ON A WALD’S EQUATION AND AVERAGE SAMPLE NUMBER IN A SEQUENTIAL SIGNAL DETECTION

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ON A WALD'S EQUATION AND AVERAGE SAMPLE NUMBER IN A SEQUENTIAL SIGNAL DETECTION

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§ 1. Summary.

It is shown that in detecting sequentially a deterministic signal \( \phi(t) \) in white noise \( \eta(t) \) a similar identity (iii) in theorem 2.1, to the Wald's holds concerning a stopping time \( \tau \) determined by making use of a likelihood ratio. It is also shown that \( \tau \) has finite moments of any order under quite weak conditions over the signal. The exact A.S.N. \( E\{\tau\} \) in a constant signal case has been obtained and given by (2, 8).

It is also considered a detection problem of a constant signal \( \phi(t) \equiv \alpha \) in a coloured noise based on a sub-optimal statistic which become optimal when the noise were white. Similar properties of a stopping time \( \tau \) to those in the white noise case have been obtained in theorem 3.1.

§ 2. Detection of a deterministic signal in a white noise.

We consider the following detection problem of a signal \( \phi(t) \) in the white noise \( \eta(t) \);

\[
\begin{align*}
H_0; \quad & x(t) = W(t) \\
H_1; \quad & x(t) = m(t) + W(t),
\end{align*}
\]

(2.1)

where \( m(t) = \int_0^t \phi(s) ds \) is the integrated signal and \( \{W(t), 0 \leq t < \infty\} \) is the Wiener process which is considered to be the integrated form of the white noise \( \eta(t) \).

By \( H_0 \) we mean that there is no signal in the (integrated) observation \( x(t) \) whose distribution is induced from the Wiener measure \( P_0 \) and by \( H_1 \) the observation \( x(t) \) is the sum of the signal \( m(t) \) and the noise \( W(t) \) whose distribution is induced by \( P_1 \), i.e. a shift of \( P_0 \) by \( m(\cdot) \).

In order for the detection problem (2.1) to be non-singular, we assume that \( \phi(\cdot) \) is square integrable on each finite interval \([0, t], 0 \leq t < \infty\).

Let us put

\[
V(t) = \int_0^t |\phi(s)|^2 ds < \infty.
\]

(2.2)

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Let $\mathcal{B}_t$, $0 \leq t < \infty$, be the $\sigma$-field generated by the observation \{x(s), 0 \leq s \leq t\} and \( P_{it}, i = 0, 1, \) restrictions of \( P_i, i = 0, 1, \) to $\mathcal{B}_t$ respectively.

$P_{0t}$ and $P_{1t}$ are equivalent for each $t$, and the logarithm of the likelihood ratio $L(x; t)$ of $P_{1t}$ with respect to $P_{0t}$ is given (See [5], [6]) by

$$L(x; t) = \int_0^t \phi(s) dx(s) - \frac{1}{2} V(t). \quad (2.3)$$

The statistic $L(x; t)$ is optimal in the sense that it will give the most powerful critical region in this detection problem for testing $H_0$ vs. $H_1$ based on \{x(s), 0 \leq s \leq t\}, (See [3]).

At first we set error probabilities to be equal to the prescribed value $\gamma$, $0 < \gamma \leq 1/2$, that is,

$$P(\text{to accept } H_i | H_i) = P(\text{to accept } H_i | H_{3-i}) = \gamma. \quad (2.4)$$

We define a stopping time $\tau$ by

$$\tau = \inf \{ t > 0; L(x; t) \leq -\lambda_0 \text{ or } L(x; t) \geq \lambda_1 \}, \quad (2.5)$$

where $\lambda_0$ and $\lambda_1$ are positive constants such that our following decision rule satisfies (2.4).

