THE STABILITY INDEX OF STABLE CONFIGURATIONS UNDER LOCAL MAJORITY TRANSFORMATION ON CELL SPACE : INFORMATION SCIENCE APPROACH TO BIOMATHEMATICS, III

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THE STABILITY INDEX OF STABLE CONFIGURATIONS UNDER LOCAL MAJORITY TRANSFORMATION ON CELL SPACE

—INFORMATION SCIENCE APPROACH TO BIOMATHEMATICS, III

By

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§ 1. Introduction

In a previous paper [2], we have proposed to introduce a local mapping transformation satisfying the principle of local majority, which is abbreviated by LMT (local majority transformation), in an $m \times n$ cell space, and we have discussed the limiting configurations and the stable configurations in the cell space.

Furthermore by introducting an inhibition state ϕ besides the two basic states 1 and 0, we have discussed some oscillatory phenomena to be obserbed in the configuration in some subspace or in the whole cell space.

By a stable configuration is meant a configuration in our cell space which is invariant under any application of LMT.

The purpose of this paper is to define a stability index of stable configurations and to evaluate the value of stability index to any stable configuration in our $m \times n$ cell space.

In SECTION 2 we shall prepare ourselves with some basic definitions regarding each cell such as stability index and recovery function and then in order to give a sequence of firing points which are indispensable for our present formulation an independent stochastic process is introduced to sequence of firing points. After these preparations we define a stability index of a stable configuration in an $m \times n$ cell space.

In SECTION 3 the main THEOREM 2 and 3 are given in reference to our THEOREM 1 which is given in our previous paper [2]. According to these two Theorem we can evaluate the stability index of any stable configuration for the case of equal probability for firing any central point. In view of the definition and the proof of our THEOREM 2 it is evident that results can be generalized to the case where firing probability are not equal.

In SECTION 4 several examples are given to illastrate our main results.

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§ 2. Definitions

Let us consider a finite, rectangular, two-dimentional iterative array of finite state automata, which is discussed in our previous paper [2]. The basic definitions and notations introduced there will be used here again. According to our formulation, there are $(m-1)$ $(n-1)$ firing points $\{q\}$ in the whole $m\times n$ cell space. In this paper we consider an infinite sequence of firing points $\{q_t\}$ $(t=1, 2, 3, \cdots)$ which is a sample sequence for an independent stochastic process with an identical distribution $\{X_t\}$.

DEFINITION 1. *A configuration C in an $m\times n$ cell space is said to be stable with respect to any one* LMT *if and only if* $\Pr.\{q/q : C\to C\}=1$.

Now let us introduce a notion of stability index of a stable configuration. For this purpose we shall define a stability index of each cell in our $m\times n$ cell space according to its location.

(i) For each cell (i, j) in interior of our $m\times n$ cell space, there are four 2×2 basic sequares $B(q_{h,k})$ with their respective central points $q_{h,k}$ $(h=i-1, i : k=j-1, j)$.

We are now concerned with the 3×3 cell subspace which is a restriction of our stable configuration and given by

$$C^{(3,3)}(x):\begin{pmatrix} X_{i-1,j-1} & X_{i-1,j} & X_{i-1,j+1} \\ X_{i,j-1} & X_{i,j} & X_{i,j+1} \\ X_{i+1,j-1} & X_{i+1,j} & X_{i+1,j+1} \end{pmatrix}. \tag{2.1}$$

Now let us introduce a transformation $\mathrm{T}_{ij}(X)$ in which X_{ij} is transformed into $\bar{X}_{ij}=1-X_{ij}$, where all the other $X_{h,k}$ remain the same value.

$$C^{(3,3)}(\mathrm{T}_{ij}(x)):\begin{pmatrix} X_{i-1,j-1} & X_{i-1,j} & X_{i-1,j+1} \\ X_{i,j-1} & 1-X_{ij} & X_{i,j+1} \\ X_{i+1,j-1} & X_{i+1,j} & X_{i+1,j+1} \end{pmatrix}. \tag{2.2}$$

DEFINITION 2. *A function $R(q_{h,k})$ is called to be a recovery function when it is equal to* 1 *or* 0 *according to the situation whether or not the configuration $C^{(3,3)}(\mathrm{T}_{ij}(x))$ returns back to the original configuration $C^{(3,3)}(x)$, after one application of* LMT *in $B(q_{h,k})$.*

