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A SEQUENTIAL CLASSIFICATION INTO ONE OF SEVERAL POPULATIONS

By

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1. Introduction.

In the paper [3], Samuel considered a classification problem of two populations which is sequential in the following sense. A group of n individuals is known to belong to either π_0 or π_1. The individuals of the group arrive sequentially for inspection and classification, and the classification of the i-th individual has to be made immediately after he has been inspected, $i=1, 2, \cdots, n$. In the paper [3], Samuel presented a complete class of decision rules.

In this paper, an approach to the problem from the stand point of game theory under the same formulation after generalising the problem to the classification to m populations. In § 3, we can prove the existence of optimal solution and in § 4, we present Bayes rules and essentially complete class. In § 5, we prove that the players, the nature and the statistician, have optimal solution, and thus the games have the values.

2. Formulation.

The m populations $\pi_1, \pi_2, \cdots, \pi_m$ with the known distribution functions $P_1, P_2, \cdots, P_m$ are given, and a group of n individuals is known to belong to one of π_j, $j=1, 2, \cdots, m$. The n individuals arrive sequentially for inspection and classification, and the classification of the i-th individuals has to be made immediately after he has been observed, $i=1, 2, \cdots, n$.

Assume the following loss for classifying an individual:

true \ decision	π_1	π_2	$\cdots\cdots$	π_m
π_1	$a_{11}=0$	a_{12}	$\cdots\cdots$	a_{1m}
π_2	a_{21}	$a_{22}=0$	$\cdots\cdots$	a_{2m}
$\vdots$				
π_m	a_{m1}	a_{m2}	$\cdots\cdots$	$a_{mm}=0$

(2.1)

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where a_{ij} $(i \neq j)$ are given positive numbers.

Let $X_1, X_2, \cdots, X_n$ be the independent random variables corresponding to n individuals and its observed value be $x_1, x_2, \cdots, x_n$. Let for each i, $i=1, 2, \cdots, n$, X_i has the same distribution P_θ corresponding to π_θ, $\theta=1, 2, \cdots, m$ where θ is unknown. We may without loss of generality assume that P_θ, $\theta=1, 2, \cdots, m$, are probability measure corresponding to density function $f(x, \theta)$, $\theta=1, 2, \cdots, m$ with respect to a specified measure space $(\mathfrak{X}, \mathfrak{A}, \mu)$, where $\mathfrak{X}$ is a space of outcomes x, $\mathfrak{A}$ is a σ-field of subsets of $\mathfrak{X}$ and assume, μ is σ-finite and $\mathfrak{X}$ is a Euclidean space or $\mathfrak{A}$ is with countable number of generators. Thus

$$P_\theta(A)=\int_A f(x, \theta)d\mu \qquad \text{for all } A \in \mathfrak{A}.$$

Let $\boldsymbol{x}_i=(x_1, \cdots, x_i)$, $i=1, 2, \cdots, n$, then any (randomized) compound decision rule for the above problem can be written as $\Phi_n=(\phi_1(\boldsymbol{x}_1), \phi_2(\boldsymbol{x}_2), \cdots, \phi_n(\boldsymbol{x}_n))$ with

$$\phi_i(\boldsymbol{x}_i)=(\phi_i^{(1)}(\boldsymbol{x}_i), \cdots, \phi_i^{(m)}(\boldsymbol{x}_i)), \qquad i=1, 2, \cdots, n.$$

Where $0 \leqq \phi_i^{(j)}(\boldsymbol{x}_i) \leqq 1$, $j=1, 2, \cdots, m$ and $\sum_{j=1}^m \phi_i^{(j)}(\boldsymbol{x}_i)=1$, then $\phi_i^{(j)}(\boldsymbol{x}_i)$ is the probability with which one classifies the i-th individual to π_j when $\boldsymbol{X}_i=\boldsymbol{x}_i$ is observed, and being measurable functions in the i-th product space. Set

$$f(\boldsymbol{x}_i, \theta)=\prod_{j=1}^{i} f(x_j, \theta), \qquad i=1, 2, \cdots, n;\ \theta=1, 2, \cdots, m.$$

Let

$$\Phi^{(i)}=\{\phi_i(\boldsymbol{x}_i)=(\phi_i^{(1)}(\boldsymbol{x}_i), \cdots, \phi_i^{(m)}(\boldsymbol{x}_i)) \mid 0 \leqq \phi_i^{(j)}(\boldsymbol{x}_i) \leqq 1,$$
$$j=1, 2, \cdots, m, \text{ and } \sum_{j=1}^m \phi_i^{(j)}(\boldsymbol{x}_i)=1 \text{ except } \mu^i\text{-null sets}\},$$
$$i=1, 2, \cdots, n.$$

