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A FURTHER COMPARISON OF TWO-SAMPLE NONPARAMETRIC TESTS FOR DISPERSION I

By

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0. Summary

This paper is concerned with two-sample nonparametric tests for dispersion alternatives. Various nonparametric tests such as Tamura's Q, Sukhatme's T, Mood's M and Freund-Ansari's W (Barton-David's and Siegel-Tukey's) are considered and compared from the point of view of the Bahadur asymptotic efficiency for the symmetric distribution. Final results are given in section 4.

1. Introduction

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two samples of independent observations drawn from two populations with cumulative distribution functions $F((x-\xi)/\sigma_1)$ and $F((x-\eta)/\sigma_2)$, respectively. We shall assume in what follows that $F(x)$ is absolutely continuous and $F(0)=1/2$.

For testing the hypotheses

$$(1.1)\qquad \begin{cases} H_0: \sigma_1=\sigma_2 \\ AH: \sigma_1 \neq \sigma_2 \qquad (\text{or } \sigma_1 < \sigma_2), \end{cases}$$

various attempts have been made to construct nonparametric tests by authors such as Mood [9], Sukhatme [11], Freund and Ansari [7], Barton and David [4], Tamura [12], Siegel and Tukey [10], Capon [5], Klotz [8] and others. But as Klotz [8] has pointed out, there is (at least asymptotically) an equivalence in the sense of test statistics among the test of Freund-Ansari, Barton-David and Siegel-Turkey, and hence we shall restrict our consideration to the tests of Sukhatme, Tamura, Mood and Freund-Ansari, whose test statistics are given as follows.

Let denote $min(x, y)$ and $max(x, y)$ by $x \wedge y$ and $x \vee y$, respedtively.

Sukhatme's T:

$$T^{(1)}=\frac{1}{m\ n}\sum_{i=1}^{m}\sum_{j=1}^{n}\phi(x_i, y_j),$$

$$T^{(2)}=\frac{1}{m\ n}\sum_{i=1}^{m}\sum_{j=1}^{n}\phi(y_j, x_i).$$

where

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$$\phi(x, y)=\begin{cases} 1; & \text{if } 0\wedge y<x<0\vee y \\ 0; & \text{otherwise}. \end{cases}$$

Tamura's Q:

$$Q^{(1)}=\frac{1}{\binom{m}{2}\binom{n}{2}}\sum_{\alpha_1<\alpha_2}\sum_{\beta_1<\beta_2}\phi(x_{\alpha_1}, x_{\alpha_2}; y_{\beta_1}, y_{\beta_2}),$$

$$Q^{(1)}=\frac{1}{\binom{m}{2}\binom{n}{2}}\sum_{\alpha_1<\alpha_2}\sum_{\beta_1<\beta_2}\phi(y_{\beta_1}, y_{\beta_2}; x_{\alpha_1}, x_{\alpha_2}),$$

where

$$\phi(x_1, x_2; y_1, y_2)=\begin{cases} 1; & \text{if } y_1\wedge y_2<x_1, x_2<y_1\vee y_2 \\ 0; & \text{otherwise}. \end{cases}$$

Mood's M:

$$M=\frac{1}{N}\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)^2 Z_{N,i},$$

where

$N=m+n$,

$$Z_{N,i}=\begin{cases} 1; & \text{if the } i\text{-th smallest in the combined sample is an } X, \\ 0; & \text{otherwise}. \end{cases}$$

Freund-Ansari's W:

$$W=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{N+1}{2}-\left|\frac{N+1}{2}-i\right|\right)Z_{N,i}.$$

It is noted that the Sukhatme's T and Tamura's Q are U statistics, while Mood's M and Freund-Ansari's W are rank statistics. The critical region of each test is given, respectively one sided (AH: $\sigma_1<\sigma_2$) and two sided testing problem, by $T^{(1)}>c$, $T^{(2)}<c$, $Q^{(1)}>c$, $Q^{(2)}<c$, $M>c$ or $W>c$ $|T^{(i)}|>c$, $|Q^{(i)}|>c$, $|M|>c$ or $|W|>c$, $i=1, 2$, where c is a generic constant. The asymptotic efficiencies of these tests have been so far investigated from the point of view of the Pitman efficiency. It is known that both $T^{(1)}$ (or $T^{(2)}$) and W have the same asymptotic efficiencies and $Q^{(1)}$ (or $Q^{(2)}$) and M have another same asymptotic efficiencies, and the latter is larger for certain wide class of distributions.

