

$$\leqq Const.\times\frac{\sqrt{V(n)}}{|V(n)-2\lambda_0|}\times exp\left\{-\frac{1}{8}\left[(1-2\lambda)^2V(n)-4\lambda_0+\frac{4\lambda_0^2}{V(n)}\right]\right\}$$

$$\longrightarrow 0 \quad \text{as } n\to\infty,$$

it follows immediately that for each real λ

$$1=\lim_{n\to\infty}\int_{[\tau\leqq n]} exp\left\{\lambda y(\tau)-\frac{\lambda^2}{2}V(\tau)\right\}dP$$

$$=E\left\{exp\left\{\lambda y(\tau)-\frac{\lambda^2}{2}V(\tau)\right\}\right\}. \qquad \text{Q. E. D.}$$

EXAMPLE 2.1. (c.f. [2], [7]). From theorem 2.1, it is easily obtain the A.S. N.'s $E\{\tau|H_0\}$ and $E\{\tau|H_1\}$ of our detection problem when the signal $\psi(s)$ is constant $\alpha>0$ which is in the white noise. From (2, 4), it is well known that λ_0 is given by

$$\lambda_0=log\left(\frac{1-\gamma}{\gamma}\right).$$

Let $E_1=\left\{W(\tau)=\frac{\lambda_0}{\alpha}+\frac{\alpha}{2}\right\}$ and E_2 be the complementary event of E_1. Then, since τ is define by

$$\tau=inf\left\{t>0;\left|W(t)-\frac{\alpha}{2}t\right|\geqq\lambda_0/\alpha\right\},$$

we have

$$E\{y(\tau)\}=\gamma E\left\{\lambda_0+\frac{\alpha^2}{2}\cdot\tau|E_1\right\}+(1-\gamma)E\left\{-\lambda_0+\frac{\alpha^2}{2}\cdot\tau|E_\theta\right\}$$

$$=\frac{\alpha^2}{2}E\{\tau\}-(1-2\gamma)\lambda_0=0,$$

that is,

$$E\{\tau\}=E\{\tau|H_0\}=E\{\tau|H_1\}$$

$$=\frac{2}{\alpha^2}(1-2\gamma)\,log\left(\frac{1-\gamma}{\gamma}\right). \tag{2.8}$$

EXAMPLE 2.2. Let $\Delta>0$ be a suitably chosen small interval and let us put

$$\phi_j(s)=1 \qquad \text{for } (j-1)\Delta\leqq s<j\Delta,\ j=1,2,\cdots,$$

$$=0 \qquad \text{otherwise}.$$

Let us asssume that $\psi(s)$ is a pulsed signal and is approximately expressed by

$$\psi(s)=\sum_{j=1}^{\infty}h\cdot\varepsilon_j\cdot\phi_j(s),\qquad 0\leqq s<\infty,$$

where $h>0$ is a given constant and $\varepsilon_j=0$ or 1, $j=1, 2, 3, \cdots$.

We also assume that a relative frequency of the occurrence of pulses $(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n)/n$ is close to a constant a, $(0<a\leqq 1)$, except for the first several n.

Then, we have approximately

$$V(t) \doteqdot \alpha^2 at\,, \qquad \text{and}$$

$$y(t)=h\sum_{j=1}^{n}\varepsilon_j z_j+h\varepsilon_{n+1}[W(t)-W(n\Delta)]\,, \tag{2.9}$$

where $z_j=W(j\Delta)-W((j-1)\Delta)$ and n is the largest integer not greater than t/Δ.

It is clear that $\{y(t),\ 0\leqq t<\infty\}$ is a Gaussian process with the mean value zero and its covariance function $R(s, t)$ is given approximately by

$$R(s, t) \doteqdot ah^2 min\,(s, t)\,.$$

Hence, this detection problem of the pulsed signal $\psi(s)$ in the white noise is nearly equivalent to the problem to detect a constant signal $\phi_0(s)\equiv h\cdot\sqrt{a}$ in the white noise as shown in Example 2.1.

Thus, we have the A.S.N. which are given approximately by

$$E\{\tau\,|\,H_0\}=E\{\tau\,|\,H_1\} \doteqdot \frac{2(1-2\gamma)}{ah^2}\cdot log\left(\frac{1-\gamma}{\gamma}\right).$$

§ 3. Detection of a constant signal in a non-white Gaussian noise.

Similary in § 2, let $\{W(s),\ F_s,\ 0\leqq s<\infty\}$ be the Wiener process.