Let $R(\theta, \Phi_n)$ denote the risk defined as average expected loss, incurred by using Φ_n when the group actually belongs to π_θ, $\theta=1, 2, \cdots, m$. Then risk incur red dy i-th inspection and classfication are

$$R(\theta, \phi_i)=\sum_{j=1}^m a_{\theta j}\int_{\mathfrak{X}-N} \phi_i^{(j)}(\boldsymbol{x}_i) f(\boldsymbol{x}_i, \theta)d\mu^i$$
$$=\sum_{j=1}^m a_{\theta j}\int \phi_i^{(j)}(\boldsymbol{x}_i) f(\boldsymbol{x}_i, \theta)d\mu^i \tag{2.2}$$

where $N=\{\boldsymbol{x}_i;\ \sum_{j=1}^m \phi_i^{(j)}(\boldsymbol{x}_i) \neq 1\}$ and $\mu^i(N)=0$ so that, compound risk is define by

$$R(\theta, \Phi_n)=\frac{1}{n}\sum_{i=1}^m R(\theta, \phi_i). \tag{2.3}$$

Let $p=(p_1, p_2, \cdots, p_m)$, $p_i \geqq 0$, $i=1, 2, \cdots, m$ and $\sum_{i=1}^m p_i=1$ be a prior probability distribution on $(\pi_1, \pi_2, \cdots, \pi_m)$, then Bayes risks for p, in each i-th inspection, are

$$R(p, \phi_i)=\sum_{\theta=1}^m p_\theta R(\theta, \phi_i) \tag{2.4}$$

and compound Bayes risk is

$$R(p, \Phi_n) = \sum_{\theta=1}^{m} p_\theta R(\theta, \Phi_n) = \frac{1}{n} \sum_{i=1}^{n} R(p, \phi_i) . \tag{2.5}$$

Note: If $\phi_i \in \Phi^{(i)}$ and $N = \{x_i ; \sum_{j=1}^{m} \phi_i^{(j)}(x_i) \neq 1\} \neq \phi$, $(\mu^i(N) = 0)$ we defined ψ_i as the following if $x_i \notin N$, $\psi_i^{(j)}(x_i) = \phi_i^{(j)}(x_i)$ for all j, if $x_i \in N$, we define arbitrary such that $0 \leqq \psi_i^{(j)}(x_i) \leqq 1$ for all j and $\sum_{j=1}^{m} \psi_i^{(j)}(x_i) = 1$, then, obviously, $R(\theta, \psi_i) = R(\theta, \phi_i)$ for all θ, i. e. ψ_i is as good as ϕ_i, so that, we can always find rule such that $0 \leqq \psi_i^{(j)}(x_i) \leqq 1$, $j = 1, 2, \cdots, m$ and $\sum_{j=1}^{m} \psi_i^{(j)}(x_i) = 1$ for all x_i.

In our problem, for each i, $i = 1, 2, \cdots, n$, let $a_\theta^{(i)} = R(\theta, \phi_i)$, $\theta = 1, 2, \cdots, m$, $\phi_i \in \Phi^{(i)}$ and let $S^{(i)}$ be a set of all points of $a^{(i)} = (a_1^{(i)}, \cdots, a_m^{(i)})$. The payoff function is define by

$$M(\theta, a^{(i)}) = a_\theta^{(i)} , \qquad \theta = 1, 2, \cdots, m .$$

Then $(I_m, S^{(i)}, M)$ is an S game, as defined page 47 of [1], in i-th inspection in our consideration.

Let $a_\theta = R(\theta, \Phi_n) = \frac{1}{n} \sum_{i=1}^{n} a_\theta^{(i)}$, $\theta = 1, 2, \cdots, m$ for $a^{(i)} = (a_1^{(i)}, \cdots, a_m^{(i)}) \in S^{(i)}$, $i = 1, 2, \cdots, n$. Let S be a set of all points $a = (a_1, a_2, \cdots, a_m)$ and the payoff function defined by

$$M(\theta, a) = a_\theta = \frac{1}{n} \sum_{i=1}^{n} a_\theta^{(i)} , \qquad \theta = 1, 2, \cdots, m .$$

Then (I_m, S, M) is also an S game in the compounded decision problem in our consideration.

Let $\boldsymbol{E} = \{ p = (p_1, \cdots, p_m) ; p_i \geqq 0, i = 1, 2, \cdots, m$ and $\sum_{i=1}^{m} p_i = 1\}$ be a set of all prior distributions over I_m, i. e. the mixed extension of I_m. We known $M(p, a^{(i)}) = \sum_{\theta=1}^{m} p_\theta a_\theta^{(i)}$ for $p \in \boldsymbol{E}$, $a^{(i)} \in S^{(i)}$ and $M(p, a) = \sum_{\theta=1}^{m} p_\theta a_\theta = \frac{1}{n} \sum_{i=1}^{n} M(p, a^{(i)})$ for $p \in \boldsymbol{E}$, $a \in S$.

3. Existence of optimal solution.

In the following we use the usually topology in m space, and thus the set $\boldsymbol{E}$ is closed, bounded convex subset of m-space.