Since the Pitman efficiency is a measure of efficiency concerning a power in the neighbourhood of $\theta=\theta_0$, the tests with same Pitman efficiency do not necessarily have the same efficiency for any $\theta>0$. Take Tamura's $Q^{(1)}$ and $Q^{(2)}$ for example. Though they are shown to have the same Pitman efficiency, more of our experiences and the recent paper of the author [13] shows $Q^{(1)}$ more powerful than $Q^{(2)}$ for testing problem H: $\theta=1$ *vs.* AH: $\theta>1$. Therefore further investigations are needed for the tests with the same Pitman efficiency.

The purpose of this paper is to give a further comparison to these tests based on the statistics mentioned above by means of the Bahadur asymptotic efficiency [2]. As noticed by Bahadur [3], Bahadur efficiency has some pitfalls, since it is an approximate measure of efficiency. To cover these pitfalls some Monte Carlo experiments are being prepared for the forth comming paper.

2. Asymptotic slopes of tests

We shall assume the location parameters ξ and η to be known. In the case of Tamura's Q, Mood's M or Freund-Ansari's W, it is sufficient to assume only the difference $\eta-\xi$ to be known, since they are translation invariant. Since all tests under consideration are scale invariant, we can assume without loss of generality throughout the paper that two samples, X of size m and Y of size n, are drawn from the population $F(x)$ and $G(x)=F(x/\theta)$, respectively, where $\theta=\sigma_2/\sigma_1$. Then the hypotheses (1.1) are reduced to

$$\begin{cases} H_0: \theta=1 \\ AH: \theta\neq 1 \qquad (\text{or } \theta>1). \end{cases}$$

Let $m=\rho N$, $n=(1-\rho)N$ and $0<\rho<1$. The asymptotic slope of the test statistic $T=T_N$ is defined by Bahadur [2] as follows. Let $T=\{T_N\}$ be a sequence of test statistics which satisfy the following conditions.

I. There exists a continuous distribution function F such that under H_0

$$\lim_{N\to\infty} P_r[T_N\leqq t]=F(t) \qquad \text{for any } t.$$

II. There exist a constant a, $0<a<\infty$, such that

$$\log[1-F(t)]=-\frac{at^2}{2}[1+0(1)] \qquad \text{as } t\to\infty.$$

III. There exists a function $b(\theta)$, $0<b(\theta)<\infty$ under AH such that for any $\theta\neq 1$

$$\lim_{N\to\infty} P_r\left[\left|\frac{T_N}{\sqrt{N}}-b(\theta)\right|>\varepsilon\right]=0 \qquad \text{for any } \varepsilon>0.$$

Let $K_N(x)=-2\log L_N(x)$, where $L_N(x)=1-F(T_N(x))$. Then it can be shown that under non null hypothesis $\frac{K_N}{N}$ tends in probability to the quantity $C(\theta)=a[b(\theta)]^2$ as $N\to\infty$. The quantity $C(\theta)$ is called by Bahadur as an asymptotic slope of the test statistics $T=\{T_N\}$.

Now we shall give the asymptotic slope of $T^{(1)}$. Let

$$T_N^{(1)}=\left(T^{(1)}-\frac{1}{4}\right)\Big/\left(\frac{m+n}{48mn}\right)^{1/2}.$$

Then under H_0, $T_N^{(1)}$ is known (for example see [11]) to have limiting standard normal distribution. Thus above conditions I and II are satisfied with $a=1$. By using Chebychev's inequality the quantity $b_{T^{(1)}}(\theta)$ which satisfies the condition III is easily obtained as follows.

$$b_{T^{(1)}}(\theta)=(48\rho(1-\rho))^{1/2}\left[\int_0^\infty FdG-\int_{-\infty}^0 FdG-1/4\right].$$

Thus the asymptotic slope of $T^{(1)}$ is obtained as $C_{T^{(1)}}(\theta)=b_{T^{(1)}}^2(\theta)$.

