ON A WALD’S EQUATION AND AVERAGE SAMPLE NUMBER IN A SEQUENTIAL SIGNAL DETECTION

Nagai, Takeaki
Faculty of Engineering Science, Osaka University

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ON A WALD'S EQUATION AND AVERAGE SAMPLE NUMBER IN A SEQUENTIAL SIGNAL DETECTION

By

Takeaki Nagai*

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§ 1. Summary.

It is shown that in detecting sequentially a deterministic signal \( \phi(t) \) in white noise \( \gamma(t) \) a similar identity (iii) in theorem 2.1, to the Wald's holds concerning a stopping time \( \tau \) determined by making use of a likelihood ratio. It is also shown that \( \tau \) has finite moments of any order under quite weak conditions over the signal. The exact A.S.N. \( E\{\tau\} \) in a constant signal case has been obtained and given by (2, 8).

It is also considered a detection problem of a constant signal \( \phi(t) \equiv \alpha \) in a coloured noise based on a sub-optimal statistic which become optimal when the noise were white. Similar properties of a stopping time \( \tau \) to those in the white noise case have been obtained in theorem 3.1.

§ 2. Detection of a deterministic signal in a white noise.

We consider the following detection problem of a signal \( \phi(t) \) in the white noise \( \gamma(t) \);

\[
H_0; \quad x(t) = W(t) \\
H_1; \quad x(t) = m(t) + W(t),
\]

where \( m(t) = \int_0^t \phi(s) ds \) is the integrated signal and \( \{W(t), 0 \leq t < \infty\} \) is the Wiener process which is considered to be the integrated form of the white noise \( \gamma(t) \).

By \( H_0 \) we mean that there is no signal in the (integrated) observation \( x(t) \) whose distribution is induced from the Wiener measure \( P_0 \) and by \( H_1 \) the observation \( x(t) \) is the sum of the signal \( m(t) \) and the noise \( W(t) \) whose distribution is induced by \( P_1 \), i.e. a shift of \( P_0 \) by \( m(\cdot) \).

In order for the detection problem (2.1) to be non-singular, we assume that \( \phi(\cdot) \) is square integrable on each finite interval \([0, t], 0 \leq t < \infty\).

Let us put

\[
V(t) = \int_0^t |\phi(s)|^2 ds < \infty.
\]
Let $\mathcal{B}_t$, $0 \leq t < \infty$, be the $\sigma$-field generated by the observation \{x(s), 0 \leq s \leq t\} and $P_{it}$, $i=0, 1$, restrictions of $P_i$, $i=0, 1$, to $\mathcal{B}_t$ respectively.

$P_{ot}$ and $P_{1t}$ are equivalent for each $t$, and the logarithm of the likelihood ratio $L(x \mid t)$ of $P_{1t}$ with respect to $P_{ot}$ is given (See [5], [6]) by

$$L(x \mid t) = \int_0^t \phi(s) dx(s) - \frac{1}{2} V(t). \quad (2.3)$$

The statistic $L(x \mid t)$ is optimal in the sense that it will give the most powerful critical region in this detection problem for testing $H_0$ vs. $H_1$ based on \{x(s), 0 \leq s \leq t\}, (See [3]).

At first we set error probabilities to be equal to the prescribed value $\gamma$, ($0 < \gamma \leq 1/2$), that is,

$$P(\mbox{to accept } H_1 \mid H_0) = P(\mbox{to accept } H_0 \mid H_1) = \gamma. \quad (2.4)$$

We define a stopping time $\tau$ by

$$\tau = \inf \{t > 0; L(x \mid t) \leq -\lambda_0 \mbox{ or } L(x \mid t) \geq \lambda_1\}. \quad (2.5)$$

where $\lambda_0$ and $\lambda_1$ are positive constants such that our following decision rule satisfies (2.4).

