九州大学学術情報リポジトリ
Kyushu University Institutional Repository

ON SOME ROBUST ESTIMATORS BASED ON NON-DISTRIBUTION-FREE U STATISTICS

Yanagawa, Takashi
Department of General Education, Kumamoto University

https://doi.org/10.5109/13040

出版情報：統計数理研究. 14 (1/2), pp.1-13, 1970-03. Research Association of Statistical Sciences
バージョン：
権利関係：

KYUSHU UNIVERSITY

ON SOME ROBUST ESTIMATORS BASED ON NON-DISTRIBUTION-FREE U STATISTICS

By

Takashi YANAGAWA*

Dedicated to Professor Tosio Kitagawa on his 60-th birthday

(Received January 12, 1970)

1. Introduction

Let $X_1, X_2, \cdots, X_{n_1}$ and $Y_1, Y_2, \cdots, Y_{n_2}$ be two independent sample of observations with continuous cumulative distribution function $F^{(1)}(t)$ and $F^{(2)}(t)$ respectively.

Testing the statistical hypotheses such as H: $\Delta=0$ *vs.* AH: $\Delta>0$ for $F^{(1)}(t)=F(t)$ and $F^{(2)}(t)=F(t-\Delta)$ or $H: c=\sigma_2/\sigma_1=1$ *vs.* $AH: c>1$ for $F^{(1)}(t)=F(t/\sigma_1)$ and $F^{(2)}(t)=F(t/\sigma_2)$, where assumption for F is merely its continuity, form the class of problems of two-sample nonparametric tests in the sense that two populations are involved, and (at least asymptotically) the test statistics are distribution-free, thus the rejection region is free from the unknown c. d. f., F. Various statistics proposed for these two problems are of either rank statistics or U statistics type.

For the corresponding problems of point estimation several attempts have been made to apply nonparametric test statistics in order to obtain robust estimrtors. Among them the most famous one would be the following Hodges and Lehmann's estimates.

Let $x=(x_1, x_2, \cdots, x_{n_1})$, $y=(y_1, y_2, \cdots, y_{n_2})$, $y+a=(y_1+a, y_2+a, \cdots, y_{n_2}+a)$ for any real number a and $T(x, y)$ be a test statistic for the hypothesis H; $\Delta=0$ against the alternative $\Delta>0$. Under conditions (A) $T(x, y+a)$ is a non-decreasing function of a for all x and y and (B) when $\Delta=0$, there exists $ET(X, Y)$, say μ, which is independent of F, Hodges and Lehmann [3] defined the estimator $\hat{\Delta}$ of Δ as follows.

$$\hat{\Delta}=(\Delta^{*}+\Delta^{**})/2\,, \tag{1.1}$$

where $\Delta^{*}=\sup\{\Delta\,;\,T(x, y-\Delta)>\mu\}$, and $\Delta^{**}=\inf\{\Delta: T(x, y-\Delta)<\mu\}$.

Especially the estimator $\hat{\Delta}$ based on the two-sample Wilcoxon statistic is called Hodges and Lehmann estimator and its remarkable robustness has been explored by many authors such as Hodges and Lehmann [3], Lehmann [6], [7] Høyland [5], Ramachandramurty [9], Spjøtvoll [10] and others.

Now as stated above Hodges and Lehmann considered the estimates based on rank tests, thus the statistics $T(x, y)$ in (1.1) are (at least asymptotically) "distribution-free", but the estimators $\hat{\Delta}$ based on this "distribution-free" $T(x, y)$ are not

* Department of General Education, Kumamoto University, Kumamoto, Japan.

" distribution-free ", since the variance of the asymptotic distribution of the estimator depends on the underlying distribution. Here the emphasis must be laid on " distribution-free " and it might be natural to raise the question. " How " distribution-free " of the statistic $T(x, y)$ cause to bring forth the robustness of the estimator ?" " Isn't it possible to obtain more robust estimator through extending our consideration to " non-distribution-free " $T(x, y)$?" The purpose of this paper is to investigate these questions.