From the similar computations the asymptotic slope of each test statistic dealt with in this paper is given as follows.

For $T^{(1)}$, $T^{(2)}$ and W

(2.1)
$$C_{T^{(1)}}(\theta)=C_W(\theta)=48\rho(1-\rho)\Big[\int_0^{\infty}FdG-\int_{-\infty}^{0}FdG-1/4\Big]^2,$$
$$C_{T^{(2)}}(\theta)=48\rho(1-\rho)\Big[\int_0^{\infty}GdF-\int_{-\infty}^{0}GdF-1/4\Big]^2;$$

for $Q^{(1)}$ and $Q^{(2)}$

(2.2)
$$C_{Q^{(1)}}(\theta)=180\rho(1-\rho)\Big[\int_{-\infty}^{\infty}F^2dG-\Big(\int_{-\infty}^{\infty}FdG\Big)^2-1/12\Big]^2,$$
$$C_{Q^{(2)}}(\theta)=180\rho(1-\rho)\Big[\int_{-\infty}^{\infty}G^2dF-\Big(\int_{-\infty}^{\infty}GdF\Big)^2-1/12\Big]^2;$$

and for M

(2.3)
$$C_M(\theta)=180\rho(1-\rho)\Big[(1-2\rho)/6-2(1-\rho)\int_{-\infty}^{\infty}FGdG-\rho\int_{-\infty}^{\infty}F^2dG+\int_{-\infty}^{\infty}FdG\Big]^2.$$

REMARK: M and W are rank statistics of Chernoff-Savage type. In general, under some reguality conditions given by Chernoff-Savage [6], asymptotic slopes of the rank tests of Chernoff-Savage type are given as follows;

$$C(\theta)=(\mu-\mu_0)^2/\tau^2$$

where

$$\mu=\int_{-\infty}^{\infty}J[H(x)]dF(x),$$

$$\mu_0=\int_0^1 J(x)dx,$$

$$\tau^2=\frac{1-\rho}{\rho}\Big[\int_0^1 J(x)dx-\Big(\int_0^1 J(x)dx\Big)^2\Big],$$

$H(x)$ and $J(x)$ are defined in Chernoff-Savage [6].

Let $\mathcal{F}_s$ be the class of distribution functions which are absolutely continuous and symmetric about the origin.

We need the following lemma in section 3.

LEMMA. *Let* $F(x)\in\mathcal{F}_s$ *and* $G(x)=F(x/\theta)$, *then*

(i)
$$1/4<\int_{-\infty}^{\infty}F^2dG\leqq 1/3 \quad \textit{for } 0<\theta\leqq 1,$$
$$1/3<\int_{-\infty}^{\infty}F^2dG\leqq 1/2 \quad \textit{for } 1<\theta<\infty;$$

(ii)
$$1/4<\int_{-\infty}^{\infty}FGdG\leqq 1/3 \quad \textit{for } 0<\theta\leqq 1,$$
$$1/3<\int_{-\infty}^{\infty}FGdG\leqq 3/8 \quad \textit{for } 1<\theta<\infty;$$

(iii)
$$1/4<\int_0^{\infty}FdG\leqq 3/8 \quad \textit{for } 0<\theta\leqq 1,$$

$$3/8 < \int_0^\infty F dG \leqq 1/2 \qquad \textit{for } 1 < \theta < \infty;$$

(iv)
$$\int_{-\infty}^\infty F^2 dG \leqq \int_{-\infty}^\infty FGdG \qquad \textit{for } 0 < \theta \leqq 1,$$

$$\int_{-\infty}^\infty F^2 dG > \int_{-\infty}^\infty FGdG \qquad \textit{for } 1 < \theta < \infty.$$