Our decision rule based on the observations \{x(s), 0 \leq s \leq t\} will be formulated as follows; When $L(x \mid \tau) = \lambda_i$, (or $-\lambda_0$), we stop sampling at $t = \tau$ and decide $H_i$, (or $H_0$), to be true, while as long as $-\lambda_0 < L(x \mid s) < \lambda_i$, $0 \leq s \leq t$, we continue sampling.

Since each distribution of $L(x \mid t)$ under $H_i$, $i=0, 1$, is symmetric to the other, the thresholds $-\lambda_0$ and $\lambda_1$ must be, under the condition (2.4), symmetric, that is, $\lambda_0 = \lambda_1$.

Let $F_t$ be the $\sigma$-field generated by \{W(s), 0 \leq s \leq t\} and let us put

$$y(t) = \int_0^t \phi(s) dw(s), \quad 0 \leq t < \infty. \quad (2.6)$$

Then we have

**Lemma 2.1.** $\{y(t), F_t, 0 \leq t < \infty\}$ is a Gaussian Martingale with the mean-value zero, its covariance function $R_y(t, s) = V(\min(t, s))$ and its realizations are continuous with probability one.

**Proof.** Clear.

From the symmetricity of the distribution of $L(x \mid t)$, we may and do proceed our discussion under the assumption that $H_0$ is always true.

We have the following evaluation of the tail probability of $\tau$:

**Lemma 2.2.** For sufficiently large $t$,

$$P(\tau > t) \leq \frac{2}{\pi} \frac{\sqrt{V(t)}}{(V(t) - 2\lambda_0)} \exp\left\{-\frac{V(t) - 2\lambda_0}{8V(t)}\right\}. \quad (2.7)$$

**Proof.** Since $[\tau > t] \subset \left[ |y(t)| \leq \frac{1}{2} V(t) < \lambda_0 \right]$, we have from lemma 2.1,

$$P(\tau > t) \leq \int_{-\lambda_0 + \frac{1}{2} V(t)}^{\lambda_0 + \frac{1}{2} V(t)} \frac{1}{\sqrt{2\pi V(t)}} \exp\left[-\frac{y^2}{2V(t)}\right] dy.$$
For a large $t$ such that $V(t) > 2\lambda_0$, the inequality (2.7) easily follows. Q. E. D.

**Lemma 2.3.** If there is a positive constant $\alpha > 0$, whatever small it is, such that the signal power $V(t)$ diverges to infinity with the same order as $O(t^\alpha)$ or faster, then for all positive $\beta > 0$, $E[\tau^\beta] < \infty$.

**Proof.** Let $F(t)$ be the c.d.f. of $\tau$. From the assumption that $V(t) = O(t^\alpha)$, we can find positive numbers $T_\alpha$ and $A^\alpha$ such that $\sqrt{V(t) - 2\lambda_0}/\sqrt{V(t)} > A^\alpha t^{\alpha/2}$, for all $t \geq T_\alpha$. It is enough for us to show that $\int_{T_\alpha}^{\infty} t^\beta dF(t) < \infty$, for all $\beta > 0$. Indeed, it is easily seen that the integral is dominated by a convergent series $K_0 \sum_{\nu=1}^{\infty} (1+\nu)^{3/2} e^{-K_1^{\nu/2}} < \infty$, where $K_0$ and $K_1$ are suitably chosen positive constants. Q. E. D.

Let us put

$$U(t) = y(t)^2 - V(t), \quad 0 \leq t < \infty,$$

and for each $\lambda$, $-\infty < \lambda < \infty$,

$$Z(t, \lambda) = \exp \{ \lambda y(t) - \frac{\lambda^2}{2} V(t) \}, \quad 0 \leq t < \infty.$$

Then, we have

**Lemma 2.4.** $\{U(t), t < \infty\}$ is a martingale with the mean value zero and $\{Z(t, \lambda), F_t, 0 \leq t < \infty\}$ is also a martingale with the mean value 1 for each real $\lambda$.