Our answer is roughly summerised as follows; " distribution-free " property of the test statistic has an only little contribution to the robustness of the estimator based on the test statistic and it is possible to obtain more robust estimator by considering outside " non-distribution-free " statistics.

In section 2 we shall preliminarily investigate a certain type of U statistics. In section 3 we shall exemplify new estimators of a shift parameter based on "non-distribution-free " U statistics, which are a generalization of Hodges and Lehmann estimator and are shown to be more robust than Hodges and Lehmann estimator. Following the same manner as (1.1) we shall define the estimates for a dispersion parameter in section 4 and exemplify new estimators based on " non-distribution-free " U statistics and compare it with ordinary nonparametric estimator based on " distribution-free " U statistics. New estimators appear to be especially useful, since in case the underlying distribution is normal its asymptotic relative efficiency relative to the maximum likelihood estimator is up to 0.95, whereas that of ordinary nonparametric estimator is only 0.61.

2. Preliminaries

The following theorem 2.1 (i), (ii) concerning U statistics are well known results (for example, see Sugiura [11]) and (iii) is a slight modification of (ii) and the theorem 2.2 and 2.3 are generalizations of Hoeffding [4] to the two-sample case.

Let $X_1, X_2, \cdots, X_{n_1}$ and $Y_1, Y_2, \cdots, Y_{n_2}$ be random samples with c. d. f. $F^{(1)}(t)$ and $F^{(2)}(t)$ respectively and $\Phi(x_1, \cdots, x_m; y_1, \cdots, y_m)$ be a real valued function which is symmetric in each set of variables $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$. Put

$$U(x, y) = \frac{1}{\binom{n_1}{m}\binom{n_2}{m}} \Sigma^* \Phi(x_{i_1}, \cdots, x_{i_m}; y_{j_1}, \cdots, y_{j_m}), \tag{2.1}$$

where the summation Σ^* extends over all possible sets of subscripts, $(i_1, \cdots, i_m)$, $(j_1, \cdots, j_m)$ such that $1 \leqq i_1 < i_2 < \cdots < i_m \leqq m$, $1 \leqq j_1 < j_2 < \cdots < j_m \leqq m$. The statistics U defined by (2.1) is called U statistics.

THEOREM 2.1. (i) *If* $E(\Phi) = \eta$, *then* $E(U) = \eta$.

(ii) *Suppose that* $E(\Phi) = \eta$ *and* $E(\Phi^2) < \infty$. *Further let* $n_1 = \rho N$ *and* $n_2 = (1-\rho)N$ *with* ρ, $0 < \rho < 1$, *positive constant independent of* N. *Then as* $N \to \infty$ *the statistic* $\sqrt{N}(U-\eta)$ *is distributed according as normal distribution, whose mean is* 0 *and variance is given by*

$$\gamma_m^2 = \frac{m^2}{\rho} \zeta_{10} + \frac{m^2}{1-\rho} \zeta_{01} \tag{2.2}$$

where $\zeta_{d_1d_2}$ *means the covariance of* $\Phi(x_1, \cdots, x_m; y_1, \cdots, y_m)$ *and* $\Phi(x_1, \cdots, x_{d_1}, x'_{d_1+1}, \cdots, x'_m; y_1, \cdots, y_{d_2}, y'_{d_2+1}, \cdots, y'_m)$ *with* $x_i, x'_j; y_k, y'_l$ *being independentry distributed as* $F^{(1)}(t)$; $F^{(2)}(t)$ *respectively.*

(iii) *Suppose that the function* Φ *depends on* N, *denote it by* Φ_N *and corresponding* U *and* η *by* U_N *and* η_N *respectively. Further suppose that* Φ_N *is uniformly bounded and* $\Phi_N \to \Phi$ *as* $N \to \infty$, *then the statistic* $\sqrt{N}(U_N - \eta_N)$ *has the same limiting distribution as* (ii).

PROOF. For the proof of (i), (ii) see Sugiura [11, p. 386]. (iii) is obtained by a slight modification of the proof of (ii), thus we shall omit the proof.