LEMMA 3.1. *For each i, $i = 1, 2, \cdots, n$, $S^{(i)}$ is a convex set and S is also a convex set.*

PROOF. Let $a^{(i)} = (a_1^{(i)}, \cdots, a_m^{(i)})$, $b^{(i)} = (b_1^{(i)}, \cdots, b_m^{(i)}) \in S^{(i)}$ where $a_\theta^{(i)} = R(\theta, \phi_i)$, $b_\theta^{(i)} = R(\theta, \psi_i)$, $\phi_i, \psi_i \in \Phi^{(i)}$, for any $0 < \alpha < 1$, $\alpha a^{(i)} + (1-\alpha) b^{(i)} = (\alpha a_1^{(i)} + (1-\alpha) b_1^{(i)}, \cdots, \alpha a_m^{(i)} + (1-\alpha) b_m^{(i)})$, then by (2.2)

$$\alpha a_\theta^{(i)} + (1-\alpha) b_\theta^{(i)} = \sum_{j=1}^{m} a_{\theta j} \int [\alpha \phi_i^{(j)}(x_i) + (1-\alpha) \psi_i^{(j)}(x_i)] f(x_i, \theta) d\mu^i$$

and $0 \leqq \alpha \phi_i^{(j)}(x_i) + (1-\alpha) \psi_i^{(j)}(x_i) \leqq 1$ for all j and let

$$N_1 = \{x_i ; \sum_{j=1}^{m} \phi_i^{(j)}(x_i) \neq 1\} , \qquad N_2 = \{x_i ; \sum_{j=1}^{m} \psi_i^{(j)}(x_i) \neq 1\} ,$$

$$N=\{\boldsymbol{x}_i\,;\ \sum_{j=1}^{m}[\alpha\phi_i^{(j)}(\boldsymbol{x}_i)+(1-\alpha)\psi_i^{(j)}(\boldsymbol{x}_i)]\neq 1\}\,,$$

since $\mu^i(N_k)=0$, $k=1, 2$, and $\mu^i(N)=0$, so that $\alpha a^{(i)}+(1-\alpha)b^{(i)}\in S^{(i)}$, $S^{(i)}$ is convex set. Let

$$a=(a_1, a_2, \cdots, a_m)\,, \qquad b=(b_1, b_2, \cdots, b_m)\in S\,,$$

where $a_\theta=\frac{1}{n}\sum_{i=1}^{n}a_\theta^{(i)}$, $b_\theta=\frac{1}{n}\sum_{i=1}^{n}b_\theta^{(i)}$, $a_\theta^{(i)}=R(\theta, \phi_i)$, $b_\theta^{(i)}=R(\theta, \psi_i)$, $\phi_i, \psi_i\in\Phi^{(i)}$, since, for any $0\leqq\alpha\leqq 1$, $\alpha a_\theta+(1-\alpha)b_\theta=\frac{1}{n}\sum_{i=1}^{n}[\alpha a_\theta^{(i)}+(1-\alpha)b_\theta^{(i)}]$ and from above, we can get S is a convex set.

LEMMA 3.2. *For each i, $i=1, 2, \cdots, n$, $S^{(i)}$ is a closed and S is also closed.*

PROOF. For each i, $i=1, 2, \cdots, n$, let $s_k^{(i)}=(b_{ik}^{(i)}, \cdots, b_{km}^{(i)})$. Where $b_{\theta k}^{(i)}=R(\theta, \psi_{ik})$, $\theta=1, 2, \cdots, m$, $k=1, 2, \cdots$ be any one sequence of points in $S^{(i)}$, where $\psi_{ik}=(\psi_{ik}^{(1)}, \cdots, \psi_{ik}^{(m)})\in\Phi^{(i)}$, $k=1, 2, \cdots$ we know, by [5], p. 354, weak compactness theorem, for sequence $\{\psi_{ik}^{(1)}\}$ there exists a subsequence $\{\psi_{ik_1}^{(1)}\}$ and $0\leqq\psi_i^{(1)}\leqq 1$ such that

$$\int\psi_{ik_1}^{(1)}(\boldsymbol{x}_i)f(\boldsymbol{x}_i, \theta)d\mu^i \longrightarrow \int\psi_i^{(1)}(\boldsymbol{x}_i)f(\boldsymbol{x}_i, \theta)d\mu^i$$

for all $\theta=1, 2, \cdots, m$ and for sequence $\{\psi_{ik_1}^{(2)}\}$ there exists a subsequence $\{\psi_{ik_{12}}^{(2)}\}$ and $0\leqq\psi_i^{(2)}\leqq 1$ such that

$$\int\psi_{ik_{12}}^{(2)}(\boldsymbol{x}_i)f(\boldsymbol{x}_i, \theta)d\mu^i \longrightarrow \int\psi_i^{(2)}(\boldsymbol{x}_i)f(\boldsymbol{x}_i, \theta)d\mu^i$$