**Proof.** It is clear that $E[U(t)] = 0$. Let us put $E(s, t) = \int_s^t e^{u} dW(u)$. Then

$$U(t+h) = U(t) + 2y(t) \xi(t, t+h) + \{\xi(t, t+h)\}^2 - V(t) - V(t+h).$$

Thus, we have

$$E[U(t+h)|F_t] = U(t), \quad a.s.$$

On the other hand, $E[e^{\lambda y(t)}] = \exp \{ \frac{\lambda^2}{2} V(t) \}$, and hence $E[Z(t, \lambda)] = 1$, for each real $\lambda$. Since it is written as follows:

$$Z(t+h, \lambda) = Z(t, \lambda) \exp \left\{ \lambda \xi(t, t+h) - \frac{\lambda^2}{2} [V(t+h) - V(t)] \right\},$$

we have

$$E[Z(t+h, \lambda)|F_t] = Z(t, \lambda), \quad a.s.$$

This shows that $\{Z(t, \lambda), F_t, 0 \leq t < \infty\}$ is a martingale for each real $\lambda$. Q. E. D.

By noticing that $\tau$ is the Brownian stopping time, that is $\{\tau > t\} \in F_t$ for each $t$, we have

**Theorem 2.1.** (i) $E[y(\tau)] = 0$,

(ii) $E[|y(\tau)|^2] = E[V(\tau)]$, and

(iii) $E[\exp \{ \lambda \cdot y(\tau) - \frac{\lambda^2}{2} V(\tau) \}] = 1$, for each real $\lambda$.

**Proof.** Let us define a sequence of stopping times $\tau_n$ by

$$\tau_n = \min (n, \tau), \quad n = 1, 2, \cdots.$$

Let $\bar{y}_n(t)$ be a sequence of stopped processes of $y(t)$ by $\tau_n$, that is, $\bar{y}_n(t) = y(t)$ for $t < \tau_n$; $= y(\tau_n)$ for $t \geq \tau_n$, and $\bar{F}_t^{(n)}$ the $\sigma$-field generated by $\tau_n$, that is,
the totality of measurable sets $A$ whose intersection with $\{ \min (t, \tau_n) \leq s \}$ belongs to $F_s$ for each $s$, $0 \leq s < \infty$.

Since $\tau_n$ is bounded a.s. for each $n$, it is seen that $\{ \bar{y}_n(t), \bar{y}_1^{(n)}, 0 \leq t < \infty \}$ is a martingale and $E\{ \bar{y}_n(t) \} = \sup_{t'} E\{ y(t') \} = 0$ for all $t$, $0 \leq t < \infty$. (See [1]).

Hence, for all $t > n$, $(n = 1, 2, \cdots)$,

$$E\{ \bar{y}_n(t) \} = E\{ y(\tau_n) \} = \int_{[\tau \leq n]} y(\tau) dP + \int_{[\tau > n]} y(n) dP.$$

Since $y(n) < \lambda_0 + \frac{1}{2} V(n)$, we have

$$\left| \int_{[\tau > n]} y(n) dP \right| \leq \left( \lambda_0 + \frac{1}{2} V(n) \right) P(\tau > n) \leq \text{Const.} \times \frac{\sqrt{V(n)(V(n) + 2\lambda_0)}}{V(n) - 2\lambda_0} \exp \left\{ \frac{-\left( V(n) - 2\lambda_0 \right)^2}{8V(n)} \right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, we have

$$E\{ y(\tau) \} = \lim_{n \rightarrow \infty} E\{ y(\tau_n) \} = 0.$$

Similarly, we write

$$\bar{U}_n(t) = U(t) \quad \text{for } t < \tau_n$$

$$= U(\tau_n) \quad \text{for } t \geq \tau_n$$

$$\bar{Z}_n(t, \lambda) = Z(t, \lambda) \quad \text{for } t < \tau_n$$

$$= Z(\tau_n, \lambda) \quad \text{for } t \geq \tau_n.$$

We have then new martingale processes $\{ \bar{U}_n(t), \bar{y}_1^{(n)}, 0 \leq t < \infty \}$ and $\{ \bar{Z}_n(t, \lambda), \bar{y}_1^{(n)}, 0 \leq t < \infty \}$. Therefore we have