THEOREM 2.2. *The quantity* $\zeta_{\alpha,\beta}$ *as defined in Theorem* 2.1 *satisfies the inequalities*

(2.3)
$$\text{(i)} \qquad 0 \leqq \frac{\zeta_{\alpha_1,0}}{\alpha_1} \leqq \frac{\zeta_{\alpha_2,0}}{\alpha_2} \qquad \text{if } 1 \leqq \alpha_1 < \alpha_2 \leqq m$$

$$0 \leqq \frac{\zeta_{0,\beta_1}}{\beta_1} \leqq \frac{\zeta_{0,\beta_2}}{\beta_2} \qquad \text{if } 1 \leqq \beta_1 < \beta_2 \leqq m$$

(2.4)
$$\text{(ii)} \qquad 0 \leqq \frac{1}{\alpha_1\beta_1}(\zeta_{\alpha_1,\beta_1} - \zeta_{\alpha_1,0} - \zeta_{0,\beta_1}) \leqq \frac{1}{\alpha_2\beta_2}(\zeta_{\alpha_2,\beta_2} - \zeta_{\alpha_2,0} - \zeta_{0,\beta_2})$$

$$\text{if} \quad 1 \leqq \alpha_1 < \alpha_2 \leqq m, \quad 1 \leqq \beta_1 < \beta_2 \leqq m.$$

THEOREM 2.3. *The variance* $\sigma^2(U)$ *of* U *satistics* (2.1) *satisfy the inequalities*

(2.5)
$$\frac{m^2}{n_1}\zeta_{10} + \frac{m^2}{n_2}\zeta_{01} + \frac{m^4}{n_1n_2}(\zeta_{11} - \zeta_{10} - \zeta_{01}) \leqq \sigma^2(U)$$

$$\leqq \frac{m}{n_1}\zeta_{m0} + \frac{m}{n_2}\zeta_{0m} + \frac{m^2}{n_1n_2}(\zeta_{mm} - \zeta_{m0} - \zeta_{0m}).$$

Further the quantity γ_m^2 *given in* (2.2) *satisfies the inequalities*

(2.6)
$$0 \leqq \gamma_m^2 \leqq \frac{m}{\rho}\zeta_{m0} + \frac{m}{1-\rho}\zeta_{0m}.$$

For proving Theorem 2.2 and 2.3 we shall require the following lemma.

LEMMA 2.1. *If*

(2.7)
$$\delta_{\alpha\beta} = \sum_{r=0}^{\alpha}\sum_{s=0}^{\beta}(-1)^{\alpha+\beta-r-s}\binom{\alpha}{r}\binom{\beta}{s}\zeta_{rs} \qquad \text{with } \delta_{00} = \zeta_{00} = 0,$$

we have

(2.8)
$$\zeta_{\alpha\beta} = \sum_{r=0}^{\alpha}\sum_{s=0}^{\beta}\binom{\alpha}{r}\binom{\beta}{s}\delta_{rs}$$

and

(2.9)
$$\delta_{\alpha\beta} \geqq 0.$$

PROOF. (2.8) follows (2.7) by induction. For proving (2.9) we shall first show that $\zeta_{\alpha\beta} \geqq 0$ for any pair (α, β), $\alpha, \beta \geqq 0$. Let

$$\eta = E\Phi(X_1, \cdots, X_m; Y_1, \cdots, Y_m),$$

$$\Phi_{\alpha\beta}(x_1, \cdots, x_\alpha; y_1, \cdots, y_\beta) = E\Phi(x_1, \cdots, x_\alpha, X_{\alpha+1}, \cdots, X_m; y_1, \cdots, y_\beta, Y_{\beta+1}, \cdots, Y_m),$$

$$\Psi_{\alpha\beta}(x_1, \cdots, x_\alpha; y_1, \cdots, y_\beta) = \Phi_{\alpha\beta}(x_1, \cdots, x_\alpha; y_1, \cdots, y_\beta) - \eta,$$

then we have

$$\zeta_{\alpha\beta} = E\{\Psi_{\alpha\beta}(X_1, \cdots, X_\alpha; Y_1, \cdots, Y_\beta)\}^2 \geqq 0. \tag{2.10}$$