After these preparations let us now introduce

DEFINITION 3. *A stability index of an (i, j) cell in the $m\times n$ cell space is defined by the conditional mean value of the recovery function over the 3×3 subspace consisting of four $\{B(q_{h,k})\}$ $(h=i-1, i\,;\ k=j-1, j)$ and will be denoted by Π_{ij}:*

$$\Pi_{ij}=\frac{\sum_{h=i-1}^{i}\sum_{k=j-1}^{j}\Pr.(q_{h,k})R(q_{h,k})}{\sum_{h=i-1}^{i}\sum_{h=j-1}^{j}\Pr.\{q_{h,k}\}} \tag{2.3}$$

EXAMPLE 1.

$$(2.4)\qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0^* & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0^* & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0^* & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0^* & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The stability index of the (2, 2) cell in each of the above 3×3 configurations is given by equal to 1, 1/2, 1/4 and 0 respectively, under the assumption of equal probability for 4 firing points.

(ii) For each cell (i, j) which is located in the boundary of $m\times n$ cell space, but not in the corner, there are two 2×2 basic squares which contain the (i, j) cell. Here again consider the change of X_{ij} into its conjugate $\bar{X}_{ij}=1-X_{ij}$, and we shall be concerned with recovery function confined over the joint set of two basic squares, which is either 3×2 or 2×3 cell subspace. This leads us again to the definition of stability index of this cell as the conditional probability of recovery function in this subspace as it was in (i).

EXAMPLE 2.

$$(2.5)\qquad \begin{pmatrix} 0 & 0 \\ 0^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0^* & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0^* & 1 \\ 1 & 0 \end{pmatrix}.$$

The stability index of the (2.1) cell in each of the above 3×2 configuraions is given by equal to 1, 1/2, 0 and 0 respectively under the assumption of equal probability for 2 firing points.

(iii) For each cell (i, j), which is located in the corner in $m\times n$ cell space, there is one 2×2 basic square which contains the cell (i, j). Here we consider the change of X_{ij} into its conjugate $\bar{X}_{ij}=1-X_{ij}$, and we shall be concerned with the recovery function confined over this basic square. This lead us to the definition of stability index of the cell as the conditional probability of recovery function in this basic square, which is now equal to the value of the recovery function in this basic square.

EXAMPLE 3.

$$(2.6)\qquad \begin{pmatrix} 0^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0^* & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0^* & 1 \\ 1 & 0 \end{pmatrix}.$$

The stability index of the (1, 1) cell in each of the above 2×2 configurations is given by equal to 1, 0 and 0 respectively.

Incidentally we have introduced the notion of a surrounding subspace to each cell whose size is 9, 6 and 4 according to the situation where the cell is located in the interior, the boundary and the corner respectively. This notion of surrounding subspace will be used in the proof of our main result in the following SECTION 3.

After these preparations we introduce

DEFINITION 4. *A stability index of a stable configuration C in an $m\times n$ cell space is defined by*

$$(2.7)\qquad \Pi=\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Pi_{ij}.$$

EXAMPLE 4. Let us consider a configuration defined in a 3×4 cell space

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

Let us assume the identical probability distribution defined over the 3×4 cell space as the basic of the independent stochastic process is uniform, that is,

$$\text{Pr. }\{q_{i,j}\} = \frac{1}{6}, \qquad \text{for } i=1, 2,\ j=1, 2, 3. \tag{2.9}$$

Let us consider two cells (2, 3) and (1, 1). Then we have

$$\Pi_{1,2} = \frac{\sum_{i=1}^{3}\sum_{j=2}^{2}\frac{1}{6}\times R(q_{ij})}{4\times\frac{1}{6}} \tag{2.10}$$

$$\Pi_{1,1} = \frac{\frac{1}{6}\times R(q_{1,1})}{\frac{1}{6}}$$

$$R(q_{1,1})=1, \qquad R(q_{1,3})=0, \qquad R(q_{1,2})=1, \tag{2.11}$$

$$R(q_{2,2})=0, \qquad R(q_{2,3})=0.$$

Consequently we have

$$\Pi_{2,3} = \frac{1}{4} \tag{2.12}$$

$$\Pi_{1,1} = 1. \tag{2.13}$$

The distribution of Π_{ij} $(i=1, 2, 3;\ j=1, 2, 3, 4)$ is given by

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.14}$$

which gives us

$$\Pi = \frac{1}{12}\sum_{i=1}^{3}\sum_{j=1}^{4}\Pi_{i,j} = \frac{5}{16}. \tag{2.15}$$

§ 3. Main results

The pattern of a stable configuration in an $m\times n$ cell space is given by Theorem 3 in our previous paper [2] which reads:

THEOREM 1 (Kitagawa and Yamaguchi [2]): *A stable configuration in an* $m\times n$

cell space has the following pattern:

