A NOTE ON THE EFFICIENCY OF TAMURA'S $ Q $

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By

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1. Introduction

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots Y_n$ be random samples from the symmetric and continuous c. d. f. $F(x)$ and $G(x) = F(x/\theta)$ respectively.

For testing the statistical hypothesis $H: \theta = 1$ against the alternative $AH: \theta > 1$ Tamura [3] has proposed the following test statistics.

(1)
$$Q_s^{(1)} = \frac{1}{\binom{m}{s}\binom{n}{2}} \Sigma^* \phi(x_{\alpha_1}, \cdots, x_{\alpha_s}; y_{\beta_1}, y_{\beta_2})$$

where

$$\phi(x_1, x_2, \cdots, x_s; y_1, y_2) = \begin{cases} 1 & \text{for } y_1 < x_1, \cdots, x_s < y_2 \text{ or } y_2 < x_1, x_2, \cdots, x_s < y_1, \\ 0 & \text{otherwise}. \end{cases}$$

and the summation Σ^* extends over all subscripts α, β such that $1 \leq \alpha_1 < \cdots < \alpha_s \leq m$, $1 \leq \beta_1 < \beta_2 \leq n$.

Among the statistics $Q_s^{(1)}$, $s = 1, 2, \cdots$, the interesting one would be $Q_1^{(1)}$ and $Q_2^{(1)}$. It has been proved that these two statistics have the same Pitman efficiencies. Thus $Q_1^{(1)}$ would be more practical than $Q_2^{(1)}$, since it is very easy to compute.

To make our investigation more precise we shall also consider the following statistics which are the same types of $Q_s^{(1)}$.

(2)
$$Q_s^{(2)} = \frac{1}{\binom{n}{s}\binom{m}{2}} \Sigma^{**} \phi(y_{\alpha_1}, \cdots, y_{\alpha_s}; x_{\beta_1}, x_{\beta_2}), \qquad s = 1, 2,$$

where Σ^{**} extends over all subscripts α, β such that $1 \leq \alpha_1 < \cdots < \alpha_s \leq n$, $1 \leq \beta_1 < \beta_2 \leq m$.

The purpose of this paper is to make further comparisons to these test statistics, which will make us to recomend $Q_1^{(2)}$ instead of $Q_1^{(1)}$ or $Q_2^{(1)}$ in practical situations.

In section 2 we shall consider the comparison of $Q_s^{(i)}$, $i, s = 1, 2$ from the view point of the Bahadur asymptotic efficiency [1]. As pointed by Bahadur [2], Bahadur asymptotic efficiency has some pitfalls since it is an approximate measure of efficiency. Thus our results in section 2 might not be enough reliable. Therefore we shall in section 3 compute the small sample power of these test statistics and give comparisons of these powers for a specific distribution.

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2. Comparison of $Q_s^{(i)}$, $i, s = 1, 2$ by the Bahadur asymptotic efficiency

Let denote the mean of $Q_s^{(i)}$, $i, s = 1, 2$ by $\mu_{i,s}(\theta)$. Then we get

(4)
$$\mu_{1,s}(\theta) = 2 \iint_{x<y} [F(y)-F(x)]^s dG(x)d(y),$$

(5)
$$\mu_{2,s}(\theta) = 2 \iint_{x<y} [G(y)-G(x)]^s dF(x)dF(y).$$

Especially in the null case we get

(6)
$$\mu_{1,s}(1) = \mu_{2,s}(1) = \frac{2}{(s+1)(s+2)}.$$

Let $m = \rho N$, $n = (1-\rho)N$ and denoting the asymptotic variance of $Q_s^{(i)}$ under null hypothesis by $\sigma_{i,s}^2$, then following the manner of Tamura [3] we get

(7)
$$\sigma_{i,s}^2 = \frac{1}{(s+1)^2} \frac{8}{\rho(1-\rho)} \Big[\frac{1}{2s+3} - \frac{2}{(s+2)^2} + \frac{[(s+1)!]^2}{(2s+3)!} \Big], \qquad s = 1, 2.$$

Let normalize the statistics $Q_s^{(i)}$ as follows.

$$Q_{N,s}^{(i)} = \frac{\sqrt{N}(Q_s^{(i)} - \mu_s(1))}{\sigma_{i,s}}, \qquad i = 1, 2.$$

