A NOTE ON THE EFFICIENCY OF TAMURA'S $\$ Q \$$

Yanagawa, Takashi
Department of General Education, Kumamoto University

<https://doi.org/10.5109/13042>

出版情報：統計数理研究. 14 (1/2), pp.25-30, 1970-03. Research Association of Statistical Sciences
バージョン：
権利関係：

A NOTE ON THE EFFICIENCY OF TAMURA'S Q

By

Takashi YANAGAWA*

(Received January 12, 1970)

1. Introduction

Let $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ be random samples from the symmetric and continuous c. d. f. $F(x)$ and $G(x) = F(x/\theta)$ respectively.

For testing the statistical hypothesis $H: \theta = 1$ against the alternative $AH: \theta > 1$ Tamura [3] has proposed the following test statistics.

$$(1) \quad Q_s^{(1)} = \frac{1}{\binom{m}{s} \binom{n}{2}} \Sigma^* \phi(x_{\alpha_1}, \dots, x_{\alpha_s}; y_{\beta_1}, y_{\beta_2})$$

where

$$\phi(x_1, x_2, \dots, x_s; y_1, y_2) = \begin{cases} 1 & \text{for } y_1 < x_1, \dots, x_s < y_2 \text{ or } y_2 < x_1, x_2, \dots, x_s < y_1, \\ 0 & \text{otherwise.} \end{cases}$$

and the summation Σ^* extends over all subscripts α, β such that $1 \leq \alpha_1 < \dots < \alpha_s \leq m$, $1 \leq \beta_1 < \beta_2 \leq n$.

Among the statistics $Q_s^{(1)}$, $s = 1, 2, \dots$, the interesting one would be $Q_1^{(1)}$ and $Q_2^{(1)}$. It has been proved that these two statistics have the same Pitman efficiencies. Thus $Q_1^{(1)}$ would be more practical than $Q_2^{(1)}$, since it is very easy to compute.

To make our investigation more precise we shall also consider the following statistics which are the same types of $Q_s^{(1)}$.

$$(2) \quad Q_s^{(2)} = \frac{1}{\binom{n}{s} \binom{m}{2}} \Sigma^{**} \phi(y_{\alpha_1}, \dots, y_{\alpha_s}; x_{\beta_1}, x_{\beta_2}), \quad s = 1, 2,$$

where Σ^{**} extends over all subscripts α, β such that $1 \leq \alpha_1 < \dots < \alpha_s \leq n$, $1 \leq \beta_1 < \beta_2 \leq m$.

The purpose of this paper is to make further comparisons to these test statistics, which will make us to recommend $Q_1^{(2)}$ instead of $Q_1^{(1)}$ or $Q_2^{(1)}$ in practical situations.

In section 2 we shall consider the comparison of $Q_s^{(i)}$, $i, s = 1, 2$ from the view point of the Bahadur asymptotic efficiency [1]. As pointed by Bahadur [2], Bahadur asymptotic efficiency has some pitfalls since it is an approximate measure of efficiency. Thus our results in section 2 might not be enough reliable. Therefore we shall in section 3 compute the small sample power of these test statistics and give comparisons of these powers for a specific distribution.

* Department of General Education, Kumamoto University.

2. Comparison of $Q_s^{(i)}$, $i, s = 1, 2$ by the Bahadur asymptotic efficiency

Let denote the mean of $Q_s^{(i)}$, $i, s = 1, 2$ by $\mu_{i,s}(\theta)$. Then we get

$$(4) \quad \mu_{1,s}(\theta) = 2 \iint_{x < y} [F(y) - F(x)]^s dG(x) d(y),$$

$$(5) \quad \mu_{2,s}(\theta) = 2 \iint_{x < y} [G(y) - G(x)]^s dF(x) dF(y).$$

Especially in the null case we get

$$(6) \quad \mu_{1,s}(1) = \mu_{2,s}(1) = \frac{2}{(s+1)(s+2)}.$$

Let $m = \rho N$, $n = (1-\rho)N$ and denoting the asymptotic variance of $Q_s^{(i)}$ under null hypothesis by $\sigma_{i,s}^2$, then following the manner of Tamura [3] we get

$$(7) \quad \sigma_{i,s}^2 = \frac{1}{(s+1)^2} \frac{8}{\rho(1-\rho)} \left[\frac{1}{2s+3} - \frac{2}{(s+2)^2} + \frac{[(s+1)!]^2}{(2s+3)!} \right], \quad s = 1, 2.$$

Let normalize the statistics $Q_s^{(i)}$ as follows.