Our decision rule based on the observations \{x(s), 0 \leq s \leq t\} will be formulated as follows; When $L(x; \tau) = \lambda_1$ (or $-\lambda_0$), we stop sampling at $t = \tau$ and decide $H_i$, (or $H_0$), to be true, while as long as $-\lambda_0 < L(x; s) < \lambda_1$, $0 \leq s \leq t$, we continue sampling.

Since each distribution of $L(x; t)$ under $H_i$, $i = 0, 1$, is symmetric to the other, the thresholds $-\lambda_0$ and $\lambda_1$ must be, under the condition (2.4), symmetric, that is, $\lambda_0 = \lambda_1$.

Let $F_t$ be the $\sigma$-field generated by \{W(s), 0 \leq s \leq t\} and let us put

$$y(t) = \int_0^t \phi(s) dw(s), \quad 0 \leq t < \infty. \quad (2.6)$$

Then we have

**Lemma 2.1.** \{y(t), F_t, 0 \leq t < \infty\} is a Gaussian Martingale with the mean-value zero, its covariance function $R_y(t, s) = V(\min(t, s))$ and its realizations are continuous with probability one.

**Proof.** Clear.

From the symmetricity of the distribution of $L(x; t)$, we may and do proceed our discussion under the assumption that $H_0$ is always true.

We have the following evaluation of the tail probability of $\tau$:

**Lemma 2.2.** For sufficiently large $t$,

$$P(\tau > t) \leq \frac{2}{\pi} \frac{\sqrt{V(t)}}{(V(t) - 2\lambda_0)} \exp\left\{ - \frac{V(t) - 2\lambda_0}{8V(t)} \right\}. \quad (2.7)$$

**Proof.** Since $[\tau > t] \subset \left[ |y(t)| - \frac{1}{2} V(t)| < \lambda_0 \right]$, we have from lemma 2.1,

$$P(\tau > t) \leq \int_{-\lambda_0}^{\lambda_0} \frac{1}{\sqrt{2\pi V(t)}} \exp\left\{ - \frac{y^2}{2V(t)} \right\} dy.$$
For a large $t$ such that $V(t) > 2\lambda_0$, the inequality (2.7) easily follows. Q. E. D.

**Lemma 2.3.** If there is a positive constant $\alpha > 0$, whatever small it is, such that the signal power $V(t)$ diverges to infinity with the same order as $O(t^\alpha)$ or faster, then for all positive $\beta > 0$, $E\{e^{\beta x}\} < \infty$.

**Proof.** Let $F(t)$ be the c.d.f. of $\tau$. From the assumption that $V(t) = O(t^\alpha)$, we can find positive numbers $T_0$ and $A^*$ such that $\sqrt{V(t)} - 2\lambda_0/\sqrt{V(t)} \geq A^* t^{\alpha/2}$, for all $t \geq T_0$. It is enough for us to show that $\int_{T_0}^\infty t^\beta dF(t) < \infty$, for all $\beta > 0$. Indeed, it is easily seen that the integral is dominated by a convergent series $K_0 \sum_{\nu=1}^\infty (1+\nu)^{\beta/2} e^{-K_1 \nu}$.

Let us put

$$U(t) = y(t)^2 - V(t), \quad 0 \leq t < \infty,$$

and for each $\lambda$, $-\infty < \lambda < \infty$,

$$Z(t, \lambda) = \exp \left\{ \lambda y(t) - \frac{\lambda^2}{2} V(t) \right\}, \quad 0 \leq t < \infty.$$

Then, we have

**Lemma 2.4.** $\{U(t), \tau_t, 0 \leq t < \infty\}$ is a martingale with the mean value zero and $\{Z(t, \lambda), \tau_t, 0 \leq t < \infty\}$ is also a martingale with the mean value 1 for each real $\lambda$.

