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APPLICATION OF KITAGAWA'S FUNCTIONAL INTEGRAL TO SOLUTIONS OF NON-LINEAR INTEGRAL EQUATIONS OF TWO VARIABLES

By

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1. Introduction

R. H. Cameron and W. T. Martin derived an expression [1], in terms of Wiener integrals, for the solution of a class of non-linear integral equations of a single variable. This note shows that by employing a lemma of J. D. Kuelbs [3], the result of Cameron and Martin may be easily extended (Theorem 1) to an expression for the solution of non-linear integral equations of two variables. The extended expression is in terms of the integral over the space of continuous functions of two voriables defined by T. Kitagawa [2], and extended by J. Yeh [7]. The essential properties of the Wiener integral (over 1-dimensional space) required for the proof given by Cameron and Martin are a Fubini theorem and a lemma concerning the Wiener measure of functions of one varioble in a small neighborhood. This note shows that both of these properties are also possessed by the integral over the space of continuous functions of two variables.

2. Extended Kitagawa Integral in Function Space of Two Variables

Let C_2 be the collection of continuous functions $\{x(t, \tau)\}$ on the unit square $0 \leqq t, \tau \leqq 1$ satisfying $x(0, \tau) = x(t, 0) = 0$. Integration on this space of functionals of the type $H[x(t_1, \tau_1), \cdots x(t_r, \tau_s)$ where $H[\eta_{11}, \cdots \eta_{rs}]$ is a function of rs real variables $\{\eta_{hk}\}$, $h = 1, 2, \cdots, r$; $k = 1, 2, \cdots, s$; and $\{t_h\{$, $\{\tau_k\}$ are preassigned division points of the unit intervals $0 \leqq t \leqq 1$, $0 \leqq \tau \leqq 1$ satisfying $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_r \leqq t_{r+1} = 1$, $0 = \tau_0 \leqq \tau_1 \leqq \cdots \tau_s \leqq \tau_{s+1} = 1$ was defined by T. Kitagawa [2] to be

$$\int_{C_2}^{w} H[x(t_1, \tau_1), \cdots x(t_r, \tau_s)] d_w x$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H[\eta_{11}, \cdots \eta_{rs}] \prod_{h=1}^{r} \prod_{k=1}^{s} p(\Delta_{hk}) d\eta_{11} \cdots d\eta_{rs}$$

where

$$p(\Delta_{hk}) = [\pi(t_h - t_{h-1})(\tau_k - \tau_{k-1})]^{-\frac{1}{2}} \exp\left\{-\frac{(\eta_{h,k} - \eta_{h,k-1} - \eta_{h-1,k} + \eta_{h-1,k-1})^2}{(t_h - t_{h-1})(\tau_k - \tau_{k-1})}\right\}$$

with the understanding that $\eta_{0,j} = \eta_{i,0} = 0$.

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J. Yeh showed [7] that this integration is with respect to a probability measure w on an interval class (Boolean algebra of sets) in C_2 and that therefore the integral could be extended to more general functionals. The measure w (also called Wiener measure in C_2) was defined as follows. Let E be a Lebesgue measurable subset of the rs-dimensional Euclidean space R_{rs}. Let $\{t_h\}$, $\{\tau_k\}$ and $p(\Delta_{hk})$ be defined as above and let a subset (interval) I of C_2 be defined as $I\{t_1, t_2, \cdots t_r, \tau_1, \cdots \tau_s, E\} = \{x \in C_2 : (x(t_1, \tau_1), \cdots x(t_r, \tau_s)) \in E\}$. Then the Wiener measure of I is defined as

$$w(I) = \int_E^{(rs)} \cdots \int \prod_{h=1}^{r} \prod_{k=1}^{s} p(\Delta_{hk}) d\eta_{11} \cdots d\eta_{rs}\,.$$

The collection $\mathfrak{I}$ of all sets of the form I was shown to be an algebra of sets and w a probability measure on that algebra. Thus there is a whole class of functionals $\Phi(x)$ which are integrable over the space C_2 with respect to this measure and we denote the integral by $\int_{C_2}^{w} \Phi(x) d_w x$. Also, as in the case of Wiener integrals over the space of functions of one variable, if $\Phi(x)$ is integrable and S is a measurable subset of C_2, we define

$$\int_S^w \Phi(x) d_w x = \int_{C_2}^w \phi(x) d_w x$$

where

$$\phi(x) = \begin{cases} \Phi(x) & \text{for } x \text{ in } S \\ 0 & \text{otherwise}\,. \end{cases}$$