Then we get under non null hypothesis

(8)
$$E\Big[\frac{Q_{N,s}^{(i)}}{\sqrt{N}} \Big] = \frac{1}{\sigma_{i,s}} (\mu_{i,s}(\theta) - \mu_{i,s}(1)).$$

Thus by using Chebychev's inequality and the definition of the asymptotic slope, the asymptotic slope $C(Q_s^{(i)}; \theta)$ of statistic $Q_s^{(i)}$ is obtained, after some calculations, as follows.

$$C(Q_s^{(i)} : \theta) = \frac{1}{\sigma_{i,s}^2} (\mu_{i,s}(\theta) - \mu_{i,s}(1))^2.$$

Then from (4), (5) and (7) we get

(9)
$$C(Q_1^{(1)} : \theta) = 720\rho(1-\rho)\Big(\int FGdG - \frac{1}{3} \Big)^2,$$

(10)
$$C(Q_2^{(1)} : \theta) = 180\rho(1-\rho)\Big(\int F^2 dG - \frac{1}{3} \Big)^2,$$

(11)
$$C(Q_1^{(2)} : \theta) = 720\rho(1-\rho)\Big(\int FGdF - \frac{1}{3} \Big)^2,$$

(12)
$$C(Q_2^{(2)} : \theta) = 180\rho(1-\rho)\Big(\int G^2 dF - \frac{1}{3} \Big)^2.$$

By integration by part it can be easily shown

$$C(Q_1^{(1)} : \theta) = C(Q_2^{(2)} : \theta), \qquad C(Q_2^{(1)} : \theta) = C(Q_1^{(2)} : \theta).$$

Further

$$C(Q_2^{(1)} : \theta) - C(Q_1^{(1)} : \theta) = 180\rho(1-\rho)\Big[\int F^2 dG - 2\int FG dG + \frac{1}{3}\Big]\Big[\int F^2 dG + 2\int FG dG - 1\Big]$$

$$= 180\rho(1-\rho)\Big[\int (F-G)^2 dG\Big]\Big[\Big(\int F^2 Gd - \frac{1}{3}\Big) + 2\Big(\int GF dG - \frac{1}{3}\Big)\Big].$$

But it is seen that $\int F^2 dG$ is an increasing function of $\theta > 0$ for $G(x) = F(x/\theta)$. Thus we get

$$\int F^2 dG > \frac{1}{3} \qquad \text{for } \theta > 1.$$

Similarly

$$\int FG dG > \frac{1}{3} \qquad \text{for } \theta > 1.$$

Thus we get

$$C(Q_2^{(1)} : \theta) > C(Q_1^{(1)} : \theta) \qquad \text{for } \theta > 1.$$

Namely it has been proved that for testing the hypothesis $H: \theta = 1$ against the alternative $AH: \theta > 1$,

   (i) *Bahadur asymptotic efficiency of $Q_2^{(1)}$ and $Q_1^{(1)}$ are respectively equivalent to that of $Q_1^{(2)}$ and $Q_2^{(2)}$,*

   (ii) *$Q_2^{(1)}$ (or $Q_1^{(2)}$) is more efficient than $Q_1^{(1)}$ (or $Q_2^{(2)}$)*

or equivalently

   (ii)' *$Q_1^{(2)}$ (or $Q_2^{(1)}$) is more efficient than $Q_1^{(1)}$ (or $Q_2^{(2)}$).*

Thus against Tamura's proposal, $Q_1^{(2)}$ instead of $Q_1^{(1)}$ would be recomended in the practical situations.

3. Small sample comparisons of $Q_s^{(i)}$, $i, s = 1, 2$

Since the results given in the section 2 are asymtotic and approximate, behaviours of statistics $Q_{N,s}^{(i)}$, $i, s = 1, 2$ must be discussed in small sample. We cannot unfortunately deal with them in the general form, therefore we only check in the simple and special cases. When $m = n = 4$, the orderings of X's and Y's which have larger values of $Q_i^{(i)}$, $i = 1, 2$ are respectively given in the following table.