for all $\theta=1, 2, \cdots, m$ and so on, finally, for sequence $\{\psi_{ik_{12\cdots(m-1)}}^{(m)}\}$ there exists a subsequence $\{\psi_{ik_{12\cdots m}}^{(m)}\}$ and $0\leqq\psi_i^{(m)}\leqq 1$ such that

$$\int\psi_{ik_{12\cdots m}}^{(m)}(\boldsymbol{x}_i)f(\boldsymbol{x}_i, \theta)d\mu^i \longrightarrow \int\psi_i^{(m)}(\boldsymbol{x}_i)f(\boldsymbol{x}_i, \theta)d\mu^i$$

for all $\theta=1, 2, \cdots, m$ also, in the proof of the weak Compactness theorem, we know

$$\lim_{k_{12\cdots m}\to\infty}\int_A\psi_{ik_{12\cdots m}}^{(j)}(\boldsymbol{x}_i)d\mu^i=\int_A\psi_i^{(j)}(\boldsymbol{x}_i)d\mu^i$$

for all $A\in\mathfrak{A}^i$ and $j=1, 2, \cdots, m$.

Since, for each $k_{12\cdots m}$, $\sum_{j=1}^{m}\psi_{ik_{12\cdots m}}^{(j)}(\boldsymbol{x}_i)=1$ *a.s.* so that $\int_A d\mu^i=\int_A\sum_{j=1}^{m}\psi_{ik_{12\cdots m}}^{(j)}(\boldsymbol{x}_i)d\mu^i$ for all $A\in\mathfrak{A}^i$ and all $k_{12\cdots m}$ therefore $\int_A d\mu^i=\int_A\sum_{j=1}^{m}\psi_i^{(j)}(\boldsymbol{x}_i)d\mu^i$ for all $A\in\mathfrak{A}^i$ and we also have $\sum_{j=1}^{m}\psi_i^{(j)}(\boldsymbol{x}_i)=1$ *a.s.* so that $\psi_2=(\psi_i^{(1)}, \cdots, \psi_i^{(m)})\in\Phi^{(i)}$ and $\lim_{k_{12\cdots m}\to\infty}b_{\theta k_{12\cdots m}}^{(i)}=R(\theta, \psi_i)$ for $\theta=1, 2, \cdots, m$ therefore $\lim_{k_{12\cdots m}\to\infty}s_{k_{12\cdots m}}^{(i)}\in S^{(i)}$, which establishes the closedness of $S^{(i)}$.

Let $s_k=(b_{1k}, \cdots, b_{mk})$, $k=1, 2, \cdots$ be any one sequence of points in S where $b_{\theta k}=\frac{1}{n}\sum_{i=1}^{n}b_{\theta k}^{(i)}$, $b_{\theta k}^{(i)}=R(\theta, \psi_{ik})$, $\theta=1, 2, \cdots, m$, $\psi_{ik}=(\psi_{ik}^{(1)}, \cdots, \psi_{ik}^{(m)})\in\Phi^{(i)}$.

From above, we know, for sequence $\{b_{\theta k}^{(1)}\}$, there exists subsequence $\{b_{\theta k_1}^{(1)}\}$ and $\psi_1=(\psi_1^{(1)}, \cdots, \psi_1^{(m)})\in\Phi^{(1)}$ such that $\lim_{k_1\to\infty}b_{\theta k_1}^{(1)}=R(\theta, \psi_1)$ for all $\theta=1, 2, \cdots, m$ and for sequence $\{b_{\theta k_1}^{(2)}\}$, there exists subsequence $\{b_{\theta k_{12}}^{(2)}\}$ and $\psi_2=(\psi_2^{(1)}, \cdots, \psi_2^{(m)})\in\Phi^{(2)}$ such that $\lim_{k_{12}\to\infty} b^{(2)}_{\theta k_{12}} = R(\theta, \phi_2)$ for all $\theta = 1, 2, \cdots, m$ and so on, finally, for sequence $\{b^{(m)}_{\theta k_{12\cdots(m-1)}}\}$ there exists subsequence $\{b^{(m)}_{\theta k_{12\cdots m}}\}$ and $\phi_m = (\phi^{(1)}_m, \cdots, \phi^{(m)}_m) \in \Phi^{(m)}$ such that $\lim_{k_{12\cdots m}\to\infty} b^{(m)}_{\theta k_{12\cdots m}} = R(\theta, \phi_m)$ for all $\theta = 1, 2, \cdots, m$, therefore $\lim_{k_{12\cdots m}\to\infty} b^{(i)}_{\theta k_{12\cdots m}} = R(\theta, \phi_i)$ for all $\theta = 1, 2, \cdots, m$, $i = 1, 2, \cdots, m$ and $\lim_{k_{12\cdots m}\to\infty} b_{\theta k_{12\cdots m}} = \frac{1}{n}\sum_{i=1}^{n} R(\theta, \phi_i)$ for all $\theta = 1, 2, \cdots, m$ so that $\lim_{k_{12\cdots m}\to\infty} s_{k_{12\cdots m}} \in S$, and finally S is a closed set.