$$E\{ U_n(t) \} = \int_{[\tau \leq n]} U(\tau) dP + \int_{[\tau > n]} U(n) dP$$

$$= \int_{[\tau \leq n]} \{ y^2(\tau) - V(\tau) \} dP + \int_{[\tau > n]} \{ y(n)^2 - V(n) \} dP$$

$$= 0.$$

Since,

$$\left| \int_{[\tau > n]} \{ y(n)^2 - V(n) \} dP \right| \leq \left\{ V(n) + \frac{1}{4}(2\lambda_0 + V(n))^2 \right\} \cdot P(\tau > n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

We have

$$\lim_{n \rightarrow \infty} \int_{[\tau \leq n]} \{ y^2(\tau) - V(\tau) \} dP = E\{ (y(\tau))^2 \} - E\{ V(\tau) \} = 0.$$

Similarly we have
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\[ 1 = E(\tilde{Z}_0(t, \lambda)) = \int_{\{r \leq n\}} e^{\lambda y(r) - \frac{\lambda^2}{2} V(r)} dP + \int_{\{r > n\}} e^{\lambda y(r) - \frac{\lambda^2}{2} V(r)} dP. \]

Since, for each real \( \lambda \),

\[
\left| \int_{\{r > n\}} e^{\lambda y(n) - \frac{\lambda^2}{2} V(n)} dP \right| \leq \text{Const.} \times \frac{\sqrt{V(n)}}{|V(n)| - 2\lambda_0} \times \exp \left\{ -\frac{1}{8} \left[ (1 - 2\lambda) V(n) - 4\lambda_0 + \frac{4\lambda_0^2}{V(n)} \right] \right\} \to 0 \quad \text{as } n \to \infty ,
\]

it follows immediately that for each real \( \lambda \)

\[
1 = \lim_{n \to \infty} \int_{\{r \leq n\}} e^{\lambda y(r) - \frac{\lambda^2}{2} V(r)} dP = E\{ \exp \{ \lambda y(r) - \frac{\lambda^2}{2} V(r) \} \} . \quad \text{Q. E. D.}
\]

**Example 2.1.** (c.f. [2], [7]). From theorem 2.1, it is easily obtain the A.S. N.’s \( E(\tau|H_0) \) and \( E(\tau|H_1) \) of our detection problem when the signal \( 0(s) \) is constant \( \alpha > 0 \) which is in the white noise. From (2, 4), it is well known that \( \lambda_0 \) is given by

\[ \lambda_0 = \log \left( \frac{1-r}{r} \right) . \]

Let \( E_i = \{ W(\tau) = \frac{\lambda_0}{\alpha} + \frac{\alpha}{2}\} \) and \( E_0 \) be the complementary event of \( E_i \). Ther, since \( \tau \) is define by

\[ \tau = \inf \left\{ t > 0 ; \left| W(t) - \frac{\alpha}{2} t \right| \geq \lambda_0/\alpha \right\} , \]

we have

\[
E(y(\tau)) = \tau E\{ \lambda_0 + \frac{\alpha^2}{2} \cdot \tau | E_i \} + (1 - \tau) E\{ -\lambda_0 + \frac{\alpha^2}{2} \cdot \tau | E_0 \}
\]

\[ = \frac{\alpha^2}{2} E(\tau) - (1 - 2\tau) \lambda_0 = 0 , \]

that is,

\[ E(\tau) = E(\tau|H_0) = E(\tau|H_1) \]

\[ = \frac{2}{\alpha^2} (1 - 2\tau) \log \left( \frac{1-r}{r} \right) . \quad (2.8) \]