(1°) *There is a pair of partition such that*

$$\begin{cases} m = m_1 + m_2 + \cdots + m_k \\ n = n_1 + n_2 + \cdots + n_l \end{cases} \tag{3.1}$$

with positive integers m_i *and* n_j, $1 \leqq i \leqq k$, $1 \leqq j \leqq l$, *where* k *and* l *are subject to*

$$1 \leqq k \leqq m, \quad 1 \leqq l \leqq n. \tag{3.2}$$

(2°) *In correspondence with the pair of partitions given in* (1°) *the whole* $m \times n$ *cell space is devided into* kl *subspaces each of which will be denoted by* $S(m_i, n_j)$ $(i=1, \cdots, k, j=1, \cdots, l)$.

(3°) (*a*) *The elements of each subspace* $S(m_i, n_j)$ *are entirely either* 1 *or* 0. *A subspace* $S(m_i, n_j)$ *all of whose elements are equal to* 1 *is called to be type* I, *while a subspace all of whose elements are equal to* 0 *to be type* O.

(*b*) *Subspace of these two types occur alternatively in an* $m \times n$ *cell space.* (*See* Figure 1.)

	n_1	n_2	n_3	n_4	n_5
m_1	I	O	I	O	I
m_2	O	I	O	I	O
m_3	I	O	I	O	I
m_4	O	I	O	I	O
m_5	I	O	I	O	I

Fig. 1. Pattern of stable configuration.

For the sake of convenience let us say that our stable configuration in the assertion of THEOREM 1 has the pattern structure of the type $(m_1 + m_2 + \cdots m_k) \times (n_1 + n_2 + \cdots + n_l)$.

In order to discuss the stability index of stable configuration with a partition structure of such a type it is indispensable to define a pair of two non-negative integers which have crucial importances to the following analysis. By k_1 let us denote the number of rows each of which is complement of row(s) adjacent to its row. By l_1 let us denote the number of columns each of which is complement of column(s) adjacent to its column.

The main result in this paper is given by the following

THEOREM 2. *For the equal probability for firing each central point in an* $m \times n$ *cell space, a stability index of a stable configuration which has a partition structure of the type* $(m_1 + m_2 + \cdots + m_k) \times (n_1 + n_2 + \cdots + n_l)$ *is given by the following formula:*

$$\text{(3.3)} \qquad \Pi=\begin{cases} \dfrac{\{m-(k-1)\}\{n-(l-1)\}}{mn}, & \textit{if } k_1<k-1 \textit{ and } l_1<l-1 \\ \dfrac{\{m-(k-1)\}\left\{n-\left(l-\frac{1}{2}\right)\right\}}{mn}, & \textit{if } k_1<k-1 \textit{ and } l_1=l-1 \\ \dfrac{\left\{m-\left(k-\frac{1}{2}\right)\right\}\{n-(l-1)\}}{mn}, & \textit{if } k_1=k-1 \textit{ and } l_1<l-1 \\ \dfrac{\left\{m-\left(k-\frac{1}{2}\right)\right\}\left\{n-\left(l-\frac{1}{2}\right)\right\}}{mn}, & \textit{if } k_1=k-1 \textit{ and } l_1=l-1 \\ 0 \qquad\qquad\qquad , & \textit{if } k_1=m \textit{ or } l_1=n\,. \end{cases}$$

PROOF: In order to find out the stability index of a stable configuration C in an $m\times n$ cell space, let us begin with the stability index of each cell c. As we have shown in SECTION 2 there corresponds to each cell c a certain definite surrounding subspace with respective to which our configuration C can be restricted. It is sufficient to consider such a restriction because the restriction implies a uniquely determined subconfiguration according to which the stability index of the cell c can be uniquely defined. Now let us consider the following three cases: (See EXAMPLE 1, 2 and 3 in SECTION 2).

(1°) Stability index of a cell c is equal to 1, when there are no cells having different state in the surrounding subspace.

(2°) Stability index of a cell c is equal to 1/2, when the following two condition are satisfied.

(i) There is one and only one adjacent cell which has different state.

(ii) The cell c belongs to a $r\times s$ subspace in which all the cells have the same state, where $(r-1)(s-1)>0$.

(3°) Stability index of a cell c is equal to 1/4, when the following two condition are satisfied.

(i) There are just two adjacent cells which have the state different for that of the cell c.

(ii) The cell c belongs to a $r\times s$ subspace in which all the cells have the same state, where $(r-1)(s-1)>0$.