Let the size α of test be $1/70$, then the critical regions of $Q_s^{(1)}$, $s = 1, 2$ contain only an ordering $YYXXXXYY$ and that of $Q_1^{(2)}$ and $Q_2^{(2)}$ are constructed by above five and seventeen orderings in the table respectively in the randomized form. In the case $\alpha = 5/70$, the critical regions of both $Q_2^{(1)}$ and $Q_1^{(2)}$ are constructed by above five orderings and that of $Q_1^{(1)}$ and $Q_2^{(2)}$ are constructed respectively by above six and seventeen ordering in the randomized form. When $F(x)$ is symmetrical, symmetric orderings have the same probability, for example

$$P_r(YYXXXYXY) = P_r(YXYXXXYY).$$

Now we assume that $F(x)$ be the uniform distribution in $(-1/2, 1/2)$, then after some computation we get

$$P_r(YYXXXXYY) = 72\int\int G^2(x)[F(y)-F(x)]^2[1-G(y)]^2 dF(x) dF(y)$$

$$= \frac{1}{70\theta^4}\Big[\frac{105}{4}(\theta-1)^4 + 42(\theta-1)^3 + 28(\theta-1)^2 + 8(\theta-1) + 1\Big] \quad \text{for } \theta > 1.$$

Table. Ordering of $Q_i^{(s)}$, $i, s = 1, 2$, $m = n = 4$.

$Q_1^{(1)}$		$Q_2^{(1)}$		$Q_1^{(2)}$		$Q_2^{(2)}$	
ordering	value of $4\binom{4}{2}Q_1^{(1)}$	ordering	value of $\binom{4}{2}\binom{4}{2}Q_2^{(1)}$	ordering	value of $4\binom{4}{2}Q_1^{(2)}$	ordering	value of $\binom{4}{2}\binom{4}{2}Q_2^{(2)}$
$YYXXXXYY$	16	$YYXXXXYY$	24	$YYXXXXYY$	0	$YYXXXXYY$	0
$YYXXXYXY$	15	$YYXXXYXY$	18	$YYYXXXXY$	0	$YYXXXYXY$	0
$YXYXXXYY$	15	$YYYXXXXY$	18	$YXXXXYYY$	0	$YYXXYXXY$	0
$YYXXYXXY$	14	$YXYXXXYY$	18	$XXXXYYYY$	0	$YYXYXXXY$	0
$YXXYXXYY$	14	$YXXXXYYY$	18	$YYYYXXXX$	0	$YYYXXXXY$	0
$YXYXXYXY$	14	$YYXXYXXY$	15	$YYXXXYXY$	3	$YXYXXXYY$	0
$YYXYXXXY$	13	$YYXYXXXY$	15	$YYXYXXXY$	3	$YXXYXXYY$	0
$YXXXYXYY$	13	$YXXYXXYY$	15	$YXYXXXYY$	3	$YXXXYXYY$	0
$YXXYXYXY$	13	$YXXXYXYY$	15	$YXXXYXYY$	3	$YXXXXYYY$	0
$YXYXYXXY$	13			$XYXXXYYY$	3	$XYXXXYYY$	0
$YYYXXXXY$	12			$YYYXXXYX$	3	$YYYXXXYX$	0
$YXYYXXXY$	12			$XXXYXYYY$	3	$XXYXXYYY$	0
$YXXYYXXY$	12			$YYYXYXXX$	3	$YYYXXYXX$	0
$YXXXYYXY$	12			$YYXXYXXY$	4	$XXXYXYYY$	0
$YXXXXYYY$	12			$YXXYXXYY$	4	$YYYYXYXXX$	0
$XYYXXXYY$	12			$XXYXXYYY$	4	$XXXXYYYY$	0
$YYXXXYYX$	12			$YYYXXYXX$	4	$YYYYXXXX$	0
⋮		⋮		⋮		⋮	

From the similar computations we get for $\theta > 1$

$$P_r(YYYXXXXY) = \frac{1}{70\theta^4}\left(\frac{35}{2}a^4 + 28a^3 + 21a^2 + 8a + 1\right),$$

$$\left.\begin{array}{l} P_r(YYXXXYXY) \\ P_r(YYXXYXXY) \\ P_r(YYXYXXXY) \end{array}\right\} = \frac{1}{70\theta^4}\ (21a^3 + 21a^2 + 8a + 1),$$

$$P_r(YXYXXYXY) = \frac{1}{70\theta^4}\ (14a^2 + 8a + 1),$$

$$P_r(YYYYXXXX) = \frac{1}{70\theta^4}\left(\frac{35}{8}a^4 + 7a^3 + 7a^2 + 4a + 1\right),$$

$$\left.\begin{array}{l} P_r(YYYXXXYX) \\ P_r(YYYXXYXX) \\ P_r(YYYXYXXX) \end{array}\right\} = \frac{1}{70\theta^4}\ (7a^3 + 7a^2 + 4a + 1),$$

where $a = \theta - 1$.