PROOF. Since

$$\alpha(\theta) \equiv \int_{-\infty}^\infty F^2 dG = 1/2 - 2\int_0^\infty F(\theta t)F(-\theta t)dF(t)$$

and

$$\alpha(\theta_1) - \alpha(\theta_2) = 2\int_0^\infty (F(\theta_1 t) - F(\theta_2 t))(F(\theta_1 t) + F(\theta_2 t) - 1)dF(t) > 0$$

for any $\theta_1 > \theta_2 > 0$, we find that $\alpha(\theta) = \int F^2 dG$ is an increasing function of $\theta > 0$. Thus we obtain (i). (ii) is immediately obtained from (i). (iii) is obvious. By a similar argument as (i), we can easily prove (iv). (Q. E. D.)

3. Comparison of the tests

Let $T_N^{(1)}$, $T_N^{(2)}$ be two sequences of test statistics with asymptotic slopes $C_{T_N^{(1)}}(\theta)$ and $C_{T_N^{(2)}}(\theta)$, respectively, then Bahadur asymptotic efficiency of $T_N^{(1)}$ relative to $T_N^{(2)}$ is defined as $\varphi_{T_N^{(1)}, T_N^{(2)}}(\theta) = C_{T_N^{(1)}}(\theta)/C_{T_N^{(2)}}(\theta)$. Now we shall consider the comparison of each test.

THEOREM 1. *($T^{(1)}$ vs. $T^{(2)}$, $Q^{(1)}$ vs. $Q^{(2)}$ and $Q^{(1)}$ vs. M)*

(i) *For arbitrary F,*

$$\varphi_{T^{(2)}, T^{(1)}}(\theta) = \varphi_{W, T^{(1)}}(\theta) = 1 \qquad \textit{for any } \theta > 0,\ \theta \neq 1.$$

(ii) *For any $F \in \mathcal{F}_s$,*

$$\varphi_{Q^{(2)}, Q^{(1)}}(\theta) > \varphi_{M, Q^{(1)}}(\theta) > 1 \qquad \textit{for any } 0 < \theta < 1,$$

$$\varphi_{Q^{(2)}, Q^{(1)}}(\theta) < \varphi_{M, Q^{(1)}}(\theta) < 1 \qquad \textit{for any } \theta > 1.$$

PROOF. By integration by part (i) can be immediately obtained from (2.1). (ii) First we shall consider $Q^{(1)}$ *vs.* M. Since $\int_{-\infty}^\infty FdG = 1/2$ for any $F \in \mathcal{F}_s$ and $G(x) = F(x/\theta)$, thus from (2.2) and (2.3) it follows that

$$C_{Q^{(1)}}(\theta) - C_M(\theta) = 180\rho(1-\rho)\Big\{\Big(\int_{-\infty}^\infty F^2 dG - 1/3\Big)^2 - \Big(2\rho/3 - 2(1-\rho)\int_{-\infty}^\infty FGdG - \rho\int_{-\infty}^\infty F^2 dG\Big)^2\Big\}$$

$$= 180\rho(1-\rho)A_1(\theta)A_2(\theta),$$

where

$$A_1(\theta) = (1-\rho)\Big(\int F^2 dG - 2\int FGdG + 1/3\Big) = (1-\rho)\int (F-G)^2 dG > 0$$

$$\text{for any } \theta > 0,\ \theta \neq 1,$$

$$A_2(\theta)=(1+\rho)\left(\int F^2dG-1/3\right)+2(1-\rho)\left(\int FGdG-1/3\right).$$

Since from the lemma $A_2(\theta)<0$ (>0) for $0<\theta<1$ $(1<\theta<\infty)$, we have $C_{Q^{(1)}}(\theta)-C_M(\theta)<0$ (>0) for $0<\theta<1$ $(1<\theta<\infty)$. Since $C_{Q^{(1)}}(\theta)\neq 0$ for any $\theta>0$, $\theta\neq 1$, we get the right hand side inequality. Next, from the above results, it is immediately obtained that $\varphi_{M,Q^{(2)}}(\theta)<1$ (>1) for $0<\theta<1$ $(1<\theta<\infty)$. Thus we get the left hand side inequality. (Q. E. D.)