**Example 2.2.** Let \( \Delta > 0 \) be a suitably chosen small interval and let us put

\[ \phi_j(s) = 1 \quad \text{for } (j-1)\Delta \leq s < j\Delta, \ j = 1, 2, \cdots , \]

\[ = 0 \quad \text{otherwise} . \]

Let us assume that \( \phi(s) \) is a pulsed signal and is approximately expressed by

\[ \phi(s) = \sum_{j=1}^{\infty} h \cdot \epsilon_j \cdot \phi_j(s) , \quad 0 \leq s < \infty , \]
where \( h > 0 \) is a given constant and \( \varepsilon_j = 0 \) or 1, \( j = 1, 2, 3, \ldots \).

We also assume that a relative frequency of the occurrence of pulses \((\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n)/n\) is close to a constant \( a \), \((0 < a \leq 1)\), except for the first several \( n \).

Then, we have approximately

\[
V(t) \approx \alpha^2 at, \quad \text{and}
\]

\[
y(t) = h \sum_{j=1}^{n} \varepsilon_j z_j + h \varepsilon_{n+1} [W(t) - W(n\mathcal{A})],
\]

(2.9)

where \( z_j = W(j\mathcal{A}) - W((j-1)\mathcal{A}) \) and \( n \) is the largest integer not greater than \( t/\mathcal{A} \).

It is clear that \( \{y(t), 0 \leq t < \infty\} \) is a Gaussian process with the mean value zero and its covariance function \( R(s, t) \) is given approximately by

\[
R(s, t) \approx ah^2 \min(s, t).
\]

Hence, this detection problem of the pulsed signal \( \phi(s) \) in the white noise is nearly equivalent to the problem to detect a constant signal \( \phi_0(s) \equiv h \cdot \sqrt{a} \) in the white noise as shown in Example 2.1.

Thus, we have the A.S.N. which are given approximately by

\[
E[\tau \mid H_0] = E[\tau \mid H_1] \approx \frac{2(1-2\gamma)}{a h^2} \cdot \log \left( \frac{1-\gamma}{\gamma} \right).
\]

§ 3. Detection of a constant signal in a non-white Gaussian noise.

Similary in § 2, let \( \{W(s), F_s, 0 \leq s < \infty\} \) be the Wiener process.

Let \( \{e(s), 0 \leq s < \infty\} \) be a Gaussian noise process defined by

\[
e(t) = e^{-(t-s)\beta} dW(u), \quad 0 \leq t < \infty, \ldots
\]

(3.1)

where \( \beta \geq 0 \) is a non-negative constant.

We shall consider the following detection problem;

\[
H_0: \quad x(t) = \xi(t),
\]

\[
H_1: \quad x(t) = \alpha t + \xi(t),
\]

(3.2)

where \( \alpha > 0 \), is a constant signal to be detected.

Let \( P_{it}, i = 0, 1 \), be the distribution of the observation \( \{x(s), 0 \leq s \leq t\} \) under \( H_i \), \( i = 0, 1 \), respectively.

Then, the detection problem (3.2) is nonsingular and the logarithm of the likelihood ratio of \( P_{it} \) with respect to \( P_{ot} \) is given by

\[
L(x; t) = \log \frac{dP_{it}(x)}{dP_{ot}(x)}
\]

\[
= L_0(x; t) + \frac{\beta}{2} \cdot \{2m(t)x(t) - (m(t))^2\}
\]

\[
+ \frac{\beta^2}{2} \left[ 2 \int_0^t m(s)x(s) ds - \int_0^t (m(s))^2 ds \right],
\]

(3.3)
where $m(t) = at$, and

$$L_0(x; t) = a x(t) - \frac{\alpha^2}{2-t}.$$  

(3.4)

Suppose that we adopt $L_0(x; t)$ as a statistic for the problem (3.2) instead of the log-likelihood ratio $L(x; t)$.