Let

$$\eta_{00} = \eta^2, \qquad \eta_{\alpha\beta} = E\{\Phi_{\alpha\beta}(X_1, \cdots, X_\alpha; Y_1, \cdots, Y_\beta)\}^2.$$

Then

$$\zeta_{\alpha\beta} = \eta_{\alpha\beta} - \eta_{00},$$

and on substituting this in (2.7) we have

$$\delta_{\alpha\beta} = \sum_{r=0}^{\alpha}\sum_{s=0}^{\beta}(-1)^{\alpha+\beta-r-s}\binom{\alpha}{r}\binom{\beta}{s}\eta_{rs}.$$

From (2.7) and (2.10) we have $\delta_{10} = \zeta_{10} \geqq 0$ and $\delta_{01} = \zeta_{01} \geqq 0$. Thus (2.9) is true for $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$. Suppose that (2.9) is true for any pair (α, β). Then (2.9) will be shown to hold for $(\alpha+1, \beta)$. Let

$$\bar{\Phi}_{00}(x_1) = \Phi_{10}(x_1) - \eta,$$

$$\bar{\Phi}_{rs}(x_1, \cdots, x_{r+1}; y_1, \cdots, y_s) = \Phi_{r+1,s}(x_1, \cdots, x_{r+1}; y_1, \cdots, y_s) - \Phi_{r,s}(x_2, \cdots, x_{r+1}; y_1, \cdots, y_s).$$

For arbitrary x_1, let

$$\bar{\eta}_{rs}(x_1) = E\bar{\Phi}_{rs}^2(x_1, X_2, \cdots, X_{r+1}; Y_1, \cdots, Y_s), \qquad r = 0, 1, 2, \cdots, \alpha-1, \quad s = 0, 1, 2, \cdots, \beta.$$

Then by induction hypothesis

$$\bar{\delta}_{\alpha-1,\beta}(x_1) = \sum_{r=1}^{\alpha-1}\sum_{s=0}^{\beta}(-1)^{\alpha+\beta-1-r-s}\binom{\alpha-1}{r}\binom{\beta}{s}\bar{\eta}_{rs}(x_1) \geqq 0$$

for any x_1. Now $E\bar{\eta}_{rs}(X_1) = \eta_{r+1,s} - \eta_{rs}$, and hence

$$E[\bar{\delta}_{\alpha-1,\beta}(X_1)] = \sum_{r=0}^{\alpha-1}\sum_{s=1}^{\beta}(-1)^{\alpha+\beta-r-s-1}\binom{\alpha-1}{r}\binom{\beta}{s}(\eta_{r+1,s} - \eta_{rs})$$

$$= \sum_{r=0}^{\alpha}\sum_{s=0}^{\beta}(-1)^{\alpha+\beta-r-s}\binom{\alpha}{r}\binom{\beta}{s}\eta_{rs} = \delta_{\alpha\beta}.$$

Thus we have $\delta_{\alpha+1,\beta} \geqq 0$. Similarly it follows that $\delta_{\alpha,\beta+1} \geqq 0$. Thus the proof of the lemma is complete.

PROOF of Theorem 2.2.