Let $\{\xi(s),\ 0\leqq s<\infty\}$ be a Gaussian noise process defined by

$$\xi(t)=\int_0^t e^{-\beta(t-u)}dW(u)\,, \qquad 0\leqq t<\infty\,, \cdots \tag{3.1}$$

where $\beta\geqq 0$ is a non-negative constant.

We shall consider the following detection problem;

$$\begin{aligned} H_0:&\quad x(t)=\xi(t)\,,\\ H_1:&\quad x(t)=\alpha t+\xi(t)\,, \end{aligned} \tag{3.2}$$

where $\alpha>0$, is a constant signal to be detected.

Let P_{it}, $i=0, 1$, be the distribution of the observation $\{x(s),\ 0\leqq s\leqq t\}$ under H_i, $i=0, 1$, respectively.

Then, the detection problem (3.2) is nonsingular and the logarithm of the likelihood ratio of P_{1t} with respect to P_{0t} is given by

$$\begin{aligned} L(x\,;\,t)&=log\,\frac{dP_{1t}}{dP_{0t}}(x)\\ &=L_0(x\,;\,t)+\frac{\beta}{2}\cdot\{2m(t)x(t)-(m(t))^2\}\\ &\quad+\frac{\beta^2}{2}\left\{2\int_0^t m(s)x(s)ds-\int_0^t(m(s))^2ds\right\}, \end{aligned} \tag{3.3}$$

where $m(t)=\alpha t$, and

$$L_0(x\,;\,t)=\alpha x(t)-\frac{\alpha^2}{2}t\,. \tag{3.4}$$

Suppose that we adopt $L_0(x\,;\,t)$ as a statistic for the problem (3.2) instead of the log-likelihood ratio $L(x\,;\,t)$.

$L_0(x\,;\,t)$ is actually not optimal for the problem (3.2) (See [3]) but it has several sub-optimal properties because it become optimal when $\beta=0$, that is, the noise $\xi(t)$ were white, and $L_0(x\,;\,t)$ does not contain the parameter β which is intrinsic in the noise.

We set error probabilities to the equal to the prescribed value γ, that is,

$$P(\text{to } accept\ H_1|H_0)=P(\text{to } accept\ H_0|H_1)=\gamma\,, \tag{3.5}$$

and consider only such decision rules that (3.5) holds.

We define a stopping τ^* by

$$\tau^*=inf\,\{t>0\,;\ L_0(x\,;\,t)\leqq-\lambda_0 \text{ or } L_0(x\,;\,t)\geqq\lambda_1\} \tag{3.6}$$

where λ_0, λ_1 are positive constants such that our following decision rule satisfies (3.5).

Our decision rule based on the observation $\{x(s),\,0\leqq s\leqq t\}$ will be formulated as follows; When $L_0(x\,;\,\tau^*)=\lambda_1$, (or $-\lambda_0$), we stop sampling at $t=\tau^*$ and decide H_1, (or H_0) to be true, while as long as $-\lambda_0<L_0(x,\,s)<\lambda_1$, $0\leqq s\leqq t$, we continue sampling.

Since each distribution of $L_0(x\,;\,t)$ under H_i, $i=0,1$, is symmetric to the other, the constants λ_0 and λ_1 must be equal under the condition (3.5).

Let us put

$$V(t)=E\{\xi(t)^2\}=(1-e^{-2\beta t})/2\beta\,.$$

Let us consider a continuous function $f(t)$, $0\leqq t<\infty$, such that

(i) $\qquad f(0)=-\lambda^*<0\,,$

(ii) $\qquad f(t)=O(t^{\alpha})\nearrow\infty \qquad \text{as } t\to\infty\,,$

for some positive constant $\alpha>0$.

We shall now define a stopping time τ as follows;

$$\tau=inf\,\{t>0\,;\ \xi(t)\leqq f(t) \text{ or } \xi(t)\geqq 2\lambda^*+f(t)\}\,. \tag{3.7}$$

The stopping time τ^* defined by (3.6) is a special case of τ by (3.7). Indeed, τ^* corresponds to the case where $f(t)=\alpha t/2-\lambda^*$ and $\lambda^*=\lambda_0/\alpha$.

For a large t, it is easily seen that

$$\begin{aligned} P(\tau\geqq t)&\leqq P(f(t)\leqq\xi(t)\leqq 2\lambda^*+f(t))\\ &\leqq P(\xi(t)\geqq f(t))\\ &\leqq\frac{1}{2\sqrt{\pi\beta}}\times\frac{1}{|f(t)|}\times exp\,\{-\beta\times[f(t)]^2\}\,. \end{aligned}$$

Thus, we have

LEMMA 3.1. *For all real* $k \geqq 0$,

$$E\{\tau^k\} < \infty .$$

Proof of lemma 3.1 is analogous to that of lemma 2.3.

