APPLICATION OF KITAGAWA’S FUNCTIONAL INTEGRAL TO SOLUTIONS OF NON-LINEAR INTEGRAL EQUATIONS OF TWO VARIABLES

Strait, Peggy
Queens College of The City University of New York

https://doi.org/10.5109/13052
APPLICATION OF KITAGAWA'S FUNCTIONAL INTEGRAL TO SOLUTIONS OF NON-LINEAR INTEGRAL EQUATIONS OF TWO VARIABLES

By

Peggy Strait*

(Received January 15, 1971)

1. Introduction

R. H. Cameron and W. T. Martin derived an expression [1], in terms of Wiener integrals, for the solution of a class of non-linear integral equations of a single variable. This note shows that by employing a lemma of J. D. Kuelbs [3], the result of Cameron and Martin may be easily extended (Theorem 1) to an expression for the solution of non-linear integral equations of two variables. The extended expression is in terms of the integral over the space of continuous functions of two variables defined by T. Kitagawa [2], and extended by J. Yeh [7]. The essential properties of the Wiener integral (over 1-dimensional space) required for the proof given by Cameron and Martin are a Fubini theorem and a lemma concerning the Wiener measure of functions of one variable in a small neighborhood. This note shows that both of these properties are also possessed by the integral over the space of continuous functions of two variables.

2. Extended Kitagawa Integral in Function Space of Two Variables

Let $C_2$ be the collection of continuous functions $\{x(t, \tau)\}$ on the unit square $0 \leq t, \tau \leq 1$ satisfying $x(0, \tau) = x(t, 0) = 0$. Integration on this space of functionals of the type $H[x(t_1, \tau_1), \ldots, x(t_r, \tau_s)]$ where $H[\eta_{11}, \ldots, \eta_{rs}]$ is a function of $rs$ real variables $\{\eta_{hk}\}, h = 1, 2, \ldots, r; k = 1, 2, \ldots, s$; and $\{t_h\}, \{\tau_k\}$ are preassigned division points of the unit intervals $0 \leq t \leq 1, 0 \leq \tau \leq 1$ satisfying $0 = t_0 \leq t_1 \leq \cdots \leq t_r \leq t_{r+1} = 1, 0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_s \leq \tau_{s+1} = 1$ was defined by T. Kitagawa [2] to be

$$\int_{C_2} H[x(t_1, \tau_1), \ldots, x(t_r, \tau_s)] d\omega x$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H[\eta_{11}, \ldots, \eta_{rs}] \prod_{h=1}^{s} \prod_{k=1}^{r} p(D_{hk}) d\eta_{11} \cdots d\eta_{rs}$$

where

$$p(D_{hk}) = \left[\pi(t_h-t_{h-1})(\tau_k-\tau_{k-1})\right]^{-1} \exp \left\{ \frac{1}{2} \left( \frac{(\eta_{h,k} - \eta_{h,k-1} - \eta_{h-1,k} + \eta_{h-1,k-1})^2}{(t_h-t_{h-1})(\tau_k-\tau_{k-1})} \right) \right\}$$

with the understanding that $\eta_{0,j} = \eta_{1,0} = 0$.

* Queens College of The City University of New York.

57
J. Yeh showed [7] that this integration is with respect to a probability measure \( w \) on an interval class (Boolean algebra of sets) in \( C_2 \) and that therefore the integral could be extended to more general functionals. The measure \( w \) (also called Wiener measure in \( C_2 \)) was defined as follows. Let \( E \) be a Lebesgue measurable subset of the \( rs \)-dimensional Euclidean space \( R^r \). Let \( \{t_k\}, \{\tau_k\} \), and \( p(\mathcal{A}_{nk}) \) be defined as above and let a subset (interval) \( I \) of \( C_2 \) be defined as \( I = \{x(t_1, \tau_1), \ldots, x(t_s, \tau_s) \in E \} \). Then the Wiener measure of \( I \) is defined as

\[
w(I) = \int_E \prod_{k=1}^{r} \prod_{s=1}^{s} p(\mathcal{A}_{nk}) d\eta_{t_k} \cdots d\eta_{s_k}.
\]

The collection \( \mathcal{S} \) of all sets of the form \( I \) was shown to be an algebra of sets and \( w \) a probability measure on that algebra. Thus there is a whole class of functionals \( \Phi(x) \), which are integrable over the space \( C_2 \) with respect to this measure and we denote the integral by \( \int_{C_2} \Phi(x) d_w x \). Also, as in the case of Wiener integrals over the space of functions of one variable, if \( \Phi(x) \) is integrable and \( S \) is a measurable subset of \( C_2 \), we define

\[
\int_{S} \Phi(x) d_w x = \int_{C_2} \Phi(x) d_w x
\]

where

\[
\Phi(x) = \begin{cases} 
\Phi(x) & \text{for } x \text{ in } S \\
0 & \text{otherwise}.
\end{cases}
\]