**Proof.** It is clear that $E\{U(t)\} = 0$. Let us put $E(s, t) = \int_s^t \phi(u)dW(u)$. Then

$$U(t+h) = U(t) + 2y(t)\xi(t, t+h) + \{\xi(t, t+h)\}^2 - V(t+h) + V(t).$$

Thus, we have

$$E\{U(t+h)|F_t\} = U(t), \quad a. s.$$

On the other hand, $E\{e^{\lambda y(t)}\} = \exp \left\{ \frac{\lambda^2}{2} V(t) \right\}$, and hence $E\{Z(t, \lambda)\} \equiv 1$, for each real $\lambda$. Since it is written as follows:

$$Z(t+h, \lambda) = Z(t, \lambda) \exp \left\{ \lambda \xi(t, t+h) - \frac{\lambda^2}{2} [V(t+h) - V(t)] \right\},$$

we have

$$E\{Z(t+h, \lambda)|F_t\} = Z(t, \lambda), \quad a. s.$$

This shows that $\{Z(t, \lambda), \tau_t, 0 \leq t < \infty\}$ is a martingale for each real $\lambda$. Q. E. D.

By noticing that $\tau$ is the Brownian stopping time, that is $\{\tau > t\} \in F_t$ for each $t$, we have

**Theorem 2.1.** (i) $E\{y(\tau)\} = 0$,

(ii) $E\{|y(\tau)|^2\} = E\{V(\tau)\}$, and

(iii) $E\{\exp \left\{ \lambda y(\tau) - \frac{\lambda^2}{2} V(\tau) \right\} \} = 1$, for each real $\lambda$.

**Proof.** Let us define a sequence of stopping times $\tau_n$ by

$$\tau_n = \min (n, \tau), \quad n = 1, 2, \ldots.$$

Let $\tilde{y}_n(t), n = 1, 2, \ldots$ be a sequence of stopped processes of $y(t)$ by $\tau_n$, that is, $\tilde{y}_n(t) = y(t)$ for $t < \tau_n$; $= y(\tau_n)$ for $t \geq \tau_n$ and $\tilde{y}(\tau_n)$ the $\sigma$-field generated by $\tau_n$, that is,
the totality of measurable sets \( A \) whose intersection with \( \{ \min(t, \tau_n) \leq s \} \) belongs to \( F_s \) for each \( s, 0 \leq s < \infty \).

Since \( \tau_n \) is bounded a.s. for each \( n \), it is seen that \( \{ \bar{y}_n(t), \bar{\mathcal{S}}_{t}^{(n)}, 0 \leq t < \infty \} \) is a martingale and \( E\{y(t')\} = 0 \) for all \( t, 0 \leq t < \infty \). (See [1]).

Hence, for all \( t > n, (n = 1, 2, \cdots) \),
\[
E\{\bar{y}_n(t)\} = E\{y(\tau_n)\} = \int_{[\tau \leq n]} y(\tau)d\mathbb{P} + \int_{[\tau > n]} y(n)d\mathbb{P}.
\]
\[
= E\{y(t)\} = 0.
\]

Since \( \tau > n, |y(n)| \leq \lambda_0 + \frac{1}{2} V(n) \), we have
\[
\left| \int_{[\tau > n]} y(n)d\mathbb{P} \right| \leq \left( \lambda_0 + \frac{1}{2} V(n) \right) P(\tau > n)
\]
\[
\leq \text{Const.} \times \frac{\sqrt{V(n)(V(n) + 2\lambda_0)}}{V(n) - 2\lambda_0} \exp \left\{ \frac{-(V(n) - 2\lambda_0)^2}{8V(n)} \right\}
\]
\[
\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

Thus, we have
\[
E\{y(t)\} = \lim_{n \rightarrow \infty} E\{y(\tau_n)\} = 0.
\]

Similarly, we write
\[
\bar{U}_n(t) = U(t) \quad \text{for } t < \tau_n
\]
\[
= U(\tau_n) \quad \text{for } t \geq \tau_n
\]
\[
\bar{Z}_n(t, \lambda) = Z(t, \lambda) \quad \text{for } t < \tau_n
\]
\[
= Z(\tau_n, \lambda) \quad \text{for } t \geq \tau_n.
\]

