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ON A CLASS OF MULTISAMPLE RANK TESTS BASED ON TRIMMED SAMPLES

By

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1. Introduction.

Hettmansperger [5] has shown that the Mann-Whitney test has larger asymptotic relative efficiencies in Pitman's sense for some distributions with heavy tails by using the trimmed samples instead of the complete samples. Recently Tamura [8] has generalized Hettmansperger's results to Bhapkar's test [3] for the c -sample problem. Along the same line as [8], we shall propose a class of rank tests based on the trimmed samples and discuss in detail about Kruskal-Wallis test [6] as its special case.

Let $X_{i1} < \dots < X_{in_i}$ be order statistics from absolutely continuous cdf $F_i(x) = F(x - \theta_i)$, $i = 1, \dots, c$ where $F(x)$ has symmetric density $f(x)$ of unknown functional form. We further assume for $0 < \alpha < \frac{1}{2}$ that $f(x)$ is continuously differentiable in some neighborhood of the unique population quantiles b_α and $b_{1-\alpha}$ of order α and $1-\alpha$, respectively. The hypothesis H_0 , to be tested, is specified by $\theta_1 = \dots = \theta_c$ against the alternatives that not all θ 's are equal. For this problem, a class of test statistics will be proposed on the basis of only the middle $n_i - 2k_i$ random variables $X_{ik_{i+1}} < \dots < X_{in_i - k_i}$, $i = 1, \dots, c$, where $k_i = [n_i \alpha]$ denotes the largest integer not exceeding $n_i \alpha$. Throughout this paper, we assume that the sample size n_i , $i = 1, \dots, c$, increases in such a way that $\lim_{N \rightarrow \infty} n_i/N = \lambda_i$, $0 < \lambda_i < 1$ where $N = \sum_{i=1}^c n_i$. Some definitions and assumptions are given in Section 2. In Section 3, we derive the asymptotic distributions of the proposed statistics. Section 4 is concerned with the test of Kruskal-Wallis type.

2. Definitions and assumptions.

Let us define for $i = 1, \dots, c$

$$(2.1) \quad T_i = (n_i - 2k_i)^{-1} \sum_{\beta=1}^{N-2k} E_{\beta}^{(p)} Z_{\beta}^{(p)}, \quad k = \sum_{i=1}^c k_i$$

where

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$$Z_{\beta}^{(\beta)} = \begin{cases} 1 & \text{if the } \beta\text{th smallest among the combined trimmed} \\ & \text{samples is from } F_i(x) \\ 0 & \text{otherwise} \end{cases}$$

and the $E_{\beta}^{(\beta)}$'s are some given constants. Then we can represent T_i by

$$(2.2) \quad T_i = \int_{-\infty}^{\infty} J_N(H_N(x)) dG_i^*(x, n_i - 2k_i)$$

where we set $J_N(\beta/(N-2k+1)) = E_{\beta}^{(\beta)}$ and $G_i^*(x, n_i - 2k_i)$ is the empirical cdf based on the i -th trimmed sample and

$$H_N(x) = \sum_{i=1}^c \lambda_i G_i^*(x, n_i - 2k_i).$$

ASSUMPTION (A).

(i) $\lim_{N \rightarrow \infty} J_N(t) = J(t)$ exists for $0 < t < 1$ and is not constant.

(ii) $\int_{I_N} [J_N(H_N) - J(H_N)] dG_i^*(x, n_i - 2k_i) = o_p(N^{-\frac{1}{2}})$, $I_N = \{x; 0 < H_N(x) < 1\}$

(iii) $J_N(1) = o(N^{\frac{1}{2}})$

(iv) $|d^j J(t)/dt^j| \leq M(t(1-t))^{-j-\frac{1}{2}+\delta}$, $j = 0, 1, 2$

and some $\delta > 0$ where M is a generic constant.

The form (2.2) and the assumption (A) have been dealt by Chernoff-Savage [4] and Puri [7]. Further we define for $i = 1, \dots, c$

$$(2.3) \quad \begin{aligned} Y_{i1} &= n_i^{\frac{1}{2}}(X_{ik_i+1} - \theta_i - b_{\alpha}), & Y_{i2} &= n_i^{\frac{1}{2}}(X_{in_i-k_i} - \theta_i - b_{1-\alpha}) \\ Y' &= (Y_{11}, Y_{12}, \dots, Y_{c1}, Y_{c2}). \end{aligned}$$