For lemma 3.1., lemma 3.2. and the theorem 2.4.2 of page 49. [1], we have the following theorem.

THEOREM 3.1. *For our problem, in each i-th inspection i.e. for the S game* $(I_m, S^{(i)}, M)$, $i = 1, 2, \cdots, n$, *we have a value and player I (nature) has a good strategy and also, player II (statistician) has a good strategy in* $S^{(i)}$. *Also, in compound decision problem, i.e. for the S game* (I_m, S, M), *we have a value and player II has a good strategy in S and player I has a good strategy.*

Let, for each $i = 1, 2, \cdots, n$, the value of the S game $(I_m, S^{(i)}, M)$ be v_i and the value of the S game (I_m, S, M) is v.

Let, for each $i = 1, 2, \cdots, n$, a good strategy for player I be $p^{(i)} = (p^{(i)}_1, \cdots, p^{(i)}_m) \in \boldsymbol{E}$ and a good strategy for player II be $a^{(i)} = (a^{(i)}_1, \cdots, a^{(i)}_m) \in S^{(i)}$, where $a^{(i)}_\theta = R(\theta, \phi_i)$, $\theta = 1, 2, \cdots, m$. Let, in compound decision problem, a good strategy for player I be $q = (q_1, \cdots, q_m) \in \boldsymbol{E}$ and a good strategy for player II be $b = (b_1, \cdots, b_m) \in S$ where $b_\theta = \frac{1}{n}\sum_{i=1}^{n} b^{(i)}_\theta$ and $b^{(i)}_\theta = R(\theta, \phi^{(0)}_i)$, $i = 1, 2, \cdots, n$, so that $v_i = M(p^{(i)}, a^{(i)})$, $i = 1, 2, \cdots, n$ and $v = M(q, b)$. Since, for each i, $M(p, a^{(i)}) \leqq v_i \leqq M(p^{(i)}, c^{(i)})$ for all $p \in \boldsymbol{E}$, $c^{(i)} \in S^{(i)}$. and $M(p, b) \leqq v \leqq M(q, c)$ for all $p \in \boldsymbol{E}$, $c \in S$. Then $M(q, a^{(i)}) \leqq v_i$ for $i = 1, 2, \cdots, n$ so that, we have

$$v \leqq \frac{1}{n}\sum_{i=1}^{n} v_i .$$

Let $\Phi = \{\Phi_n = (\phi_1(\boldsymbol{x}_1), \cdots, \phi_n(\boldsymbol{x}_n))\,;\ \phi_i(\boldsymbol{x}_i) \in \Phi^{(i)},\ i = 1, 2, \cdots, n\}$.

From [1], p. 11, definition 1.4.1. and definition 1.4.2. we know, for each i, $i = 1, 2, \cdots, n$, the game $(I_m, \Phi^{(i)}, R)$ and the S game $(I_m, S^{(i)}, M)$ is equivalent. Also the game (I_m, Φ, R) and the S game (I_m, S, M) is equivalent.

By [1], p. 16, theorem 1.6.2. and p. 28, theorem 1.8.2., the values of two equivalent games are equal. So that, from theorem 3.1., for each i, the game $(I_m, \Phi^{(i)}, R)$ has a value same as the S game $(I_m, S^{(i)}, M)$, also, the game (I_m, Φ, R) has a value same as the S game (I_m, S, M).

4. Bayes decision rule and essentially complete class.

We consider the games $(I_m, \Phi^{(i)}, R)$, $i = 1, 2, \cdots, n$ and (I_m, Φ, R).

THEOREM 4.1. *Let* $p = (p_1, p_2, \cdots, p_m) \in \boldsymbol{E}$. *If* $R(p, \phi^{(0)}_i) = \inf_{\phi_i \in \Phi^{(i)}} R(p, \phi_i)$ *for each* i, $i = 1, 2, \cdots, n$, *then* $\Phi^{(0)}_n = (\phi^{(0)}_1, \cdots, \phi^{(0)}_n)$ *is a compound Bayes rule with respect to* p, *i.e.*

$$R(p, \Phi^{(0)}_n) = \inf_{\Phi_n \in \Phi} R(p, \Phi_n) .$$

PROOF. By (2·5) and assumption

$$R(p, \Phi_n^{(0)}) = \frac{1}{n} \sum_{i=1}^{n} R(p, \phi_i^{(0)}) \leqq \frac{1}{n} \sum_{i=1}^{n} R(p, \phi_i) = R(p, \Phi_n)$$

for all $\Phi_n = (\phi_1, \cdots, \phi_n) \in \Phi$, so that $R(p, \Phi_n^{(0)}) = \inf_{\Phi_n \in \Phi} R(p, \Phi_n)$. The theorem is thus proved.