(i) By (2.8) we have for $\alpha_1 < \alpha_2$

$$\alpha_1\zeta_{\alpha_2 0} - \alpha_{\alpha_2}\zeta_{\alpha_1 0} = \alpha_1\sum_{r=1}^{\alpha_2}\binom{\alpha_2}{r}\delta_{r0} - \alpha_2\sum_{r=1}^{\alpha_1}\binom{\alpha_1}{r}\delta_{r0}$$

$$= \sum_{r=1}^{\alpha_1}\left[\alpha_1\binom{\alpha_2}{r} - \alpha_2\binom{\alpha_1}{r}\right]\delta_{r0} + \alpha_1\sum_{r=\alpha_1+1}^{\alpha_2}\binom{\alpha_2}{r}\delta_{r0}.$$

From (2.9), and since $\alpha_1\binom{\alpha_2}{r}-\alpha_2\binom{\alpha_1}{r}\geqq 0$ if $1\leqq r\leqq\alpha_1<\alpha_2$, it follows $\alpha_1\zeta_{\alpha_2 0}\geqq\alpha_2\zeta_{\alpha_1 0}$, and similarly we obtain $\beta_1\zeta_{0\beta_2}\geqq\beta_2\zeta_{0,\beta_1}$. Thus we have right hand side inequalities of (2.3). The left hand side inequalities are obtained from (2.10).

(ii) By (2.8) it follows

$$\zeta_{\alpha\beta}-(\zeta_{\alpha 0}+\zeta_{0\beta})=\sum_{r=1}^{\alpha}\sum_{s=1}^{\beta}\binom{\alpha}{r}\binom{\beta}{s}\delta_{rs}.$$

Thus from (2.9) we have left hand side inequalities of (2.4). Moreover we have for $\alpha_1<\alpha_2$ and $\beta_1<\beta_2$

$$\begin{aligned}&\alpha_1\beta_1(\zeta_{\alpha_2\beta_2}-\zeta_{\alpha_2 0}-\zeta_{0\beta_2})-\alpha_2\beta_2(\zeta_{\alpha_1\beta_1}-\zeta_{\alpha_1 0}-\zeta_{0\beta_1})\\&=\sum_{r=1}^{\alpha_1}\sum_{s=1}^{\beta_1}\Big[\alpha_1\beta_1\binom{\alpha_2}{r}\binom{\beta_2}{s}-\alpha_2\beta_2\binom{\alpha_1}{r}\binom{\beta_1}{s}\Big]\delta_{rs}\\&\quad+\sum_{r=1}^{\alpha_2}\sum_{s=\beta_1+1}^{\beta_2}\alpha_1\beta_1\binom{\alpha_2}{r}\binom{\beta_2}{s}\delta_{rs}+\sum_{r=\alpha_1+1}^{\alpha_2}\sum_{s=1}^{\beta_1}\alpha_1\beta_1\binom{\alpha_2}{r}\binom{\beta_2}{s}\delta_{rs}.\end{aligned}$$

From (2.9), and since

$$\alpha_1\beta_1\binom{\alpha_2}{r}\binom{\beta_2}{s}-\alpha_2\beta_2\binom{\alpha_1}{r}\binom{\beta_1}{s}\geqq 0\qquad\text{if } 1\leqq r<\alpha_1<\alpha_2,\ 1\leqq s\leqq\beta_1<\beta_2,$$

then if follows

$$\alpha_1\beta_1(\zeta_{\alpha_2\beta_2}-\zeta_{\alpha_2 0}-\zeta_{0\beta_2})\geqq\alpha_2\beta_2(\zeta_{\alpha_1\beta_1}-\zeta_{\alpha_1 0}-\zeta_{0\beta_1}).$$

Thus the proof of the theorem 2.2 is complete.

PROOF of Theorem 2.3. The variance $\sigma^2(U)$ of U statistics (2.1) is given as follows.

$$\sigma^2(U)=\frac{1}{\binom{n_1}{m}\binom{n_2}{m}}\sum_{r=0}^{m}\sum_{s=0}^{m}\binom{m}{r}\binom{m}{s}\binom{n_1-m}{m-r}\binom{n_2-m}{m-s}\zeta_{rs}. \tag{2.11}$$

From (2.3) and (2.4) we have

$$\begin{aligned}&\alpha\zeta_{10}+\beta\zeta_{01}+\alpha\beta(\zeta_{11}-\zeta_{10}-\zeta_{01})\leqq\zeta_{\alpha\beta}\\&\qquad\leqq\frac{\alpha}{m}\zeta_{m0}+\frac{\beta}{m}\zeta_{0m}+\frac{\alpha\beta}{m^2}(\zeta_{mm}-\zeta_{m0}-\zeta_{0m}).\end{aligned}$$