We have then new martingale processes \( \{\bar{U}_n(t), \bar{\mathcal{S}}_{t}^{(n)}, 0 \leq t < \infty \} \) and \( \{\bar{Z}_n(t, \lambda), \bar{\mathcal{S}}_{t}^{(n)}, 0 \leq t < \infty \} \). Therefore we have
\[
E\{\bar{U}_n(t)\} = \int_{[\tau \leq n]} U(\tau)d\mathbb{P} + \int_{[\tau > n]} U(n)d\mathbb{P}
\]
\[
= \int_{[\tau \leq n]} \{ y^2(\tau) - V(\tau) \}d\mathbb{P} + \int_{[\tau > n]} \{ y(n)^2 - V(n) \}d\mathbb{P}
\]
\[
= 0.
\]

Since,
\[
\left| \int_{[\tau > n]} \{ y(n)^2 - V(n) \}d\mathbb{P} \right| \leq \left\{ V(n) + \frac{1}{4}(2\lambda_0 + V(n))^2 \right\} \cdot P(\tau > n)
\]
\[
\rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]

We have
\[
\lim_{n \rightarrow \infty} \int_{[\tau \leq n]} [ y^2(\tau) - V(\tau) ]d\mathbb{P} = E\{(y(\tau))^2\} - E\{V(\tau)\} = 0.
\]

Similarly we have
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\[ 1 = E\{ Z(t, \lambda) \} = \int_{\{r \leq n\}} e^{\lambda y(r) - \frac{\lambda^2}{2} V(r)} dP + \int_{\{r > n\}} e^{\lambda y(r) - \frac{\lambda^2}{2} V(r)} dP. \]

Since, for each real \( \lambda \),

\[
\left| \int_{\{r > n\}} \exp \left\{ \lambda y(n) - \frac{\lambda^2}{2} V(n) \right\} dP \right| \leq \text{Const.} \times \frac{\sqrt{V(n)}}{|V(n) - 2\lambda^2|} \times \exp \left\{ -\frac{1}{8} \left[ (1-2\lambda)^2 V(n) - 4\lambda^2 + \frac{4\lambda^2}{V(n)} \right] \right\} \to 0 \quad \text{as } n \to \infty ,
\]

it follows immediately that for each real \( \lambda \)

\[
1 = \lim_{n \to \infty} \int_{\{r \leq n\}} \exp \left\{ \lambda y(r) - \frac{\lambda^2}{2} V(r) \right\} dP = E\{ \exp \{ \lambda y(\tau) - \frac{\lambda^2}{2} V(\tau) \} \} . \quad \text{Q. E. D.}
\]

**Example 2.1.** (c.f. [2], [7]). From theorem 2.1, it is easily obtain the A.S. N.'s \( E\{\tau|H_0\} \) and \( E\{\tau|H_1\} \) of our detection problem when the signal \( \phi(s) \) is constant \( \alpha > 0 \) which is in the white noise. From (2, 4), it is well known that \( \lambda_0 \) is given by

\[ \lambda_0 = \log \left( \frac{1 - \tau}{\tau} \right). \]

Let \( E_1 = \{ W(\tau) = \frac{\lambda_0}{\alpha} + \frac{\alpha^2}{2} \} \) and \( E_8 \) be the complementary event of \( E_1 \). Ther,

\[ \tau = \inf \{ t > 0 ; |W(t) - \frac{\alpha^2}{2} t| \geq \lambda_0/\alpha \} , \]

we have

\[ E\{ y(\tau) \} = \tau E\{ \lambda_0 + \frac{\alpha^2}{2} \cdot \tau | E_1 \} + (1-\tau) E\{ -\lambda_0 + \frac{\alpha^2}{2} \cdot \tau | E_8 \} \]

\[ = \frac{\alpha^2}{2} E(\tau) - (1-2\tau)\lambda_0 = 0 , \]

that is,

\[ E(\tau) = E\{\tau|H_0\} = E\{\tau|H_1\} \]

\[ = \frac{2}{\alpha^2} (1-2\tau) \log \left( \frac{1 - \tau}{\tau} \right) . \quad (2.8) \]