$$Q_{N,s}^{(i)} = \frac{\sqrt{N}(Q_s^{(i)} - \mu_s(1))}{\sigma_{i,s}}, \quad i = 1, 2.$$

Then we get under non null hypothesis

$$(8) \quad E \left[\frac{Q_{N,s}^{(i)}}{\sqrt{N}} \right] = \frac{1}{\sigma_{i,s}} (\mu_{i,s}(\theta) - \mu_{i,s}(1)).$$

Thus by using Chebychev's inequality and the definition of the asymptotic slope, the asymptotic slope $C(Q_s^{(i)}; \theta)$ of statistic $Q_s^{(i)}$ is obtained, after some calculations, as follows.

$$C(Q_s^{(i)}; \theta) = \frac{1}{\sigma_{i,s}^2} (\mu_{i,s}(\theta) - \mu_{i,s}(1))^2.$$

Then from (4), (5) and (7) we get

$$(9) \quad C(Q_1^{(1)}; \theta) = 720\rho(1-\rho) \left(\int FGdG - \frac{1}{3} \right)^2,$$

$$(10) \quad C(Q_2^{(1)}; \theta) = 180\rho(1-\rho) \left(\int F^2dG - \frac{1}{3} \right)^2,$$

$$(11) \quad C(Q_1^{(2)}; \theta) = 720\rho(1-\rho) \left(\int FGdF - \frac{1}{3} \right)^2,$$

$$(12) \quad C(Q_2^{(2)}; \theta) = 180\rho(1-\rho) \left(\int G^2dF - \frac{1}{3} \right)^2.$$

By integration by part it can be easily shown

$$C(Q_1^{(1)}; \theta) = C(Q_2^{(2)}; \theta), \quad C(Q_2^{(1)}; \theta) = C(Q_1^{(2)}; \theta).$$

Further

$$\begin{aligned} C(Q_2^{(1)} : \theta) - C(Q_1^{(1)} : \theta) &= 180\rho(1-\rho) \left[\int F^2 dG - 2 \int FG dG + \frac{1}{3} \right] \left[\int F^2 dG + 2 \int FG dG - 1 \right] \\ &= 180\rho(1-\rho) \left[\int (F-G)^2 dG \right] \left[\left(\int F^2 dG - \frac{1}{3} \right) + 2 \left(\int GF dG - \frac{1}{3} \right) \right]. \end{aligned}$$

But it is seen that $\int F^2 dG$ is an increasing function of $\theta > 0$ for $G(x) = F(x/\theta)$. Thus we get

$$\int F^2 dG > \frac{1}{3} \quad \text{for } \theta > 1.$$

Similarly

$$\int FG dG > \frac{1}{3} \quad \text{for } \theta > 1.$$

Thus we get

$$C(Q_2^{(1)} : \theta) > C(Q_1^{(1)} : \theta) \quad \text{for } \theta > 1.$$

Namely it has been proved that for testing the hypothesis $H: \theta = 1$ against the alternative $AH: \theta > 1$,

(i) Bahadur asymptotic efficiency of $Q_2^{(1)}$ and $Q_1^{(1)}$ are respectively equivalent to that of $Q_1^{(2)}$ and $Q_2^{(2)}$,

(ii) $Q_2^{(1)}$ (or $Q_1^{(2)}$) is more efficient than $Q_1^{(1)}$ (or $Q_2^{(2)}$) or equivalently

(ii)' $Q_1^{(1)}$ (or $Q_2^{(1)}$) is more efficient than $Q_1^{(1)}$ (or $Q_2^{(2)}$).

Thus against Tamura's proposal, $Q_1^{(2)}$ instead of $Q_1^{(1)}$ would be recommended in the practical situations.

3. Small sample comparisons of $Q_s^{(i)}$, $i, s = 1, 2$

Since the results given in the section 2 are asymptotic and approximate, behaviours of statistics $Q_{N,s}^{(i)}$, $i, s = 1, 2$ must be discussed in small sample. We cannot unfortunately deal with them in the general form, therefore we only check in the simple and special cases. When $m = n = 4$, the orderings of X 's and Y 's which have larger values of $Q_i^{(j)}$, $i = 1, 2$ are respectively given in the following table.