(4°) Stability index of a cell c is equal to 0, when it belongs to a $r\times s$ subspace in which all the cell have the same state, where $(r-1)(s-1)=0$.

Let us denote by $N(\mu)$ the number of all cells in a stable configuration C each of whose stability index is μ, where $\mu=0$, 1/4, 1/2, 1.

In order to prove the assertion to THEOREM 2, let us calculate $N(\mu)$, $\mu=0$, 1/4, 1/2, 1 for each individual case defined in the right hand side of (3, 3).

Case (1°) $k_1<k-1$ and $l_1<l-1$.

By direct observation,

$$N(0)=k_1n+l_1m-kl_1$$

$$N\left(\frac{1}{4}\right)=4(l-l_1-1)(k-k_1-1)$$

$$N\left(\frac{1}{2}\right)=2(m-k_1)(l-l_1-1)+2(n-l_1)(k-k_1-1)$$
$$-8(l-l_1-1)(k-k_1-1)$$

$$N(1)=mn-2(m-k_1)(l-l_1-1)-2(n-l_1)(k-k_1-1)$$
$$+4(l-l_1-1)(k-k_1-1)-k_1n-l_1m+kl_1\,.$$

Consequently we have

$$\Pi=\frac{1}{mn}\left\{ON(0)+\frac{1}{4}\times N\left(\frac{1}{4}\right)+\frac{1}{2}\times N\left(\frac{1}{2}\right)+1\times N(1)\right\}$$
$$=\frac{1}{mn}\{m-(k-1)\}\{n-(l-1)\}\,.$$

Case (2°) $k_1<k-1$ and $l_1=l-1$.
By direct observation

$$N(0)=l_1m$$

$$N\left(\frac{1}{4}\right)=2(k-1)$$

$$N\left(\frac{1}{2}\right)=m+2(k-1)(n-l_1-2)$$

$$N(1)=mn-2(n-l_1-1)(k-1)-(l+1)m\,.$$

Consequently we have

$$\Pi=\frac{1}{mn}\{m-(k-1)\}\left\{n-\left(l-\frac{1}{2}\right)\right\}.$$

Case (3°) $k_1=k-1$ and $l_1<l-1$.
The similar computation of the case (2°) yield us

$$\Pi=\frac{1}{mn}\left\{m-\left(k-\frac{1}{2}\right)\right\}\{n-(l-1)\}\,.$$

Case (4°) $k_1=k-1$ and $l_1=l-1$.
By direct observation we have

$$N(0)=k_1n+l_1m-k_1l_1$$

$$N\left(\frac{1}{4}\right)=1$$

$$N\left(\frac{1}{2}\right)=(m-k_1-1)+(n-l_1-1)$$

$$N(1)=(m-k_1-1)(n-l_1-1)\,.$$

Consequentry we have

$$\Pi=\frac{1}{mn}\left\{m-\left(k-\frac{1}{2}\right)\right\}\left\{n-\left(l-\frac{1}{2}\right)\right\}.$$

Case (5°) $k_1=k=m$ or $l_1=l=n$.
By direct observation we have

$$N(0)=mn$$

$$N\left(\frac{1}{4}\right)=N\left(\frac{1}{2}\right)=N(1)=0.$$

Consequentry we have $\Pi=0$.

Summing up the results in these five cases, we reach the conclusion to be proved. q. e. d.

In view of our proof to THEOREM 2, it is immediate to observe

THEOREM 3. *The stability index of a stable configuration in an $m\times n$ cell space is equal to that of its complementary configuration which is stable.*

In order to illustrate several basic notions and the result of THEOREM 2, let us give the following.

EXAMPLE 5. Let us consider

$$C_1=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\quad C_2=\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}\quad C_3=\begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix} \tag{4.1}$$

$$C_4=\begin{pmatrix}1&1&0\\0&0&1\\0&0&1\end{pmatrix}\quad C_5=\begin{pmatrix}1&0&1\\0&1&0\\0&1&0\end{pmatrix}.$$

The application of THEOREM 2 gives us Π.

The partition structure and the value of Π for each of these five configuration are given by Table 1.

Table 1.

	c_1	c_2	c_3	c_4	c_5
k	1	2	3	2	2
l	1	1	1	2	3
k_1	0	1	3	1	1
l_1	0	0	0	1	3
m_1	3	1	1	1	1
m_2	—	2	1	2	2
m_3	—	—	1	—	—
n_1	3	3	3	2	1
n_2	—	—	—	1	1
n_3	—	—	—	—	1
Π	1	$\frac{5}{9}$	0	$\frac{4}{9}$	0

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