Now, we obtain

THEOREM 3.1. *Let* τ *be defined by* (3.7). *Then we have*

(i) $\quad E\{\xi(\tau)\} = 0,$

(ii) $\quad E\{\xi(\tau)^2\} = E\{V(\tau)\}$, *and*

(iii) *for each real* λ,

$$E\left\{exp\left[\lambda\xi(\tau) - \frac{\lambda^2}{2} V(\tau)\right]\right\} \equiv 1 . \tag{3.8}$$

PROOF. It is clear that τ is a stopping time with respect to F_t, $t \geqq 0$. The stochastic process $\{\xi(t),\ 0 \leqq t < \infty\}$ is the unique non-anticipated solution of a stochastic differential equation;

$$d\xi(t) = -\beta\xi(t)dt + dW(t), \tag{3.9}$$

with $\xi(0) = 0$. Hence, it enjoys the strong Markov property with respect to a Brownian stopping time, for example, say, τ. (See [4]).

For any random variable g and any measurable set A, we will write

$$E_A\{g\} = E\{I_A \cdot g\} ,$$

where I_A is the indicator function of A.

Let $\mathfrak{B}_\tau$ be the σ-field generated by τ, that is, the totality of sets whose intersections with $[\tau > t]$ belong to F_t for every t, $0 \leqq t < \infty$. Then, we have from the strong Markov property,

$$\begin{aligned} E_{[\tau \leqq t]}\{\xi(t)\} &= E\{E\{I_{[\tau \leqq t]} \cdot \xi(t) \mid \mathfrak{B}_\tau\}\} \\ &= E\{I_{[\tau \leqq t]} \cdot E\{\xi(t) \mid \tau, \xi(\tau)\}\} \\ &= E_{[\tau \leqq t]}\{\xi(\tau)\} . \end{aligned}$$

Since for $\tau > t$, $f(t) < \xi(t) < 2\lambda^* + f(t)$, we have

$$\begin{aligned} |E_{[\tau > t]}\{\xi(t)\}| &\leqq [2\lambda^* + f(t)] \cdot P(\tau > t) \\ &\leqq \frac{1}{2\sqrt{\pi\beta}} \times \frac{|2\lambda^* + f(t)|}{|f(t)|} \times exp\,\{-\beta |f(t)|^2\} \\ &\longrightarrow 0 \quad \text{as } t \to \infty . \end{aligned}$$

Thus, it is seen that

$$\begin{aligned} E\{\xi(t)\} &= \lim_{t \to \infty} E_{[\tau \leqq t]}\{\xi(\tau)\} + \lim_{t \to \infty} E_{[\tau > t]}\{\xi(t)\} \\ &= E\{\xi(\tau)\} = 0 . \end{aligned}$$

We have shown that (i) holds.

Let us write

$$U(t) = \xi(t)^2 - V(t).$$

Then, $U(t)$ is a functional of the Markov process $\{\xi(s), 0 \leqq s < \infty\}$ and hence we have

$$\begin{aligned} E_{[\tau \leqq t]}\{U(t)\} &= E\{E\{I_{[\tau \leqq t]} \cdot U(t) \mid \mathfrak{B}_\tau\}\} \\ &= E\{I_{[\tau \geqq t]} \cdot E\{U(t) \mid \tau, \xi(\tau)\}\} \\ &= E_{[\tau \leqq t]}\{U(\tau)\} = E_{[\tau \leqq t]}\{\xi(\tau)^2 - V(\tau)\}. \end{aligned}$$

Since, for $\tau > t$, $|\xi(t)| \leqq 2\lambda^* + f(t)$, it follows that

$$\begin{aligned} |E_{[\tau > t]}\{U(t)\}| &\leqq [V(t) + (2\lambda^* + f(t))^2] \cdot P(\tau > t) \\ &\leqq \frac{1}{2\sqrt{\pi}\beta} \cdot \frac{[V(t) + (2\lambda^* + f(t))^2]}{|f(t)|} \cdot e^{-\beta |f(t)|^2} \\ &\longrightarrow 0 \quad \text{as } t \to \infty. \end{aligned}$$

Thus, we have

$$\begin{aligned} E\{U(t)\} &= \lim_{t\to\infty} E_{[\tau \leqq t]}\{U(\tau)\} + \lim_{t\to\infty} E_{[\tau > t]}\{U(t)\} \\ &= E\{\xi(\tau)^2 - V(\tau)\} = 0. \end{aligned}$$

We have shown that $E\{\xi(\tau)^2\} = E\{V(\tau)\}$.