We here notice that the statistics T_i , given Y , are the the rank statistics of Chernoff-Savage type [4] based on the trimmed samples from the cdf $G_i(x)$ with density $g_i(x)$,

$$(2.4) \quad g_i(x) = \begin{cases} f(x - \theta_i) / [F(b_{1-\alpha} + Y_{i2}/n_i^{\frac{1}{2}}) - F(b_{\alpha} + Y_{i1}/n_i^{\frac{1}{2}})] \\ \quad \text{for } b_{\alpha} + \theta_i + Y_{i1}/n_i^{\frac{1}{2}} \leq x \leq b_{1-\alpha} + \theta_i + Y_{i2}/n_i^{\frac{1}{2}} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define for $i = 1, \dots, c$

$$(2.5) \quad \begin{aligned} R_i &= (N - 2k)^{\frac{1}{2}}(T_i - E(T_i | Y)), & \mathbf{R}' &= (R_1, \dots, R_{c-1}) \\ W_i &= (N - 2k)^{\frac{1}{2}}\left(T_i - \int_0^1 J(t) dt\right), & \mathbf{W}' &= (W_1, \dots, W_{c-1}) \end{aligned}$$

where $E(*|Y)$ is the expected value of the statistic $*$, given Y .

3. Asymptotic distributions.

Now we shall consider the asymptotic distributions of the proposed statistics under the hypothesis H_0 and the following sequence of alternatives

$$(3.1) \quad H_N: F_i(x) = F(x - \nu_i / N^{\frac{1}{2}}), \quad i = 1, \dots, c$$

where not all ν 's are equal.

LEMMA 3.1. *The random vector $(\mathbf{R}, \mathbf{Y})$ is asymptotically normally distributed if the assumption (A) holds.*

PROOF. It has been shown by Puri [7] that the random vector $\mathbf{R}$ has the joint normal distribution $N(\mathbf{0}, \Sigma^{(v)})$, given $\mathbf{Y}$, if the assumption (A) holds where $\Sigma^{(v)} = \|\sigma_{ij}^{(v)}\|$ $i, j = 1, \dots, c-1$

$$(3.2) \quad \sigma_{ij}^{(v)} = \lambda_i^{-1}(\delta_{ij} - \lambda_i) \left[\int_0^1 J^2(t) dt - \left(\int_0^1 J(t) dt \right)^2 \right]$$

where $\delta_{ij} = 1$ or 0 for $i = j$ or $i \neq j$. It is also well known that $\mathbf{Y}$ has the asymptotic normal cdf $N(\mathbf{0}, \mathbf{\Omega})$ where

$$(3.3) \quad \mathbf{\Omega} = \begin{bmatrix} \mathbf{\Omega}_1 & \dots & \mathbf{0} \\ \dots & \dots & \dots \\ \mathbf{0} & \dots & \mathbf{\Omega}_1 \end{bmatrix} \quad \mathbf{\Omega}_1 = f^{-2}(b_\alpha) \begin{bmatrix} \alpha(1-\alpha) & \alpha^2 \\ \alpha^2 & \alpha(1-\alpha) \end{bmatrix}$$

These facts establish the asymptotic normality of $(\mathbf{R}, \mathbf{Y})$.

THEOREM 3.1. *The random vector $\mathbf{W}$ has the asymptotic normal distribution under the assumption (A) and $J(1) < \infty$, $J(0) < \infty$.*

PROOF. By expanding the Puri's results [7]

$$E(T_i | \mathbf{Y}) = \int_{-\infty}^{\infty} J \left[\sum_{k=1}^c \lambda_k G_k(x) \right] dG_i(x) + O_p(N^{-1})$$

in a Taylor series, we get the following (3.4).

$$(3.4) \quad E(T_i | \mathbf{Y}) = \int_0^1 J(t) dt + (\nu_i - \bar{\nu}) \int_{b_\alpha}^{b_{1-\alpha}} J'[(F(t) - \alpha)/(1 - 2\alpha)] f(t) dF(t) \\ \div N^{\frac{1}{2}}(1 - 2\alpha)^2 + \sum_{k=1}^c \lambda_k n_k^{-\frac{1}{2}} (r_1 Y_{k1} - r_2 Y_{k2}) + n_i^{-\frac{1}{2}} (s_1 Y_{i1} - s_2 Y_{i2}) + O_q(N^{-1})$$

where

$$r_1 = (1 - 2\alpha)^{-3} f(b_\alpha) \int_{b_\alpha}^{b_{1-\alpha}} J'[(F(t) - \alpha)/(1 - 2\alpha)] (F(t) - 1 + \alpha) dF(t)$$

$$r_2 = (1 - 2\alpha)^{-3} f(b_{1-\alpha}) \int_{b_\alpha}^{b_{1-\alpha}} J'[(F(t) - \alpha)/(1 - 2\alpha)] (F(t) - \alpha) dF(t)$$