For each i, $i=1, 2, \cdots, n$, and each $p \in \boldsymbol{E}$ we can find a Bayes rule: as follows. Since

$$R(p, \phi_i) = \sum_{\theta=1}^{m} p_\theta R(\theta, \phi_i)$$

$$= \sum_{j=1}^{m} \int \left[\sum_{\theta=1}^{m} p_\theta a_{\theta j} f(\boldsymbol{x}_i, \theta) \right] \phi_i^{(j)}(\boldsymbol{x}_i) d\mu^i ,$$

let

$$A_j(\boldsymbol{x}_i) = \sum_{\theta=1}^{m} p_\theta a_{\theta j} f(\boldsymbol{x}_i, \theta) , \qquad j=1, 2, \cdots, m .$$

Define the decision rule by,

(4.1) $$0 \leqq \phi_{ip}^{(k)}(\boldsymbol{x}_i) \leqq 1 \qquad \text{for } k=1, 2, \cdots, m \text{ and } \sum_{k \in I(x_i)} \phi_{ip}^{(k)}(\boldsymbol{x}_i) = 1$$

when

$$I(\boldsymbol{x}_i) = \{k\,;\, A_k(\boldsymbol{x}_i) = \min_{1 \leqq j \leqq m} A_j(\boldsymbol{x}_i)\} .$$

Let

$$\phi_{ip}(\boldsymbol{x}_i) = (\phi_{ip}^{(1)}(\boldsymbol{x}_i), \cdots, \phi_{ip}^{(m)}(\boldsymbol{x}_i)) .$$

As we have

$$R(p, \phi_i) - R(p, \phi_{ip}) = \sum_{j=1}^{m} \sum_{k=1}^{m} \int [A_j(\boldsymbol{x}_i) - A_k(\boldsymbol{x}_i)] \phi_i^{(j)}(\boldsymbol{x}_i) \phi_{ip}^{(k)}(\boldsymbol{x}_i) d\mu^i$$

$$\geqq 0 \qquad \text{for all } \phi_i \in \Phi^{(i)} ,$$

therefore the decision rule defined by (4.1) is Bayes with respect to p, i.e. $R(p, \phi_{ip}) = \inf_{\phi_i \in \Phi^{(i)}} R(p, \phi_i)$. We thus have the following

THEOREM 4.2. *For each i, $i=1, 2, \cdots, n$, decision rule $\phi_i(\boldsymbol{x}_i) = (\phi_i^{(1)}(\boldsymbol{x}_i), \cdots, \phi_i^{(m)}(\boldsymbol{x}_i))$ defined by* (4.1) *is a Bayes decision rule with respect to p and $\Phi_n = \phi_1, \cdots, \phi_n)$ is a compound Bayes decision rule with respect to p.*

We also have the following.

THEOREM 4.3. *The essentially complete class of our compound decision problem is give by*

$$\boldsymbol{C} = \{\Phi_n^p\,;\, p = (p_1, \cdots, p_n) \in \boldsymbol{E}\}$$

where

$$\Phi_n^p = (\phi_{1p}(\boldsymbol{x}_1), \cdots, \phi_{np}(\boldsymbol{x}_n))$$

and

$$\phi_{ip}(\boldsymbol{x}_i) = (\phi_{ip}^{(1)}(\boldsymbol{x}_i), \cdots, \phi_{ip}^{(m)}(\boldsymbol{x}_i))$$

is defined by (4.1).

PROOF. For each $p \in \boldsymbol{E}$ and $\Phi_n \in \Phi - \boldsymbol{C}$ there exist $\Phi_n^p \in \boldsymbol{C}$ such that

$$R(p, \Phi_n^{(p)}) \leqq R(p, \Phi_n) .$$

5. Optimal solution.

In the following, we consider the optimal solution for the game (I_m, Φ, M).

METHOD 1. Since for each $\Phi_n=(\phi_1(\boldsymbol{x}_1), \cdots, \phi_n(\boldsymbol{x}_n))\in\Phi$, we have $R(p, \Phi_n) =\frac{1}{n}\sum_{i=1}^{n}\sum_{\theta=1}^{m}p_\theta R(\theta, \phi_i)$ and $|R(\theta, \phi_i)|\leqq\sum_{j=1}^{m}a_{\theta j}$. So that, by [1], p. 40, theorem 2.2.7. $\inf_{\Phi_n\in\Phi} R(p, \Phi_n)$ is continuous function of $p(\in E)$. Since $\boldsymbol{E}$ is closed and bounded, $\boldsymbol{C}$ is a essentially complete, therefore there exists $q\in\boldsymbol{E}$ such that

$$\sup_{p\in E}\inf_{\Phi_n\in\Phi} R(p, \Phi_n)=\inf_{\Phi_n\in\Phi} R(q, \Phi_n)$$

and there exists $\Phi_n^q\in\boldsymbol{C}$ corresponding to q such that

$$\inf_{\Phi_n\in\Phi} R(q, \Phi_n)=R(q, \Phi_n^q)\,.$$

By section 3, we know the game (I_m, Φ, M) has a value v. So that, we get the following theorem.