The theorem means that the efficiency of Tamura's Q, unlike that of Sukhatme's T, depends on the direction of the alternative hypothesis; $Q^{(1)}$ is more efficient than $Q^{(2)}$ for the one sided case $AH: \theta>1$, but less efficient for $AH: \theta<1$.

In the case one might expect to get higher efficiency by adding $Q^{(1)}$ and $-Q^{(2)}$ for the two sided testing problem. The new statistic $Q^{(12)}=Q^{(1)}-Q^{(2)}$ is also a U statistic given as follows.

$$Q^{(12)}=\frac{1}{\binom{m}{2}\binom{n}{2}}\sum_{\substack{\alpha_1<\alpha_2\\ \beta_1<\beta_2}}\omega(x_{\alpha_1}, x_{\alpha_2}; y_{\beta_1}, y_{\beta_2}),$$

where

$$\omega(x_1, x_2; y_1, y_2)=\begin{cases} 1 & \text{if } y_1\wedge y_2<x_1,\ x_2<y_1\vee y_2 \\ -1 & \text{if } x_1\wedge x_2<y_1,\ y_2<x_1\vee x_2 \\ 0 & \text{if otherwise.}\end{cases}$$

Denoting the mean value of $Q^{(12)}$ by $\mu_{12}(\theta)$, then we get

$$\mu_{12}(\theta)=2\iint_{x<y}[F(y)-F(x)]^2dG(x)dG(y)-2\iint_{x<y}[G(y)-G(x)]^2dF(x)dF(y).$$

From the general theory of U statistics $N^{1/2}(Q^{(12)}-\mu_{12}(\theta))$ has limiting normal distribution with mean 0 and its asymptotic variance under null hypothesis is easily calculated as follows.

$$\sigma_{12}^2(1)=\frac{4}{45\rho(1-\rho)}.$$

Thus the efficacy of $Q^{(12)}$ in the Mood's sense is given by

$$\left(\frac{d\mu_{12}(\theta)}{d\theta}\Big|_{\theta=1}/\sigma_{12}(1)\right)^2=180\rho(1-\rho)\left[\int_{-\infty}^{\infty}(2F(x)-1)xf(x)dF(x)\right]^2,$$

which is equivalent to that of $Q^{(1)}$. Thus we get the following theorem.

THEOREM 2. *$Q^{(12)}$ has the same Pitman efficiency as $Q^{(1)}$.*

THEOREM 3. *For any $F\in\mathcal{F}_s$, we get*

(i)
$$\varphi_{Q^{(2)},Q^{(12)}}(\theta)>\varphi_{Q^{(12)},Q^{(1)}}(\theta)>1 \quad \textit{for } 0<\theta<1,$$
$$\varphi_{Q^{(2)},Q^{(12)}}(\theta)<\varphi_{Q^{(12)},Q^{(1)}}(\theta)<1 \quad \textit{for } \theta>1;$$

(ii) (a) *if* $0<\rho<1/2$, *then* $\varphi_{Q^{(12)},M}(\theta)<1$ (>1) *for* $0<\theta<1$ $(1<\theta<\infty)$,

(b) *if* $0<\rho<1/2$, *then* $\varphi_{Q^{(12)},M}(\theta)=1$ *for any* $\theta>0$, $\theta\neq 1$,

(c) *if* $1/2<\rho<1$, *then* $\varphi_{Q^{(12)},M}(\theta)>1$ (<1) *for* $0<\theta<1$ $(1<\theta<\infty)$.