Thus the power of $Q_s^{(i)}$, denoted by $\gamma_s^{(i)}$, is given for $\alpha = 1/70$

(13) $$\gamma_1^{(1)} = \gamma_2^{(1)} = \left(\frac{105}{4}a^4 + 42a^3 + 28a^2 + 8a + 1\right)/70\theta^4 \quad \text{for } \theta > 1,$$

(14) $$\gamma_1^{(2)} = (70a^4 + 112a^3 + 84a^2 + 32a + 5)/350\theta^4 \quad \text{for } \theta > 1,$$

(15) $$\gamma_2^{(2)} = (70a^4 + 280a^3 + 252a^2 + 104a + 17)/1190\theta^4 \quad \text{for } \theta > 1.$$

Comparing (13), (14) and (15) we get

(16) $$\gamma_1^{(1)} = \gamma_2^{(1)} > \gamma_1^{(2)} > \gamma_2^{(2)} \quad \text{for } \theta > 1.$$

In the case $\alpha = 5/70$ we get

(17) $$\gamma_1^{(1)} = \left(\frac{315}{4}a^4 + 336a^3 + 322a^2 + 120a + 15\right)/210\theta^4 \quad \text{for } \theta > 1,$$

(18) $$\gamma_1^{(2)} = (70a^4 + 112a^3 + 84a^2 + 32a + 5)/70\theta^4 \quad \text{for } \theta > 1,$$

(19) $$\gamma_2^{(1)} = \left(\frac{245}{4}a^4 + 140a^3 + 112a^2 + 40a + 5\right)/70\theta^4 \quad \text{for } \theta > 1,$$

(20) $$\gamma_2^{(2)} = (70a^4 + 280a^3 + 252a^2 + 104a + 17)/238\theta^4 \quad \text{for } \theta > 1.$$

Comparing (17), (18), (19) and (20) we get

(21)
$$\left\{\begin{array}{ll} \gamma_1^{(2)} > \gamma_2^{(1)} > \gamma_1^{(1)} > \gamma_2^{(2)} & \text{for } \theta > 5.047, \\ \gamma_2^{(1)} > \gamma_1^{(2)} > \gamma_1^{(1)} > \gamma_2^{(2)} & \text{for } 1.863 < \theta \leq 5.047, \\ \gamma_2^{(1)} > \gamma_1^{(1)} > \gamma_1^{(2)} > \gamma_2^{(2)} & \text{for } 1 < \theta \leq 1.863, \end{array}\right.$$

(16) and (21) support the results in section 2. Namely, let denote by $B(T^{(1)} : T^{(2)})$ the Bahadur asymptotic efficiency of $T^{(1)}$ relative to $T^{(2)}$, then we have following correspondence between the results of section 2 and section 3.

Results in section 2	Results in section 3
$B(Q_1^{(1)} : Q_2^{(1)}) < 1$	$\gamma_1^{(1)} < \gamma_2^{(1)}$
$B(Q_1^{(2)} : Q_2^{(2)}) > 1$	$\gamma_1^{(2)} > \gamma_2^{(2)}$
$B(Q_1^{(1)} : Q_1^{(2)}) < 1$	$\gamma_1^{(1)} < \gamma_1^{(2)}$
$B(Q_2^{(1)} : Q_2^{(2)}) > 1$	$\gamma_2^{(1)} > \gamma_2^{(2)}$
$B(Q_1^{(1)} : Q_2^{(2)}) = 1$	$\gamma_1^{(1)} > \gamma_2^{(2)}$
$B(Q_2^{(1)} : Q_1^{(2)}) = 1$	$\gamma_2^{(1)} \gtreqless \gamma_1^{(2)}$.

4. Acknowledgements.

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