$L_0(x; t)$ is actually not optimal for the problem (3.2) (See [3]) but it has several sub-optimal properties because it become optimal when $\beta = 0$, that is, the noise $\xi(t)$ were white, and $L_0(x; t)$ does not contain the parameter $\beta$ which is intrinsic in the noise.

We set error probabilities to the equal to the prescribed value $\gamma$, that is,

$$P(\text{to accept } H_1|H_0) = P(\text{to accept } H_0|H_1) = \gamma,$$

(3.5)

and consider only such decision rules that (3.5) holds.

We define a stopping $\tau^*$ by

$$\tau^* = \inf \{ t > 0 ; L_0(x; t) \leq -\lambda_o \text{ or } L_0(x; t) \geq \lambda_i \}$$

(3.6)

where $\lambda_o$, $\lambda_i$ are positive constants such that our following decision rule satisfies (3.5).

Our decision rule based on the observation $\{x(s), 0 \leq s \leq t\}$ will be formulated as follows; When $L_0(x; \tau^*) = \lambda_i$, (or $-\lambda_o$), we stop sampling at $t = \tau^*$ and decide $H_i$, (or $H_o$) to be true, while as long as $-\lambda_o < L_0(x, s) < \lambda_i$, $0 \leq s \leq t$, we continue sampling.

Since each distribution of $L_0(x; t)$ under $H_i$, $i = 0, 1$, is symmetric to the other, the constants $\lambda_o$ and $\lambda_i$ must be equal under the condition (3.5).

Let us put

$$V(t) = E\left[\xi(t)^2\right] = \frac{1-e^{-2\beta t}}{2\beta}.$$ 

Let us consider a continuous function $f(t)$, $0 \leq t < \infty$, such that

(i) $f(0) = -\lambda^* < 0$,

(ii) $f(t) = O(t^a)$ $\exp \infty$ as $t \to \infty$,

for some positive constant $a > 0$.

We shall now define a stopping time $\tau$ as follows;

$$\tau = \inf \{ t > 0 ; \xi(t) \leq f(t) \text{ or } \xi(t) \geq 2\lambda^*+f(t)\}.$$  

(3.7)

The stopping time $\tau^*$ defined by (3.6) is a special case of $\tau$ by (3.7). Indeed, $\tau^*$ corresponds to the case where $f(t) = at/2 - \lambda^*$ and $\lambda^* = \lambda_o/\alpha$.

For a large $t$, it is easily seen that

$$P(\tau \geq t) \leq P(f(t) \leq \xi(t) \leq 2\lambda^*+f(t))$$

$$\leq P(\xi(t) \geq f(t))$$

$$\leq \frac{1}{\sqrt{\pi \beta}} \times \frac{1}{|f(t)|} \times \exp \{ -\beta \times [\xi(t)]^2 \}.$$ 

Thus, we have
LEMMA 3.1. For all real $k \geq 0$,

$$E(\tau^k) < \infty.$$ 

Proof of lemma 3.1 is analogous to that of lemma 2.3.

Now, we obtain

THEOREM 3.1. Let $\tau$ be defined by (3.7). Then we have

(i) $E(\xi(\tau)) = 0$,

(ii) $E(\xi(\tau)^2) = E(V(\tau))$, and

(iii) for each real $\lambda$,

$$E\left[\exp\left\{\lambda\xi(\tau) - \frac{\lambda^2}{2} V(\tau)\right\}\right] = 1.$$  

PROOF. It is clear that $\tau$ is a stopping time with respect to $F_t$, $t \geq 0$. The stochastic process $\{\xi(t), 0 \leq t < \infty\}$ is the unique non-anticipated solution of a stochastic differential equation;

$$d\xi(t) = -\beta \xi(t)dt + dW(t),$$  \hspace{1cm} (3.9)

with $\xi(0) = 0$. Hence, it enjoys the strong Markov property with respect to a Brownian stopping time, for example, say, $\tau$. (See [4]).