PROOF. The asymptotic slope of $Q^{(12)}$ is easily obtained as follows.

$$C_{Q^{(12)}}(\theta)=45\rho(1-\rho)\Big[\int_{-\infty}^{\infty}F^2dG-\Big(\int_{-\infty}^{\infty}FdG\Big)^2-\int_{-\infty}^{\infty}G^2dF+\Big(\int_{-\infty}^{\infty}GdF\Big)^2\Big]^2. \tag{2.4}$$

Then we get from (2.2) and (2.3)

$$C_{Q^{(1)}}(\theta)-C_{Q^{(12)}}(\theta)=45\rho(1-\rho)\Big[4\Big(\int_{-\infty}^{\infty}F^2dG-\frac{1}{3}\Big)^2-\Big(\int_{-\infty}^{\infty}F^2dG-\int_{-\infty}^{\infty}G^2dF\Big)^2\Big]$$
$$=45\rho(1-\rho)B_1(\theta)B_2(\theta)\,,$$
$$C_{Q^{(12)}}(\theta)-C_M(\theta)=45\rho(1-\rho)\Big[\Big(\int_{-\infty}^{\infty}F^2dG-\int_{-\infty}^{\infty}G^2dF\Big)^2$$
$$-4\Big(\frac{2-\rho}{3}-2(1-\rho)\int_{-\infty}^{\infty}FGdG-\rho\int_{-\infty}^{\infty}F^2dG\Big)^2\Big]$$
$$=45\rho(1-\rho)D_1(\theta)D_2(\theta)$$

for any $F\in\mathcal{F}_s$, where

$$B_1(\theta)=\int_{-\infty}^{\infty}F^2dG+\int_{-\infty}^{\infty}G^2dF-2/3=\int_{-\infty}^{\infty}(F-G)^2dG\,,$$
$$B_2(\theta)=3\Big(\int_{-\infty}^{\infty}F^2dG-1/3\Big)+2\Big(\int_{-\infty}^{\infty}FGdG-1/3\Big)\,,$$
$$D_1(\theta)=(1-2\rho)\int_{-\infty}^{\infty}(F-G)^2dG\,,$$
$$D_2(\theta)=(1+2\rho)\Big(\int_{-\infty}^{\infty}F^2dG-1/3\Big)+(6-4\rho)\Big(\int_{-\infty}^{\infty}FGdG-1/3\Big)$$

Thus by the lemma and the similar argument as in the theorem 1 (ii) we complete the proof. (Q. E. D.)

Since $T^{(1)}$, $T^{(2)}$ and W are seen to have the same efficiency in the sence of Bahadur as well as Pitman, we shall in the sequel consider only W among them and compare it with $Q^{(1)}$ and M.

Let denote by f the p. d. f. of F, and define the class of distribution functions $\mathcal{F}_{ss}$, $\mathcal{F}_{s+}$, $\mathcal{F}_{s0}$ and $\mathcal{F}_{s-}$ as follows.

$$\mathcal{F}_{ss}=\{F\,;\,F\in\mathcal{F}_s,\ xf(\theta x)\text{ is bounded and continuous in }\theta>0\text{ for any }x>0\text{ a. e. }(F)\}\,,$$
$$\mathcal{F}_{s+}=\{F\,;\,F\in\mathcal{F}_{ss},\ \int_0^{\infty}xf(x)F(x)dF(x)>a\int_0^{\infty}xf(x)dF(x)\}\,,$$
$$\mathcal{F}_{s0}=\{F\,;\,F\in\mathcal{F}_{ss},\ \int_0^{\infty}xf(x)F(x)dF(x)=a\int_0^{\infty}xf(x)dF(x)\}\,,$$
$$\mathcal{F}_{s-}=\{F\,;\,F\in\mathcal{F}_{ss},\ \int_0^{\infty}xf(x)F(x)dF(x)<a\int_0^{\infty}xf(x)dF(x)\}\,,$$

where $a=(15+2\sqrt{15})/30$.