3. Fubini Theorem for Extended Kitagawa Integrals in \( C_2 \)

Cameron and Martin stated in [1] that the Fubini theorem holds for two Wiener integrals or for Wiener and Lebesgue integrals since the Wiener mapping takes function space into a linear interval to which the ordinary Fubini theorem applies. Although we can show in exactly the same manner (i.e. by means of a mapping to \( C_2 \) into the unit square) that the Fubini theorem holds for extended Kitagawa integrals over the space \( C_2 \), we need not use this method since we already know from section 2 that \( w \) is a finite measure in \( C_2 \). This implies that the Fubini theorem holds for two Kitagawa integrals over \( C_2 \) or for Kitagawa integral over \( C_2 \) and Lebesgue integral over the unit square.

4. Wiener Measure of Functions in Small Neighborhoods

LEMMA 1. For each \( x_0(t, \tau) \) in \( C_2 \) and each \( \eta > 0 \) the set \( \mathcal{T}_\eta \) consisting of all functions \( x(t, \tau) \) in \( C_2 \) satisfying

\[
\int_{t_0}^{t_1} \int_{\tau_0}^{\tau_1} (x(t, \tau) - x_0(t, \tau))^2 dt d\tau < \eta
\]

has positive Wiener measure

\[
\int_{\mathcal{T}_\eta} d_w x > 0.
\]
PROOF. Lemma 3, page 358, of Kuelbs paper [3] states that if \( E \) is an open subset of \( C_2 \), then \( w(E) > 0 \). Observe that

\[
\begin{align*}
\{ x : \int_0^1 \int_0^1 \{ x(t, \tau) - x_0(t, \tau) \}^2 dt d\tau < \eta \} \supseteq \left\{ x : \sup_{0 \leq \tau \leq 1} |x(x) - x_{0}(x)| < \sqrt{\eta} \right\}
\end{align*}
\]

The set on the right of the inequality is open. Thus,

\[
\int_{C_2} d_w x > 0.
\]

5. An Expression for the Solution of Non-Linear Integral Equations of Two Variables

THEOREM 1. Let \( G(t, r, \xi, \eta, u) \) be continuous in \( 0 \leq t \leq 1, 0 \leq r \leq 1, 0 \leq \xi \leq 1, 0 \leq \eta \leq 1, -\infty < u < \infty \) and let it satisfy there the uniform Lipschitz condition

\[
|G(t, u_2) - G(t, r, \xi, \eta, u_1)| < M|u_2 - u_1|.
\]

Then if \( y(t, \tau) \) is any continuous function in \( 0 \leq t \leq 1, 0 \leq r \leq 1 \) and vanishing at \( t = 0 \) and at \( \tau = 0 \), the integral equation \( x(t, r) = y(t, \tau) + \int_0^1 \int_0^1 G(t, r, \xi, \eta, x(\xi, \eta)) d\xi d\eta \) has a unique solution \( x_0(t, r) \) given by

\[
x_0(s, \sigma) = \lim_{\rho \to \infty} \int_{C_2} \exp \left[ -\rho \int_0^s \int_0^1 \left( y(t, \tau) - x(t, \tau) + \int_0^1 \int_0^1 G(t, r, \xi, \eta, x(\xi, \eta)) d\xi d\eta \right)^2 dt d\tau \right] x(s, \sigma) d_w x
\]

where the limit is taken in the \( L_2 \) sense for ordinary Lebesgue integrals, \( 0 \leq s \leq 1, 0 \leq \sigma \leq 1 \), and the two integrals \( \int_{C_2} \) are extended Kitagawa integrals as defined in section 2 above.

This theorem is a direct generalization of Theorem 1 in [1]. We shall not include here a detailed proof because now that it has been ascertained in sections 2 through 4 that the essential properties of the Wiener integral required for the proof given by Cameron and Martin are also possessed by the extended Kitagawa integral over \( C_2 \), a proof for the two variable case may be obtained by referring to the original proof in [1]. One should begin with Section 3, The General Theorem, on page 285 of [1], and continue through section 7 which concludes at the top of page 294.

As a final remark it should be noted that all results here are valid for the case of \( n \) variables.

References