**Example 2.2.** Let \( \Delta > 0 \) be a suitably chosen small interval and let us put

\[ \phi_j(s) = 1 \quad \text{for } (j-1)\Delta \leq s < j\Delta, \; j = 1, 2, \cdots , \]

\[ = 0 \quad \text{otherwise} . \]

Let us assume that \( \phi(s) \) is a pulsed signal and is approximately expressed by

\[ \phi(s) = \sum_{j=1}^{\infty} h_j \cdot \phi_j(s) , \quad 0 \leq s < \infty , \]
where \( h > 0 \) is a given constant and \( \varepsilon_j = 0 \) or 1, \( j = 1, 2, 3, \ldots \).

We also assume that a relative frequency of the occurrence of pulses \( (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)/n \) is close to a constant \( \alpha \), \( 0 < \alpha \leq 1 \), except for the first several \( n \).

Then, we have approximately
\[
V(t) \doteq \alpha^2 a t, \quad \text{and}
\]
\[
y(t) = h \sum_{j=1}^{n} \varepsilon_j z_j + h \varepsilon_{n+1} [W(t) - W(n \Delta)],
\]
where \( z_j = W(j \Delta) - W((j-1) \Delta) \) and \( n \) is the largest integer not greater than \( t/\Delta \).

It is clear that \( \{y(t), 0 \leq t < \infty\} \) is a Gaussian process with the mean value zero and its covariance function \( R(s, t) \) is given approximately by
\[
R(s, t) \doteq ah \min(s, t).
\]

Hence, this detection problem of the pulsed signal \( \phi(s) \) in the white noise is nearly equivalent to the problem to detect a constant signal \( \phi_0(s) \equiv h \cdot \sqrt{a} \) in the white noise as shown in Example 2.1.

Thus, we have the A.S.N. which are given approximately by
\[
E \{ \tau \mid H_0 \} = E \{ \tau \mid H_1 \} \doteq \frac{2(1 - 2\gamma)}{ah^2} \cdot \log \left( \frac{1 - \gamma}{\gamma} \right).
\]

§ 3. Detection of a constant signal in a non-white Gaussian noise.

Similarly in § 2, let \( \{W(s), F_s, 0 \leq s < \infty\} \) be the Wiener process. Let \( \{\xi(s), 0 \leq s < \infty\} \) be a Gaussian noise process defined by
\[
\xi(t) = \int_0^t e^{-\beta(t-u)} dW(u), \quad 0 \leq t < \infty, \ldots
\]
where \( \beta \geq 0 \) is a non-negative constant.

We shall consider the following detection problem;
\[
H_0: \quad x(t) = \xi(t),
\]
\[
H_1: \quad x(t) = \alpha t + \xi(t), \quad \alpha > 0,
\]
where \( \alpha > 0 \), is a constant signal to be detected.

Let \( P_{it}, i = 0, 1, \) be the distribution of the observation \( \{x(s), 0 \leq s \leq t\} \) under \( H_i, i = 0, 1, \) respectively.

Then, the detection problem (3.2) is nonsingular and the logarithm of the likelihood ratio of \( P_{it} \) with respect to \( P_{0t} \) is given by
\[
L(x; t) = \log \frac{dP_{it}}{dP_{0t}}(x)
\]
\[
= L_0(x; t) + \frac{\beta}{2} \cdot \left\{ 2m(t)x(t) - (m(t))^2 \right\}
\]
\[
+ \frac{\beta^2}{2} \left\{ 2 \int_0^t m(s)x(s)ds - \int_0^t (m(s))^2 ds \right\}, \quad (3.3)
\]
where \( m(t) = \alpha t \), and
\[
L_0(x; t) = \alpha x(t) - \frac{\alpha^2}{2} t. \tag{3.4}
\]

Suppose that we adopt \( L_0(x; t) \) as a statistic for the problem (3.2) instead of the log-likelihood ratio \( L(x; t) \).