THEOREM 4. ($Q^{(1)}$ vs. W and M vs. W) *There exists an* $\varepsilon>0$ *such that*

(i) *if* $F\in\mathcal{F}_{s+}$, *then* $\varphi_{Q^{(1)},W}(\theta)$, $\varphi_{M,W}(\theta)>1$ *for* $1-\varepsilon<\theta<1+\varepsilon$, $\theta\neq1$,

(ii) *if* $F \in \mathcal{F}_{s0}$, *then both* $\varphi_{Q^{(1)},W}(\theta)-1$ *and* $\varphi_{M,W}(\theta)-1$ *have the same sign for* $1-\varepsilon<\theta<1+\varepsilon$, $\theta \neq 1$,

(iii) *if* $F \in \mathcal{F}_{s-}$, *then* $\varphi_{Q^{(1)},W}(\theta)$, $\varphi_{M,W}(\theta)>1$ *for* $1-\varepsilon<\theta<1+\varepsilon$, $\theta \neq 1$.

PROOF. We shall first prove for $Q^{(1)}$ vs. W. From (2.1) and (2.2) it follows for any $F \in \mathcal{F}_s$

$$C_{Q^{(1)}}(\theta)-C_W(\theta)=4\rho(1-\rho)E_1(\theta)E_2(\theta)\,,$$

where

$$E_1(\theta)=3\sqrt{5}\Big(\int_{-\infty}^{\infty}F^2dG-1/3\Big)+4\sqrt{3}\Big(\int_{-\infty}^{\infty}FdG-3/8\Big)\,,$$

$$E_2(\theta)=3\sqrt{5}\Big(\int_{-\infty}^{\infty}F^2dG-1/3\Big)-4\sqrt{3}\Big(\int_{-\infty}^{\infty}FdG-3/8\Big)\,.$$

From the lemma mentioned above we get $E_1(\theta)<0$ (>0) for $0<\theta<1$ $(1<\theta<\infty)$. Further it follows that

$$E_2(\theta)=6\sqrt{5}\int_0^{\infty}\Big[F^2(\theta x)-\frac{2\sqrt{3}+3\sqrt{5}}{3\sqrt{5}}F(\theta x)+\frac{2\sqrt{5}+3\sqrt{3}}{7\sqrt{5}}F^2(x)\Big]dF(x)\,.$$

Thus we get $dE_2(\theta)/d\theta=12\sqrt{5}\,A(\theta)$ for any $F \in \mathcal{F}_{ss}$, where

$$A(\theta)=\int_0^{\infty}xf(\theta x)\Big[F(\theta x)-\frac{3\sqrt{5}+2\sqrt{3}}{6\sqrt{5}}\Big]dF(x)\,.$$

Since $E_2(1)=0$, then there exists an $\varepsilon>0$ such that (i) if $A(1)>0$, then $E_2(\theta)<0$ (>0) for $1-\varepsilon<\theta<1$ $(1<\theta<1+\varepsilon)$, (ii) if $A(1)=0$, then either $E_2(\theta)>0$ or <0 for $1-\varepsilon<\theta<1+\varepsilon$, $\theta\neq 1$ and (iii) if $A(1)<0$, then $E_2(\theta)>0$ (<0) for $1-\varepsilon<\theta<1$ $(1<\theta<1+\varepsilon)$. Thus it follows that (i) if $F \in \mathcal{F}_{s+}$, then $C_{Q^{(1)}}(\theta)-C_W(\theta)>0$ for $1-\varepsilon<\theta<1+\varepsilon$, $\theta\neq 1$, (ii) if $F \in \mathcal{F}_{s0}$, $C_{Q^{(1)}}(\theta)-C_W(\theta)$ has the same sign for $1-\varepsilon<\theta<1+\varepsilon$, $\theta\neq 1$ and (iii) if $F \in \mathcal{F}_{s-}$, then $C_{Q^{(1)}}(\theta)-C_W(\theta)<0$ for $1-\varepsilon<\theta<1+\varepsilon$, $\theta\neq 1$. But as $C_W(\theta)\neq 0$ for any $\theta>0$, $\theta\neq 1$, thus we get the conclusions stated in the theorem. Next we shall prove for M *vs.* W. From (2.1) and (2.3) it follows that for any $F \in \mathcal{F}_s$,

$$C_M(\theta)-C_W(\theta)=12\rho(1-\rho)E_1^*(\theta)E_2^*(\theta)\,,$$

where

$$E_1^*(\theta)=\sqrt{15}\Big[(\rho-2)/3+2(1-\rho)\int_{-\infty}^{\infty}FGdG+\rho\int_{-\infty}^{\infty}F^2dG\Big]+4\Big(\int_0^{\infty}FdG-3/8\Big)\,,$$

$$E_2^*(\theta)=\sqrt{15}\Big[(\rho-2)/3+2(1-\rho)\int_{-\infty}^{\infty}FGdG+\rho\int_{-\infty}^{\infty}F^2dG\Big]-4\Big(\int_0^{\infty}FdG-3/8\Big)\,.$$