Let the size α of test be $1/70$, then the critical regions of $Q_s^{(1)}$, $s = 1, 2$ contain only an ordering $YYXXXXYY$ and that of $Q_1^{(2)}$ and $Q_2^{(2)}$ are constructed by above five and seventeen orderings in the table respectively in the randomized form. In the case $\alpha = 5/70$, the critical regions of both $Q_2^{(1)}$ and $Q_1^{(2)}$ are constructed by above five orderings and that of $Q_1^{(1)}$ and $Q_2^{(2)}$ are constructed respectively by above six and seventeen ordering in the randomized form. When $F(x)$ is symmetrical, symmetric orderings have the same probability, for example

$$P_r(YYXXXXYY) = P_r(YXYXXXXY).$$

Now we assume that $F(x)$ be the uniform distribution in $(-1/2, 1/2)$, then after some computation we get

$$\begin{aligned} P_r(YYXXXXYY) &= 72 \iint G^2(x) [F(y) - F(x)]^2 [1 - G(y)]^2 dF(x) dF(y) \\ &= \frac{1}{70\theta^4} \left[-\frac{105}{4}(\theta-1)^4 + 42(\theta-1)^3 + 28(\theta-1)^2 + 8(\theta-1) + 1 \right] \quad \text{for } \theta > 1. \end{aligned}$$

Table. Ordering of $Q_i^{(i)}$, $i, s = 1, 2$, $m = n = 4$.

$Q_1^{(1)}$		$Q_2^{(1)}$		$Q_1^{(2)}$		$Q_2^{(2)}$	
ordering	value of $4\binom{4}{2}Q_1^{(1)}$	ordering	value of $\binom{4}{2}\binom{4}{2}Q_2^{(1)}$	ordering	value of $4\binom{4}{2}Q_1^{(2)}$	ordering	value of $\binom{4}{2}\binom{4}{2}Q_2^{(2)}$
Y Y X X X X Y Y	16	Y Y X X X X Y Y	24	Y Y X X X X Y Y	0	Y Y X X X X Y Y	0
Y Y X X X Y X Y	15	Y Y X X X Y X Y	18	Y Y Y X X X X Y	0	Y Y X X X Y X Y	0
Y X Y X X X Y Y	15	Y Y Y X X X X Y	18	Y X X X X Y Y Y	0	Y Y X X Y X X Y	0
Y Y X X Y X X Y	14	Y X Y X X X Y Y	18	X X X X Y Y Y Y	0	Y Y X Y X X X Y	0
Y X X Y X X Y Y	14	Y X X X X Y Y Y	18	Y Y Y Y X X X X	0	Y Y Y X X X X Y	0
Y X Y X X Y X Y	14	Y Y X X Y X X Y	15	Y Y X X X Y X Y	3	Y X Y X X X Y Y	0
Y Y X Y X X X Y	13	Y Y X Y X X X Y	15	Y Y X Y X X X Y	3	Y X X Y X X Y Y	0
Y X X X Y X Y Y	13	Y X X Y X X Y Y	15	Y X Y X X X Y Y	3	Y X X X Y X Y Y	0
Y X X Y X Y X Y	13	Y X X X Y X Y Y	15	Y X X X Y X Y Y	3	Y X X X X Y Y Y	0
Y X Y X Y X X Y	13	⋮		X Y X X X Y Y Y	3	X Y X X X Y Y Y	0
Y Y Y X X X X Y	12	⋮		Y Y Y X X X Y X	3	Y Y Y X X X Y X	0
Y X Y Y X X X Y	12	⋮		X X X Y X Y Y Y	3	X X Y X X Y Y Y	0
Y X X Y Y X X Y	12			Y Y Y X Y X X X	3	Y Y Y X X Y X X	0
Y X X X Y Y X Y	12			Y Y X X Y X X Y	4	X X X Y X Y Y Y	0
Y X X X X Y Y Y	12			Y X X Y X X Y Y	4	Y Y Y X Y X X X	0
X Y Y X X X Y Y	12			X X Y X X Y Y Y	4	X X X X Y Y Y Y	0
Y Y X X X Y Y X	12			Y Y Y X X Y X X	4	Y Y Y Y X X X X	0
⋮				⋮		⋮	

From the similar computations we get for $\theta > 1$

$$\begin{aligned}
 P_r(YYYYXXXXY) &= \frac{1}{70\theta^4} \left(\frac{35}{2}a^4 + 28a^3 + 21a^2 + 8a + 1 \right), \\
 P_r(YYXXXXYXY) & \\
 P_r(YYXXXYXXY) & \left. \vphantom{P_r(YYXXXXYXY)} \right) = \frac{1}{70\theta^4} (21a^3 + 21a^2 + 8a + 1), \\
 P_r(YYXYXXXXY) & \\
 P_r(YXYXXYXY) &= \frac{1}{70\theta^4} (14a^2 + 8a + 1), \\
 P_r(YYYYXXXXX) &= \frac{1}{70\theta^4} \left(\frac{35}{8}a^4 + 7a^3 + 7a^2 + 4a + 1 \right), \\
 P_r(YYYXXXYX) & \\
 P_r(YYYXXYXX) & \left. \vphantom{P_r(YYYXXXYX)} \right) = \frac{1}{70\theta^4} (7a^3 + 7a^2 + 4a + 1), \\
 P_r(YYYXYXXX) &
 \end{aligned}$$

where $a = \theta - 1$.