THEOREM 5.1. *There exists $q\in\boldsymbol{E}$ such that*

$$\inf_{\Phi_n\in\Phi} R(q, \Phi_n)=\sup_{p\in E}\inf_{\Phi_n\in\Phi} R(p, \Phi_n)$$

and let $\Phi_n^q\in\boldsymbol{C}$ be corresponding to q. Then $v=R(q, \Phi_n^q)$ is a value of the game (I_m, Φ, R) and q is a good strategy for nature and Φ_n^q is a good strategy for statistician.

METHOD 2. For each $p\in\boldsymbol{E}$ there exists $\Phi_n^p\in\boldsymbol{C}$ such that

$$R(p, \Phi_n^p)=\inf_{\Phi_n\in\Phi} R(p, \Phi_n)$$

so that,

$$\sup_{p\in E}\inf_{\Phi_n\in\Phi} R(p, \Phi_n)=\sup_{p\in E} R(p, \Phi_n^p)$$

we can easily prove, $R(p, \Phi_n^p)$ is a continuous function of p, so that, there exists $q\in\boldsymbol{E}$ such that

$$\sup_{p\in E} R(p, \Phi_n^p)=R(q, \Phi_n^q)\,.$$

Therefore, we can get the following theorem.

THEOREM 5.2. *There exists $q\in\boldsymbol{E}$ such that*

$$\sup_{p\in E} R(p, \Phi_n^p)=R(q, \Phi_n^q)$$

where $\Phi_n^p\in\boldsymbol{C}$ is corresponding to p. Then $v=R(q, \Phi_n^q)$ is a value of the game (I_m, Φ, R) and q is a good strategy for nature and Φ_n^q is a good strategy for statistician.

EXAMPLE. Now, we given one artificial example for case $m=2$, $n=2$ and $a_{11}=a_{22}=0$, $a_{12}=a_{21}=a>0$. Let

$$f(x, \theta)=\left(\frac{\theta}{3}\right)^x\left(1-\frac{\theta}{3}\right)^{1-x}, \qquad x=0, 1,\ \theta=1, 2\,.$$

For each $p=(p_1, p_2)\in\boldsymbol{E}$, by theorem 4.3, we can get a optimal rule as the following

$$\phi_1^{(1)}(x_1)=1 \quad \text{if } x_1 \leqq \frac{1}{2}-\frac{1}{2}\log_2\frac{p_2}{p_1},$$

$$\phi_1^{(2)}(x_1)=1 \quad \text{if } x_1 > \frac{1}{2}-\frac{1}{2}\log_2\frac{p_2}{p_1},$$

Let

$$\phi_1^p(x)=(\phi_1^{(1)}(x_1),\ \phi_1^{(2)}(x_1)),$$

$$\phi_2^{(1)}(x_1,x_2)=1 \quad \text{if } x_1+x_2 \leqq 1-\frac{1}{2}\log_2\frac{p_2}{p_1},$$

$$\phi_2^{(2)}(x_1,x_2)=1 \quad \text{if } x_1+x_2 > 1-\frac{1}{2}\log_2\frac{p_2}{p_1},$$

Let

$$\phi_2^p(x_1,x_2)=(\phi_2^{(1)}(x_1,x_2),\ \phi_2^{(2)}(x_1,x_2)).$$

Then $\Phi_2^p=(\phi^p(x_1),\ \phi^p(x_1,x_2))\in \boldsymbol{C}$ and

$$R(p,\Phi_2^p)=\frac{ap_1}{2}\Big[\sum_{x_1>\frac{1}{2}-\frac{1}{2}\log_2\frac{p_2}{p_1}}\Big(\frac{1}{3}\Big)^{x_1}\Big(\frac{2}{3}\Big)^{1-x_1}$$

$$+\sum_{x_1+x_2>1-\frac{1}{2}\log_2\frac{p_2}{p_1}}\Big(\frac{1}{3}\Big)^{x_1+x_2}\Big(\frac{2}{3}\Big)^{2-x_1-x_2}\Big]$$

$$+\frac{ap_2}{2}\Big[\sum_{x_1\leqq\frac{1}{2}-\frac{1}{2}\log_2\frac{p_2}{p_1}}\Big(\frac{2}{3}\Big)^{x_1}\Big(\frac{1}{3}\Big)^{1-x_1}$$

$$+\sum_{x_1+x_2\leqq 1-\frac{1}{2}\log_2\frac{p_2}{p_1}}\Big(\frac{2}{3}\Big)^{x_1+x_2}\Big(\frac{1}{3}\Big)^{2-x_1-x_2}\Big]$$

$$=\frac{a}{18}\Big[3p_1\sum_{x_1>\frac{1}{2}-\frac{1}{2}\log_2\frac{p_2}{p_1}}2^{1-x_1}+p_1\sum_{x_1+x_2>1-\frac{1}{2}\log_2\frac{p_2}{p_1}}2^{2-x_1-x_2}$$

$$+3p_2\sum_{x_1\leqq\frac{1}{2}-\frac{1}{2}\log_2\frac{p_2}{p_1}}2^{x_1}+p_2\sum_{x_1+x_2\leqq 1-\frac{1}{2}\log_2\frac{p_2}{p_1}}2^{x_1+x_2}\Big].$$