\( L_0(x; t) \) is actually not optimal for the problem (3.2) (See [3]) but it has several sub-optimal properties because it become optimal when \( \beta = 0 \), that is, the noise \( \xi(t) \) were white, and \( L_0(x; t) \) does not contain the parameter \( \beta \) which is intrinsic in the noise.

We set error probabilities to the equal to the prescribed value \( \gamma \), that is,
\[
P(\text{to accept } H_1|H_0) = P(\text{to accept } H_0|H_1) = \gamma, \tag{3.5}
\]
and consider only such decision rules that (3.5) holds.

We define a stopping \( \tau^* \) by
\[
\tau^* = \inf \{ t > 0 ; L_0(x; t) \leq -\lambda_0 \text{ or } L_0(x; t) \geq \lambda_1 \} \tag{3.6}
\]
where \( \lambda_0, \lambda_1 \) are positive constants such that our following decision rule satisfies (3.5).

Our decision rule based on the observation \( \{x(s), 0 \leq s \leq t\} \) will be formulated as follows; When \( L_0(x; \tau^*) = \lambda_1 \) (or \( -\lambda_0 \)), we stop sampling at \( t = \tau^* \) and decide \( H_1 \) (or \( H_0 \)) to be true, while as long as \( -\lambda_0 < L_0(x, s) < \lambda_1 \), \( 0 \leq s \leq t \), we continue sampling.

Since each distribution of \( L_0(x; t) \) under \( H_i, i = 0, 1 \), is symmetric to the other, the constants \( \lambda_0 \) and \( \lambda_1 \) must be equal under the condition (3.5).

Let us put
\[
V(t) = E[\xi(t)\xi] = \frac{(1 - e^{-2\beta t})}{2\beta^2}.
\]

Let us consider a continuous function \( f(t), 0 \leq t < \infty \), such that

(i) \( f(0) = -\lambda^* < 0 \),

(ii) \( f(t) = O(t^\alpha) \to \infty \) as \( t \to \infty \),

for some positive constant \( \alpha > 0 \).

We shall now define a stopping time \( \tau \) as follows;
\[
\tau = \inf \{ t > 0 ; \xi(t) \leq f(t) \text{ or } \xi(t) \geq 2\lambda^* + f(t) \}. \tag{3.7}
\]

The stopping time \( \tau^* \) defined by (3.6) is a special case of \( \tau \) by (3.7). Indeed, \( \tau^* \) corresponds to the case where \( f(t) = \frac{\alpha t}{2} - \lambda^* \) and \( \lambda^* = \lambda_0/\alpha \).

For a large \( t \), it is easily seen that
\[
P(\tau \leq t) \leq P(\xi(t) \leq 2\lambda^* + f(t)) \leq P(\xi(t) \geq f(t)) \leq \frac{1}{2\sqrt{\pi\beta}} \times \frac{1}{|f(t)|} \times \exp \{ -\beta \times [f(t)]^2 \}.
\]

Thus, we have
Lemma 3.1. For all real $k \geq 0$,

$$E(\tau^k) < \infty.$$  

Proof of lemma 3.1 is analogous to that of lemma 2.3.