Substituting these inequalities to (2.11), and using the identities

$$\sum_{k=1}^{m}\binom{m-1}{k-1}\binom{n_i-m}{m-k}=\binom{n_i-1}{k-1},\qquad i=1,2,$$

$$\sum_{k=1}^{m}\binom{m}{k}\binom{n_i-m}{m-k}=\binom{n_i}{m}-\binom{n_i-m}{m},\qquad i=1,2,$$

we obtain (2.5). The last part of the theorem follows immediately from (2.2) and by multiplying N to both sides of (2.5) and letting $N\to\infty$. Thus the proof of the theorem 2.3 is complete.

3. Estimator for a shift parameter

We shall consider in this section the robustness of the estimator for a shift parameter.

Let $x_1, x_2, \cdots, x_{n_1}$ and $y_1, y_2, \cdots, y_{n_2}$ be independent random samples with c. d. f. $F^{(1)}(t)=F(t)$ and $F^{(2)}(t)=F(t-\Delta)$ respectively, and let $h(x_1, x_2, \cdots, x_m)$ be any m variate symmetric real valued function such that

$$h(ax_1+b, \cdots, ax_m+b)=ah(x_1, \cdots, x_m)+b \tag{3.1}$$

for any real a and b.

We shall restrict the U statistics (2.1) as follows.

$$\Phi(x_1, \cdots, x_m; y_1, \cdots, y_m)=\varphi(h(x_1, \cdots, x_m); h(y_1, \cdots, y_m)) \tag{3.2}$$

where

$$\varphi(u, v)=\begin{cases} 1 & \text{if } u<v, \\ 0 & \text{otherwise}. \end{cases}$$

U statistics thus restricted are seen to be "distribution-free" for $m=1$, but "non-distribution-free" for $m\geqq 2$ and satisfy conditions (A) and (B) in section 1 and the estimator $\hat{\Delta}_m(x, y)$ of Δ obtained from (1.1) using this U statistics as $T(x, y)$ is easily given by the median of $h(y_{j_1}, \cdots, y_{j_m})-h(x_{i_1}, \cdots, x_{i_m})$, $\binom{n_1}{m}\binom{n_2}{m}$ in number, where $y_{j_1}, y_{j_2}, \cdots, y_{j_m}$, $1\leqq j_1<j_2<\cdots<j_m\leqq m$ and $x_{i_1}, x_{i_2}, \cdots, x_{i_m}, 1\leqq i_1<i_2<\cdots<i_m\leqq m$ are drawn from $x_1, x_2, \cdots, x_{n_1}$, and $y_1, y_2, \cdots, y_{n_2}$, respectively. We shall denote the estimator by

$$\hat{\Delta}_m(x, y)=\text{med}\,[h(y_{j_1}, y_{j_2}, \cdots, y_{j_m})-h(x_{i_1}, x_{i_2}, \cdots, x_{i_m})]\,.$$

Example 3.1. Suppose that $h(x_1, x_2, \cdots, x_m)=\dfrac{1}{m}\sum_{i=1}^{m} x_i$, then we have

$$\hat{\Delta}_{m,1}(x, y)=\text{med}\,[(y_{j_1}+\cdots+y_{j_m})-(x_{i_1}+\cdots+x_{i_m})]/m\,,$$

which is an extension of well known Hodges and Lehmann estimator.

Example 3.2. Suppose that $h(x_1, x_2, \cdots, x_m)=\text{med}\,(x_1, \cdots, x_m)$, then we have

$$\hat{\Delta}_{m,2}(x, y)=\text{med}\,[\text{med}\,(y_{j_1}, \cdots, y_{j_m})-\text{med}\,(x_{i_1}, \cdots, x_{i_m})]\,.$$

Science Hodges-Lehmann [3] has been investigated the characteristics of $\hat{\Delta}$ given in (1.1) from the general point of view, we shall not state it for $\hat{\Delta}_m(x, y)$, except for its asymptotic normality. To obtain limiting distribution of $\hat{\Delta}_m(x, y)$ we need the following lemma due to Hodges-Lehmann [3].