After some calculations, we can get $\sup_{p\in E} R(p,\Phi_2^p)=R(q,\Phi_2^q)=\frac{a}{3}$ where $q=\Big(\frac{1}{2},\frac{1}{2}\Big)\in \boldsymbol{E}$, $\Phi_2^q=(\phi_1^q(x_1),\ \phi_2^q(x_1,x_2))$ and $\phi_1^{(1)}(x_1)=1$ if $x_1\leqq\frac{1}{2}$, $\phi_1^{(2)}(x_1)=1$ if $x_1>\frac{1}{2}$, $\phi_1^q(x_1)=(\phi_1^{(1)}(x_1),\ \phi_1^{(2)}(x_1))$, $\phi_2^{(1)}(x_1,x_2)=1$ if $x_1+x_2\leqq 1$, $\phi_2^{(2)}(x_1,x_2)=1$ if $x_1+x_2>1$,

$$\phi_2^{(q)}(x_1,x_2)=(\phi_2^{(1)}(x_1,x_2),\ \phi_2^{(2)}(x_1,x_2)).$$

By theorem 5.2., $v=R(q,\Phi_2^q)=\frac{a}{3}$ is a value for our game (I_2,Φ,R) and $q=\Big(\frac{1}{2},\frac{1}{2}\Big)$ is a good strategy for nature, Φ_2^q is a good strategy for statistician. The following decision rules is also good strategies for statistician.

(a) $\phi_1^{(1)}(x_1)=1$ if $x_1<\frac{1}{2}$, $\phi_1^{(2)}(x_1)=1$ if $x_1\geqq\frac{1}{2}$

$$\phi_1^{(q)}(x)=(\phi_1^{(1)}(x_1),\ \phi_1^{(2)}(x_1)).$$

$\phi_2^{(1)}(x_1,x_2)=1$ if $x_1+x_2<1$, $\phi_2^{(2)}(x_1,x_2)=1$ if $x_1+x_2\geqq 1$,

$$\phi_2^{(q)}(x_1, x_2) = (\phi_2^{(1)}(x_1, x_2),\ \phi_2^{(2)}(x_1, x_2)).$$

$$\Phi_2^q = (\phi_1^{(q)}(x_1),\ \phi_2^{(q)}(x_1, x_2)) \quad \text{and} \quad v = R(q, \Phi_2^q) = \frac{a}{3}.$$

(b) $\phi_1^{(1)}(x_1) = 1$ if $x_1 < \frac{1}{2}$, $\phi_1^{(2)}(x_1) = 1$ if $x_1 \geqq \frac{1}{2}$

$$\phi_1^{(q)}(x_1) = (\phi_1^{(1)}(x_1),\ \phi_1^{(2)}(x_1)).$$

$\phi_2^{(1)}(x_1, x_2) = 1$ if $x_1 + x_2 \leqq 1$, $\phi_2^{(2)}(x_1, x_2) = 1$ if $x_1 + x_2 > 1$,

$$\phi_2^{(q)}(x_1, x_2) = (\phi_2^{(1)}(x_1, x_2),\ \phi_2^{(2)}(x_1, x_2)).$$

$$\Phi_2^{(q)} = (\phi_1^{(q)}(x_1),\ \phi_2^{(q)}(x_1, x_2)) \quad \text{and} \quad v = R(q, \Phi_2^q) = \frac{a}{3}.$$

(c) $\phi_1^{(1)}(x_1) = 1$ if $x_1 \leqq \frac{1}{2}$, $\phi_1^{(2)}(x_1) = 1$ if $x_1 > \frac{1}{2}$,

$$\phi_1^{(q)}(x_1) = (\phi_1^{(1)}(x_1),\ \phi_1^{(2)}(x_1)).$$

$\phi_2^{(1)}(x_1, x_2) = 1$ if $x_1 + x_2 < 1$, $\phi_2^{(2)}(x_1, x_2) = 1$ if $x_1 + x_2 \geqq 1$,

$$\phi_2^{(q)}(x_1, x_2) = (\phi_2^{(1)}(x_1, x_2),\ \phi_2^{(2)}(x_1, x_2)).$$

$$\Phi_2^{(q)} = (\phi_1^{(q)}(x_1),\ \phi_2^{(q)}(x_1, x_2)) \quad \text{and} \quad v = R(q, \Phi_2^q) = \frac{a}{3}.$$

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