Now, we obtain

Theorem 3.1. Let $\tau$ be defined by (3.7). Then we have

(i) $E(\xi(\tau)) = 0$,

(ii) $E(\xi(\tau)^2) = E(V(\tau))$, and

(iii) for each real $\lambda$,

$$E\left( \exp \left[ \lambda \xi(\tau) - \frac{\lambda^2}{2} V(\tau) \right] \right) = 1.$$  

(3.8)

Proof. It is clear that $\tau$ is a stopping time with respect to $F_t$, $t \geq 0$. The stochastic process $\{\xi(t), 0 \leq t < \infty\}$ is the unique non-anticipated solution of a stochastic differential equation;

$$d\xi(t) = -\beta \xi(t)dt + dW(t),$$  

(3.9)

with $\xi(0) = 0$. Hence, it enjoys the strong Markov property with respect to a Brownian stopping time, for example, say, $\tau$. (See [4]).

For any random variable $g$ and any measurable set $A$, we will write

$$E_A[g] = E[I_A \cdot g],$$

where $I_A$ is the indicator function of $A$.

Let $\mathcal{F}_\tau$ be the $\sigma$-field generated by $\tau$, that is, the totality of sets whose intersections with $[\tau > t]$ belong to $F_t$ for every $t$, $0 \leq t < \infty$. Then, we have from the strong Markov property,

$$E_{[\tau > t]}(\xi(t)) = E\{E[I_{[\tau > t]} \cdot \xi(t) | \mathcal{F}_\tau]\}$$

$$= E\{I_{[\tau > t]} \cdot E(\xi(t) | \tau, \xi(\tau))\}$$

$$= E_{[\tau > t]}(\xi(\tau)).$$

Since for $\tau > t$, $f(t) < \xi(t) < 2\lambda_0 + f(t)$, we have

$$|E_{[\tau > t]}(\xi(t))| \leq [2\lambda_0 + f(t)] \cdot P(\tau > t)$$

$$\leq \frac{1}{2\sqrt{\pi} \beta} \cdot \frac{|2\lambda_0 + f(t)|}{|f(t)|} \cdot \exp \{-\beta |f(t)|^2\}$$

$$\longrightarrow 0 \quad \text{as } t \to \infty.$$  

Thus, it is seen that

$$E(\xi(t)) = \lim_{t \to \infty} E_{[\tau > t]}(\xi(t)) + \lim_{t \to \infty} E_{[\tau > t]}(\xi(t))$$

$$= E(\xi(\tau)) = 0.$$  

We have shown that (i) holds.

Let us write
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\[ U(t) = \xi(t)^2 - V(t). \]

Then, \( U(t) \) is a functional of the Markov process \( \{ \xi(s), 0 \leq s < \infty \} \) and hence we have

\[
E_{\tau \leq t} \{ U(t) \} = E \{ E_{\tau \leq t} \cdot U(t) \mid \mathcal{F}_\tau \} \\
= E \{ I_{\tau \leq t} \cdot E \{ U(t) \mid \tau, \xi(\tau) \} \} \\
= E_{\tau \leq t} \{ U(\tau) \} = E_{\tau \leq t} \{ \xi(\tau)^2 - V(\tau) \}.
\]

Since, for \( \tau > t, |\xi(\tau)| \leq 2\lambda^* + f(t) \), it follows that

\[
\left| E_{\tau > t} \{ U(t) \} \right| \leq \left[ V(t) + (2\lambda^* + f(t))^2 \right] \cdot P(\tau > t)
\]

\[
\leq \frac{1}{2\sqrt{\pi} \beta} \left[ \frac{V(t) + (2\lambda^* + f(t))^2}{|f(t)|} \right] \cdot e^{-\frac{\lambda}{f(t)}}
\]

\[
\to 0 \quad \text{as } t \to \infty.
\]

Thus, we have

\[
E \{ U(t) \} = \lim_{t \to \infty} E_{\tau \leq t} \{ U(t) \} = E_{\tau \leq t} \{ \xi(\tau)^2 - V(\tau) \} = 0.
\]

We have shown that \( E \{ \xi(\tau)^2 \} = E \{ V(\tau) \} \).