Thus the power of $Q_s^{(i)}$, denoted by $\gamma_s^{(i)}$, is given for $\alpha = 1/70$

$$(13) \quad \gamma_1^{(1)} = \gamma_2^{(1)} = \left(\frac{105}{4}a^4 + 42a^3 + 28a^2 + 8a + 1 \right) / 70\theta^4 \quad \text{for } \theta > 1,$$

$$(14) \quad \gamma_1^{(2)} = (70a^4 + 112a^3 + 84a^2 + 32a + 5) / 350\theta^4 \quad \text{for } \theta > 1,$$

$$(15) \quad \gamma_2^{(2)} = (70a^4 + 280a^3 + 252a^2 + 104a + 17) / 1190\theta^4 \quad \text{for } \theta > 1.$$

Comparing (13), (14) and (15) we get

$$(16) \quad \gamma_1^{(1)} = \gamma_2^{(1)} > \gamma_1^{(2)} > \gamma_2^{(2)} \quad \text{for } \theta > 1.$$

In the case $\alpha = 5/70$ we get

$$(17) \quad \gamma_1^{(1)} = \left(\frac{315}{4}a^4 + 336a^3 + 322a^2 + 120a + 15 \right) / 210\theta^4 \quad \text{for } \theta > 1,$$

$$(18) \quad \gamma_1^{(2)} = (70a^4 + 112a^3 + 84a^2 + 32a + 5) / 70\theta^4 \quad \text{for } \theta > 1,$$

$$(19) \quad \gamma_2^{(1)} = \left(\frac{245}{4}a^4 + 140a^3 + 112a^2 + 40a + 5 \right) / 70\theta^4 \quad \text{for } \theta > 1,$$

$$(20) \quad \gamma_2^{(2)} = (70a^4 + 280a^3 + 252a^2 + 104a + 17) / 238\theta^4 \quad \text{for } \theta > 1.$$

Comparing (17), (18), (19) and (20) we get

$$(21) \quad \begin{cases} \gamma_1^{(2)} > \gamma_2^{(1)} > \gamma_1^{(1)} > \gamma_2^{(2)} & \text{for } \theta > 5.047, \\ \gamma_2^{(1)} > \gamma_1^{(2)} > \gamma_1^{(1)} > \gamma_2^{(2)} & \text{for } 1.863 < \theta \leq 5.047, \\ \gamma_2^{(1)} > \gamma_1^{(1)} > \gamma_1^{(2)} > \gamma_2^{(2)} & \text{for } 1 < \theta \leq 1.863, \end{cases}$$

(16) and (21) support the results in section 2. Namely, let denote by $B(T^{(1)} : T^{(2)})$ the Bahadur asymptotic efficiency of $T^{(1)}$ relative to $T^{(2)}$, then we have following correspondence between the results of section 2 and section 3.

Results in section 2

$$B(Q_1^{(1)} : Q_2^{(1)}) < 1$$

$$B(Q_1^{(2)} : Q_2^{(2)}) > 1$$

$$B(Q_1^{(1)} : Q_1^{(2)}) < 1$$

$$B(Q_2^{(1)} : Q_2^{(2)}) > 1$$

$$B(Q_1^{(1)} : Q_2^{(2)}) = 1$$

$$B(Q_2^{(1)} : Q_1^{(2)}) = 1$$

Results in section 3

$$\gamma_1^{(1)} < \gamma_2^{(1)}$$

$$\gamma_1^{(2)} > \gamma_2^{(2)}$$

$$\gamma_1^{(1)} < \gamma_1^{(2)}$$

$$\gamma_2^{(1)} > \gamma_2^{(2)}$$

$$\gamma_1^{(1)} > \gamma_2^{(2)}$$

$$\gamma_2^{(1)} \geq \gamma_1^{(2)}.$$

4. Acknowledgements.

I am grateful to Professor Tamura of Kyushu Institute of Design for suggesting this investigation.

References

- [1] BAHADUR, R. R. (1960). *Statistic comparison of tests*. Ann. Math. Statist. **31**, 276–295.
- [2] BAHADUR, R. R. (1967). *Rates of convergence of estimates and tests statistics*. Ann. Math. Statist. **38**, 303–324.
- [3] TAMURA, R. (1965). *Nonparametric tests for scale*. Bull. Math. Statist. **12**, 89–94.