Thus the similar computations as above lead the proof of the theorem. (Q. E. D.)

REMARK. The class of distribution functions $\mathcal{F}_{s+}$ is wide enough to include normal, logistic and double exponential distributions among others, while $\mathcal{F}_{s-}$, generally, includes distributions more spread than those belonging to $\mathcal{F}_{s+}$, Cauchy distribution for example.

4. Conclusions

From the theorems above we can summarise our results briefly as follows. It is noted that the comparison depends not only on the underlying distribution but also on the direction of the alternative hypothesis.

For the *right-sided testing problem* $H_0: \theta=1$ *vs.* $AH: \theta>1$, $Q^{(1)}$ is always the most efficient among $Q^{(1)}$, $Q^{(2)}$, $Q^{(12)}$ and M for any distribution $F \in \mathcal{F}_s$.

For *two-sided hypothesis* $H_0: \theta=1$ *vs.* $AH: \theta \neq 1$, unlike Sukhatme's $T^{(1)}$ and $T^{(2)}$ have the same asymptotic efficiencies for any distributions and for any $\theta>0$, $\theta \neq 1$, efficiencies of $Q^{(1)}$ and $Q^{(2)}$ depend on the alternative hypothesis. Theorem 2 and 3 lead one to the following comparisons among $Q^{(1)}$, $Q^{(2)}$, $Q^{(12)}$ and M.

(i) For $0<\rho<1/2$

$$\varphi_{Q^{(2)},Q^{(1)}}(\theta) > \varphi_{M,Q^{(1)}}(\theta) > \varphi_{Q^{(12)},Q^{(1)}}(\theta) > 1 \qquad \text{for any } 0<\theta<1,$$

$$\varphi_{Q^{(2)},Q^{(1)}}(\theta) < \varphi_{M,Q^{(1)}}(\theta) < \varphi_{Q^{(12)},Q^{(1)}}(\theta) < 1 \qquad \text{for any } \theta>1;$$

(ii) For $\rho=1/2$

$$\varphi_{Q^{(2)},Q^{(1)}}(\theta) > \varphi_{M,Q^{(1)}}(\theta) = \varphi_{Q^{(12)},Q^{(1)}}(\theta) > 1 \qquad \text{for any } 0<\theta<1,$$

$$\varphi_{Q^{(2)},Q^{(1)}}(\theta) < \varphi_{M,Q^{(1)}}(\theta) = \varphi_{Q^{(12)},Q^{(1)}}(\theta) < 1 \qquad \text{for any } \theta>1;$$

(iii) For $1/2<\rho<1$

$$\varphi_{Q^{(2)},Q^{(1)}}(\theta) > \varphi_{Q^{(12)},Q^{(1)}}(\theta) > \varphi_{M,Q^{(1)}}(\theta) > 1 \qquad \text{for any } 0<\theta<1,$$

$$\varphi_{Q^{(2)},Q^{(1)}}(\theta) < \varphi_{Q^{(12)},Q^{(1)}}(\theta) < \varphi_{M,Q^{(1)}}(\theta) < 1 \qquad \text{for any } \theta>1.$$

Thus within our present knowledge, we have no decisive choice among $Q^{(1)}$, $Q^{(2)}$, $Q^{(12)}$ and M for two-side testing problems.

For any distribution belonging to $\mathcal{F}_{s+}$, $Q^{(1)}$ and M are more efficient than W, but less efficient for $\mathcal{F}_{s-}$.

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