Let us put for each real λ,

$$Z(t, \lambda) = exp\left[\lambda \xi(t) - \frac{\lambda^2}{2} V(t)\right], \qquad 0 \leqq t < \infty.$$

Then, it is clear that $Z(t, \lambda)$ is F_t-measurable and $E\{Z(t, \lambda)\} \equiv 1$.

Thus, we have

$$\begin{aligned} E_{[\tau \leqq t]}\{Z(t, \lambda)\} &= E\{E\{I_{[\tau \leqq t]} Z(t, \lambda) \mid \mathfrak{B}_\tau\}\} \\ &= E\{I_{[\tau \leqq t]} E\{Z(t, \lambda \mid \tau, \xi(\tau)\}\} \\ &= E_{[\tau \leqq t]}\{Z(\tau, \lambda)\}. \end{aligned}$$

Now, we shall evaluate $E_{[\tau > t]}\left\{exp\left[\lambda \xi(t) - \frac{\lambda^2}{2} V(t)\right]\right\}$.

Since, for $\tau > t$, $f(t) < \xi(t) < 2\lambda^* + f(t)$, we have for each non-negative real λ,

$$\begin{aligned} E_{[\tau > t]}\{Z(t, \lambda)\} &\leqq exp\left\{\lambda[2\lambda^* + f(t)] - \frac{\lambda^2}{2} V(t)\right\} \cdot P(\tau > t) \\ &\leqq \frac{1}{2|f(t)|\sqrt{\pi}\beta} exp\left\{2\lambda\lambda^* - \beta |f(t)|^2 \left(1 - \frac{\lambda}{\beta f(t)}\right)\right\} \\ &\longrightarrow 0 \quad \text{as } t \to \infty, \end{aligned}$$

and for each negative real λ,

$$\begin{aligned} E_{[\tau > t]}\{Z(t, \lambda)\} &\leqq exp\left\{\lambda f(t) - \frac{\lambda^2}{2} V(t)\right\} \\ &\longrightarrow 0 \quad \text{as } t \to \infty. \end{aligned}$$

Thus, it follows that for each real λ,

$$E\{Z(t, \lambda)\} = \lim_{t\to\infty} E_{[\tau\leqq t]}\{Z(\tau, \lambda)\} + \lim_{t\to\infty} E_{[\tau>t]}\{Z(t, \lambda)\}$$
$$= E\{Z(\tau, \lambda\} \equiv 1 .$$

This completes the proof of theorem 3.1. Q. E. D.

The stopping time τ^* is the special case of τ in (3.7) and hence from theorem 3.1 it is seen that

$$E\{\xi(\tau^*)\} = 0 ,$$
$$E\{\xi(\tau^*)^2\} = E\{V(\tau^*)\} ,$$

and for each real λ,

$$E\left\{exp\left[\lambda\xi(\tau^*) - \frac{\lambda^2}{2} V(\tau^*)\right]\right\} \equiv 1 .$$

COROLLARY.

$$E\{\tau^* | H_0\} = E\{\tau^* | H_1\} = 2\lambda_0(1-2\gamma)/\alpha^2 ,$$

where λ_0 is such a constant that the error probabilities satisfy (3.5).

PROOF. Let $E_0 = \left\{\xi(\tau^*) = \frac{\alpha}{2}\tau^* - \frac{\lambda_0}{\alpha}\right\}$ and E_1 be the complementary event of E_0.

Then, by noticing that τ^* is equal to τ when $f(t) = \frac{\alpha}{2}t - \lambda^*$ and $\lambda^* = \lambda_0/\alpha$. and also that $P(E_1 | H_0) = \gamma$ and $P(E_0 | H_0) = 1-\gamma$, it follows from theorem 3.1 that

$$E\{\xi(\tau^*)\} = \frac{\alpha}{2}[E\{\tau^* | E_0\} \cdot P(E_0 | H_0) + E\{\tau^* | E_1\} \cdot P(E_1 | H_0)]$$
$$- \frac{\lambda_0}{\alpha} P(E_0 | H_0) + \frac{\lambda_0}{\alpha} P(E_1 | H_0)$$
$$= \frac{\alpha}{2} E\{\tau^* | H_0\} - \frac{\lambda_0}{\alpha}(1-2\gamma) = 0 .$$

Thus, we have proved corollary. Q. E. D.

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