$$s_1 = (1 - 2\alpha)^{-2} f(b_\alpha) \left[\int_{b_\alpha}^{b_{1-\alpha}} J'[(F(t) - \alpha)/(1 - 2\alpha)] dF(t) - (1 - 2\alpha) J(0) \right]$$

$$s_2 = (1 - 2\alpha)^{-2} f(b_{1-\alpha}) \left[\int_{b_\alpha}^{b_{1-\alpha}} J'[(F(t) - \alpha)/(1 - 2\alpha)] dF(t) - (1 - 2\alpha) J(1) \right]$$

$$\bar{\nu} = \sum_{i=1}^c \lambda_i \nu_i.$$

For any constant vector $\mathbf{a}' = (a_1, \dots, a_{c-1})$, the scalar product $\mathbf{a}'\mathbf{W}$, where

$$(3.5) \quad \mathbf{a}'\mathbf{W} = \mathbf{a}'\mathbf{R} + (N-2k)^{\frac{1}{2}} \sum_{i=1}^{c-1} a_i \left[E(T_i | \mathbf{Y}) - \int_0^1 J(t) dt \right],$$

becomes to a linear function of $\mathbf{R}$ and $\mathbf{Y}$ from (3.4). Thus lemma 3.1 and the theorem by Anderson [1], p. 76, lead the asymptotic normality of $\mathbf{a}'\mathbf{W}$. The asymptotic normality of the random vector $\mathbf{W}$ may be established from that of $\mathbf{a}'\mathbf{W}$ for any $\mathbf{a}$. We can easily see from (3.4) that $EW_i \sim 0$ under H_0 .

Denoting the covariance matrix of $\mathbf{W}$ by Σ , the statistic

$$(3.6) \quad V_\alpha = \mathbf{W}' \Sigma^{-1} \mathbf{W}$$

may be used as the test statistic for H_0 . As a special case of V_α , we shall, in Section 4, discuss about the test of Kruskal-Wallis type.

4. The test of Kruskal-Wallis type.

The test statistic of Kruskal-Wallis type can be obtained by setting $J(t) = t$ in (3.6). In this case we rewrite the statistics such as T_i , W_i and etc. in the previous sections by T_i^* , W_i^* and etc.

LEMMA 4.1. *The asymptotic mean vector and covariance matrix of $\mathbf{W}^*$ is given by $\boldsymbol{\mu}' = (\mu_1, \dots, \mu_{c-1})$ and $\Sigma = \|\sigma_{ij}\|$ where*

$$(4.1) \quad \mu_i = (\nu_i - \bar{\nu}) \int_{b_\alpha}^{b_{1-\alpha}} f^2(x) dx / (1-2\alpha)^{3/2},$$

$$(4.2) \quad \sigma_{ij} = (\delta_{ij} \lambda_i^{-1} - 1)(1+4\alpha)/12(1-2\alpha)$$

PROOF. From the identity

$$(4.3) \quad \begin{aligned} E(T_i | \mathbf{Y}) &= \frac{1}{2} + (\nu_i - \bar{\nu}) \int_{b_\alpha}^{b_{1-\alpha}} f^2(x) dx / N^{\frac{1}{2}} (1-2\alpha)^2 \\ &\quad + f(b_\alpha)(n_i^{-\frac{1}{2}} Z_i - \sum_{k=1}^c \lambda_k n_k^{-\frac{1}{2}} Z_k) / 2(1-2\alpha) + O_p(N^{-1}) \\ Z_j &= Y_{j1} + Y_{j2} \end{aligned}$$

which is obtained from (3.4), we first get

$$EW_i^* = (\nu_i - \bar{\nu}) \int_{b_\alpha}^{b_{1-\alpha}} f^2(x) dx / (1-2\alpha)^{3/2}.$$

Next from

$$W_i^* = R_i^* + (N-2k)^{\frac{1}{2}} \left[E(T_i^* | \mathbf{Y}) - \frac{1}{2} \right],$$

we get

$$\begin{aligned} \sigma_{ij} &= \text{cov}(R_i^*, R_j^*) + (N-2k) \text{cov}(E(T_i^* | \mathbf{Y}), E(T_j^* | \mathbf{Y})) \\ &\quad + (N-2k)^{\frac{1}{2}} [\text{cov}(R_i^*, E(T_j^* | \mathbf{Y})) + \text{cov}(R_j^*, E(T_i^* | \mathbf{Y}))]. \end{aligned}$$