Let us put for each real \( \lambda \),

\[
Z(t, \lambda) = \exp \left[ \lambda \xi(t) - \frac{\lambda^2}{2} V(t) \right], \quad 0 \leq t < \infty.
\]

Then, it is clear that \( Z(t, \lambda) \) is \( F_t \)-measurable and \( E \{ Z(t, \lambda) \} \equiv 1 \).

Thus, we have

\[
E_{\tau \leq t} \{ Z(t, \lambda) \} = E \{ E_{\tau \leq t} Z(t, \lambda) \mid \mathcal{F}_\tau \} \\
= E \{ I_{\tau \leq t} E \{ Z(t, \lambda) \mid \tau, \xi(\tau) \} \} \\
= E_{\tau \leq t} \{ Z(\tau, \lambda) \}.
\]

Now, we shall evaluate \( E_{\tau > t} \{ \exp \left[ \lambda \xi(t) - \frac{\lambda^2}{2} V(t) \right] \} \).

Since, for \( \tau > t, f(t) < \xi(t) < 2\lambda^* + f(t) \), we have for each non-negative real \( \lambda \),

\[
E_{\tau > t} \{ Z(t, \lambda) \} \leq \exp \left[ \lambda \left( 2\lambda^* + f(t) \right) - \frac{\lambda^2}{2} V(t) \right] \cdot P(\tau > t)
\]

\[
\leq \frac{1}{2|f(t)| \sqrt{\pi} \beta} \exp \left( 2\lambda \beta^* - \beta |f(t)| \left( 1 - \frac{\lambda}{\beta f(t)} \right) \right)
\]

\[
\to 0 \quad \text{as } t \to \infty,
\]

and for each negative real \( \lambda \),

\[
E_{\tau > t} \{ Z(t, \lambda) \} \leq \exp \left( \lambda f(t) - \frac{\lambda^2}{2} V(t) \right)
\]

\[
\to 0 \quad \text{as } t \to \infty.
\]

Thus, it follows that for each real \( \lambda \),
This completes the proof of theorem 3.1. Q. E. D.

The stopping time $\tau^*$ is the special case of $\tau$ in (3.7) and hence from theorem 3.1 it is seen that

$$E[\xi(\tau^*)] = 0,$$
$$E[\xi(\tau^*)^2] = E[V(\tau^*)],$$

and for each real $\lambda$,

$$E[\exp\left(\lambda \xi(\tau^*) - \frac{\lambda^2}{2} V(\tau^*)\right)] = 1.$$

**COROLLARY.**

$$E[\tau^*|H_0] = E[\tau^*|H_1] = \frac{2\lambda_0(1-2\gamma)}{\alpha^2},$$

where $\lambda_0$ is such a constant that the error probabilities satisfy (3.5).

**PROOF.** Let $E_0 = \{\xi(\tau^*) = -\frac{\alpha}{2} \tau^* - \frac{\lambda_0}{\alpha}\}$ and $E_1$ be the complementary event of $E_0$.

Then, by noticing that $\tau^*$ is equal to $\tau$ when $f(\tau) = \frac{\alpha}{2} - \tau$ and $\lambda^* = \lambda_0/\alpha$. and also that $P(E_1|H_0) = \gamma$ and $P(E_0|H_0) = 1-\gamma$, it follows from theorem 3.1 that

$$E[\xi(\tau^*)] = \frac{\alpha}{2} [E[\tau^*|E_0] \cdot P(E_0|H_0) + E[\tau^*|E_1] \cdot P(E_1|H_0)]$$

$$- \frac{\lambda_0}{\alpha} P(E_0|H_0) + \frac{\lambda_0}{\alpha} P(E_1|H_0)$$

$$= \frac{\alpha}{2} E[\tau^*|H_0] - \frac{\lambda_0}{\alpha} (1-2\gamma) = 0.$$

Thus, we have proved corollary. Q. E. D.

**References**