3. Fubini Theorem for Extended Kitagawa Integrals in C_2

Cameron and Martin stated in [1] that the Fubini theorem holds for two Wiener integrals or for Wiener and Lebesgue integrals since the Wiener mapping takes function space into a linear interval to which the ordinary Fubini theorem applies. Although we can show in exactly the same manner (i. e. by means of a mapping to C_2 into the unit square) that the Fubini theorem holds for extended Kitagawa integrals over the space C_2, we need not use this method since we already know from section 2 that w is a finite measure in C_2. This implies that the Fubini theorem holds for two Kitagawa integrals over C_2 or for Kitagawa integral over C_2 and Lebesgue integral over the unit square.

4. Wiener Measure of Functions in Small Neighborhoods

LEMMA 1. *For each $x_0(t, \tau)$ in C_2 and each $\eta > 0$ the set T_η consisting of all functions $x(t, \tau)$ in C_2 satisfying*

$$\int_0^1 \int_0^1 \{x(t, \tau) - x_0(t, \tau)\}^2 dt d\tau < \eta$$

has positive Wiener measure

$$\int_{T\eta}^w d_w x > 0\,.$$

PROOF. Lemma 3, page 358, of Kuelbs paper [3] states that if E is an open subset of C_2, then $w(E)>0$. Observe that

$$\left\{x:\int_0^1\int_0^1\{x(t,\tau)-x_0(t,\tau)\}^2dtd\tau<\eta\right\}\supseteqq\left\{x:\sup_{\substack{0\leqq t\leqq 1\\ 0\leqq\tau\leqq 1}}|x-x_0|<\sqrt{\eta}\right\}$$

The set on the right of the inequality is open. Thus,

$$\int_{T\eta}^{w}d_wx>0\,.$$

5. An Expression for the Solution of Non-Linear Integral Equations of Two Variables

THEOREM 1. *Let $G(t,\tau,\xi,\eta,u)$ be continuous in $0\leqq t\leqq 1$, $0\leqq\tau\leqq 1$, $0\leqq\xi\leqq 1$, $0\leqq\eta\leqq 1$, $-\infty<u<\infty$ and let it satisfy there the uniform Lipschitz condition*

$$|G(t,\tau,\xi,\eta,u_2)-G(t,\tau,\xi,\eta,u_1)|<M|u_2-u_1|.$$

Then if $y(t,\tau)$ is any continuous function in $0\leqq t\leqq 1$, $0\leqq\tau\leqq 1$ and vanishing at $t=0$ and at $\tau=0$, the integral equation $x(t,\tau)=y(t,\tau)+\int_0^t\int_0^\tau G(t,\tau,\xi,\eta,x(\xi,\eta))d\xi d\eta$ has a unique solution $x_0(t,\ )$ given by

$$x_0(s,\sigma)=\lim_{\rho\to\infty}\frac{\int_{C_2}^{w}\exp\left[-\rho\int_0^1\int_0^1\left\{y(t,\tau)-x(t,\tau)+\int_0^t\int_0^\tau G(t,\tau,\xi,\eta,x(\xi,\eta)d\xi d\eta\right\}^2dtd\tau\right]x(s,\sigma)d_wx}{\int_{C_2}^{w}\exp\left[-\rho\int_0^1\int_0^1\left\{y(t,\tau)-x(t,\tau)+\int_0^t\int_0^\tau G(t,\tau,\xi,\eta,x(\xi,\eta)d\xi d\eta\right\}^2dtd\tau\right]d_wx}$$

where the lim is taken in the L_2 sense for ordinary Lebesque integrals, $0\leqq s\leqq 1$, $0\leqq\sigma\leqq 1$, and the two integrals $\int_{C_2}^{w}$ are extended Kitagawa integrals as defined in section 2 above.

This theorem is a direct generalization of Theorem 1 in [1]. We shall not include here a detailed proof because now that it has been ascertained in sections 2 through 4 that the essential properties of the Wiener integral required for the proof given by Cameron and Martin are also possessed by the extended Kitagawa integral over C_2, a proof for the two variable case may be obtained by referring to the original proof in [1]. One should begin with Section 3, *The General Theorem,* on page 285 of [1], and continue through section 7 which concludes at the top of page 294.

As a final remark it should be noted that all results here are valid for the case of n variables.

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