For any random variable $g$ and any measurable set $A$, we will write

$$E_A[g] = E[I_A \cdot g],$$

where $I_A$ is the indicator function of $A$.

Let $\mathcal{F}_\tau$ be the $\sigma$-field generated by $\tau$, that is, the totality of sets whose intersections with $[\tau > t]$ belong to $F_t$ for every $t$, $0 \leq t < \infty$. Then, we have from the strong Markov property,

$$E_{\tau \leq t} \{\xi(t)\} = E\{E_{\tau \leq t} \cdot \xi(t) | \mathcal{F}_\tau\}\}
$$

$$= E\{E_{\tau \leq t} \cdot E(\xi(t) | \tau, \xi(\tau))\}\}
$$

$$= E_{\tau \leq t} \{\xi(\tau)\}.$$

Since for $\tau > t$, $f(t) < \xi(t) < 2\lambda + f(t)$, we have

$$|E_{[\tau > t]} \{\xi(t)\}| \leq [2\lambda^* + f(t)] \cdot P(\tau > t)\]
$$

$$\leq \frac{1}{2\sqrt{\pi \beta}} \times \frac{|2\lambda^* + f(t)|}{|f(t)|} \times \exp\{-\beta |f(t)|^2\}
$$

$$\rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

Thus, it is seen that

$$E(\xi(t)) = \lim_{t \rightarrow \infty} E_{\tau \leq t} \{\xi(\tau)\} + \lim_{t \rightarrow \infty} E_{[\tau > t]} \{\xi(t)\}
$$

$$= E(\xi(\tau)) = 0.$$ 

We have shown that (i) holds.

Let us write
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\[ U(t) = \xi(t)^2 - V(t). \]

Then, \( U(t) \) is a functional of the Markov process \( \{ \xi(s), 0 \leq s < \infty \} \) and hence we have

\[
E_{t \geq t} \{ U(t) \} = E \{ E_{t \geq t} \cdot U(t) | \mathcal{F}_t \} \\
= E \{ E_{t \geq t} \cdot E \{ U(t) | \tau, \xi(\tau) \} \} \\
= E_{t \geq t} \{ U(\tau) \} = E_{t \geq t} \{ \xi(\tau)^2 - V(\tau) \}.
\]

Since, for \( \tau > t, |\xi(\tau)| \leq 2\lambda^* + \beta(f), \) it follows that

\[
|E_{t \geq t} \{ U(t) \}| \leq \left[ V(t) + (2\lambda^* + \beta(f))^2 \right] \cdot P(\tau > t)
\]

\[
\leq \frac{1}{2\sqrt{\pi} \beta} \cdot \frac{[V(t) + (2\lambda^* + \beta(f))^2]}{|f(t)|} \cdot e^{-x(t)}
\]

\[
\rightarrow 0 \quad \text{as } t \rightarrow \infty.
\]

Thus, we have

\[
E \{ U(t) \} = \lim_{t \rightarrow \infty} E_{t \geq t} \{ U(t) \} + \lim_{t \rightarrow \infty} E_{t \geq t} \{ U(t) \}
\]

\[
= E \{ \xi(\tau)^2 - V(\tau) \} = 0.
\]

We have shown that \( E \{ \xi(\tau)^2 \} = E \{ V(\tau) \}. \)

Let us put for each real \( \lambda, \)

\[
Z(t, \lambda) = \exp \left[ \lambda \xi(t) - \frac{\lambda^2}{2} V(t) \right], \quad 0 \leq t < \infty.
\]

Then, it is clear that \( Z(t, \lambda) \) is \( F_t \)-measurable and \( E \{ Z(t, \lambda) \} = 1. \)