Noticing the relations (3.2), (3.3), (4.3) and the following

$$\begin{aligned}\text{cov}(R_i^*, E(T_j^* | \mathbf{Y})) &= E_r[\{E(T_j^* | \mathbf{Y}) - E(T_j^*)\} E(R_i^* | \mathbf{Y})] \\ &= 0,\end{aligned}$$

we can get

$$\begin{aligned}\sigma_{ij} &= (\delta_{ij} - \lambda_i) / 12\lambda_i + f^2(b_\alpha) E[(\lambda_i^{-\frac{1}{2}} Z_i - \sum_{k=1}^c \lambda_k^{-\frac{1}{2}} Z_k)(\lambda_j^{-\frac{1}{2}} Z_j - \sum_{k=1}^c \lambda_k^{-\frac{1}{2}} Z_k)] \\ &\quad \div 4(1-2\alpha) \\ &= (\delta_{ij} - \lambda_i)(1 + 4\alpha) / 12\lambda_i(1-2\alpha).\end{aligned}$$

Under H_0 we get $\mu = \mathbf{0}$ by setting all $\nu_i = 0$.

THEOREM 4.1. *The test statistic*

$$(4.4) \quad V_\alpha^* = 12(1-2\alpha)(1+4\alpha)^{-1} N^{-1} \sum_{i=1}^c n_i W_i^{*2}$$

is asymptotically distributed as x_{c-1}^2 with $c-1$ degree of freedom under H_0 and as non-central $x_{c-1}^2(\delta_\alpha)$ with $c-1$ degree of freedom and the noncentrality parameter δ_α under H_N where

$$(4.5) \quad \delta_\alpha = 12 \sum_{i=1}^c \lambda_i (\nu_i - \bar{\nu})^2 \left(\int_{b_\alpha}^{b_{1-\alpha}} f^2(x) dx \right)^2 / (1-2\alpha)^2 (1+4\alpha).$$

PROOF. It follows from Theorem 3.1 and Lemma 4.1 that $\mathbf{W}^*$ is asymptotically normal $N(\mathbf{0}, \Sigma)$ under H_0 and $N(\mu, \Sigma)$ under H_N . Therefore $\mathbf{W}^{*'} \Sigma^{-1} \mathbf{W}^*$ is asymptotically distributed as x_{c-1}^2 under H_0 and $x_{c-1}^2(\delta_\alpha)$ under H_N where

$$\begin{aligned}\Sigma^{-1} &= 12(1-2\alpha)(1+4\alpha)^{-1} \lambda_c^{-1} \begin{vmatrix} \lambda_1(\lambda_1 + \lambda_c) & \lambda_1 \lambda_2 & \cdots & \lambda_1 \lambda_{c-1} \\ \lambda_2 \lambda_1 & \lambda_2(\lambda_2 + \lambda_c) & \cdots & \lambda_2 \lambda_{c-1} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{c-1} \lambda_1 & \lambda_{c-1} \lambda_2 & \cdots & \lambda_{c-1}(\lambda_{c-1} + \lambda_c) \end{vmatrix} \\ \delta_\alpha &= \mu' \Sigma^{-1} \mu.\end{aligned}$$

Some calculations show that

$$\mathbf{W}^{*'} \Sigma^{-1} \mathbf{W}^* = 12(1-2\alpha)(1+4\alpha)^{-1} \sum_{i=1}^c \lambda_i W_i^{*2},$$

and

$$\mu' \Sigma^{-1} \mu = 12 \sum_{i=1}^c \lambda_i (\nu_i - \bar{\nu})^2 \left(\int_{b_\alpha}^{b_{1-\alpha}} f^2(x) dx \right)^2 / (1-2\alpha)^2 (1+4\alpha).$$

We here notice that Kruskal-Wallis's test may be denoted by V_0^* .

It has been shown by Andrews [2] that the Pitman efficiency e_α of the V_α^* test respective to the V_0^* test is given by the ratio of the noncentrality parameters in the asymptotic x_{c-1}^2 distributions of their test statistics. From (4.5), we have

$$(4.6) \quad e_\alpha = \left(\int_{b_\alpha}^{b_{1-\alpha}} f^2(x) dx \right)^2 / (1+4\alpha)(1-2\alpha)^2 \left(\int_{-\infty}^{\infty} f^2(x) dx \right)^2$$

Lastly we give the numerical values of e_α for some distributions with heavier tails than the normal distribution.

Table of e_α .

α	.1	.2	.25	.3	.35	.4
Logistic	.99	.97	.95	.92	.88	.84
D. Exp.	1.03	1.09	1.13	1.16	1.20	1.25
Cauchy	1.09	1.26	1.34	1.40	1.43	1.44

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