Thus, we have

\[
E_{t \geq t} \{ Z(t, \lambda) \} = E \{ E_{t \geq t} \cdot Z(t, \lambda) | \mathcal{F}_t \} \\
= E \{ E_{t \geq t} \cdot Z(t, \lambda) | \tau, \xi(\tau) \} \\
= E_{t \geq t} \{ Z(\tau, \lambda) \}.
\]

Now, we shall evaluate \( E_{t \geq t} \{ \exp \left[ \lambda \xi(t) - \frac{\lambda^2}{2} V(t) \right] \}. \)

Since, for \( \tau > t, \beta(f) < \xi(t) < 2\lambda^* + f(t), \) we have for each non-negative real \( \lambda, \)

\[
E_{t \geq t} \{ Z(t, \lambda) \} \leq \exp \left[ \lambda \beta(f) - \frac{\lambda^2}{2} V(t) \right] \cdot P(\tau > t)
\]

\[
\leq \frac{1}{2|f(t)| \sqrt{\pi} \beta} \cdot \exp \left[ 2\lambda \beta - \beta |f(t)|^2 \left( 1 - \frac{1}{2} \right) \right]
\]

\[
\rightarrow 0 \quad \text{as } t \rightarrow \infty,
\]

and for each negative real \( \lambda, \)

\[
E_{t \geq t} \{ Z(t, \lambda) \} \leq \exp \left[ \lambda f(t) - \frac{\lambda^2}{2} V(t) \right]
\]

\[
\rightarrow 0 \quad \text{as } t \rightarrow \infty.
\]

Thus, it follows that for each real \( \lambda, \)
\[ E(Z(t, \lambda)) = \lim_{t \to \infty} E_{\tau < t}(Z(\tau, \lambda)) + \lim_{t \to \infty} E_{\tau \geq t}(Z(t, \lambda)) \]

\[ = E(Z(\tau, \lambda)) \equiv 1. \]

This completes the proof of theorem 3.1. Q. E. D.

The stopping time \( \tau^* \) is the special case of \( \tau \) in (3.7) and hence from theorem 3.1 it is seen that

\[ E(\xi(\tau^*)) = 0, \]

\[ E(\xi(\tau^*)^2) = E(V(\tau^*)), \]

and for each real \( \lambda \),

\[ E[\exp\left(\lambda \xi(\tau^*) - \frac{\lambda^2}{2} V(\tau^*)\right)] \equiv 1. \]

**COROLLARY.**

\[ E(\tau^* | H_0) = E(\tau^* | H_1) = \frac{\lambda_0 (1 - 2\gamma)}{\alpha^2}, \]

where \( \lambda_0 \) is such a constant that the error probabilities satisfy (3.5).

**PROOF.** Let \( E_0 = \{\xi(\tau^*) = -\frac{\alpha}{2} \tau^* - \frac{\lambda_0}{\alpha}\} \) and \( E_1 \) be the complementary event of \( E_0 \).

Then, by noticing that \( \tau^* \) is equal to \( \tau \) when \( \int(t) = \frac{\alpha}{2} t - \lambda^* \) and \( \lambda^* = \frac{\lambda_0}{\alpha} \) and also that \( P(E_1 | H_0) = \gamma \) and \( P(E_0 | H_0) = 1 - \gamma \), it follows from theorem 3.1 that

\[ E(\xi(\tau^*)) = \frac{\alpha}{2} \left[E(\tau^* | E_0) \cdot P(E_0 | H_0) + E(\tau^* | E_1) \cdot P(E_1 | H_0)\right] \]

\[ - \frac{\lambda_0}{\alpha} P(E_0 | H_0) - \frac{\lambda_0}{\alpha} P(E_1 | H_0) \]

\[ = \frac{\alpha}{2} E(\tau^* | H_0) - \lambda_0 \frac{(1 - 2\gamma)}{\alpha} = 0. \]

Thus, we have proved corollary. Q. E. D.

**References**