LEMMA 3.1. *For any real number a the estimate $\hat{\Delta}_m(x, y)$ satisfy the inequalities*

$$P[U(X, Y-a)<1/2]\leqq P[\hat{\Delta}_m(X, Y)\leqq a]\leqq P[U(X, Y-a)\leqq 1/2]\,.$$

Let the c. d. f. of the statistics $S=h(Y_1, Y_2, \cdots, Y_m)-h(X_1, X_2, \cdots, X_m)$ be $G_m(s)$ and its p. d. f. be $g_m(s)$, where X_i and Y_j, $i, j=1. 2, \cdots, m$ are supposed to be independent and identically distributed with c. d. f. F.

THEOREM 3.1. *Suppose that $g_m(0)\neq 0$. Theh $\sqrt{N}(\hat{\Delta}_m(x, y)-\Delta)$ has a limiting normal* *distribution whose mean is* 0 *and variance is* $\gamma_m^2/g_m^2(0)$, *where* γ_m^2 *is given in* (2.2) *with* X_i, X_j'; Y_k, Y_l' *being independent and identically distributed as* $F(t)$.

PROOF. From the lemma 3.1, and since $\hat{\Delta}_m(x, y+c)=\hat{\Delta}_m(x, y)+c$, it follows for any a

$$\lim_{N\to\infty} P_\Delta[N^{\frac{1}{2}}(\hat{\Delta}_m(X, Y)-\Delta)\leqq a]=\lim_{N\to\infty} P_0[U(X, Y-aN^{-\frac{1}{2}})\leqq \frac{1}{2}]$$

$$=\lim_{N\to\infty} P_0\{N^{\frac{1}{2}}[U(X, Y-aN^{-\frac{1}{2}})-E_0U(X, Y-aN^{-\frac{1}{2}})]$$

$$\leqq N^{\frac{1}{2}}[\frac{1}{2}-E_0U(X, Y-aN^{-\frac{1}{2}})]\},$$

where P_0 and E_0 mean the probability and expectation under $\Delta=0$. From Theorem 2.1 (iii) it follows under P_0 that $N^{\frac{1}{2}}[U(X, Y-aN^{-\frac{1}{2}})-E_0(X, Y-aN^{-\frac{1}{2}})]$ is distributed asymptotically according as a normal distribution with mean 0 and variance γ_m^2, where γ_m^2 is given in (2.2) with X_i, X_j'; Y_k, Y_l' being independent and identically distributed as $F(t)$. Further we have as $N\to\infty$

$$N^{\frac{1}{2}}[\frac{1}{2}-E_0U(X, Y-aN^{-\frac{1}{2}})]=N^{\frac{1}{2}}[G_m(aN^{-\frac{1}{2}})-G_m(0)]\to ag_m(0).$$

Thus the proof of the theorem 3.1 is complete.

The asymptotic efficiency of $\hat{\Delta}_m(x, y)$ relative to the classical estimator $\bar{y}-\bar{x}$, denoted by $A.R.E.(\hat{\Delta}_m(x, y)|\bar{y}-\bar{x})$, in the sence of reciprocal ratio of asymptotic variance is given as follows.

$$A.R.E.(\hat{\Delta}_m(x, y)|\bar{y}-\bar{x})=\frac{\sigma_f^2 g_m^2(0)}{\rho(1-\rho)\gamma_m^2}, \tag{3.3}$$

where σ_f^2 means the variance of the statistic. Assume that the variance of $T=h(X_1, \cdots, X_m)$ with X_i, $i=1, 2, \cdots, m$ being independent and identically distributed as $F(x)$ exists and denote it by σ_m^2. Then we have

THEOREM 3.2. *For any continuous c.d.f.,* $F(x)$ *which satisfy* $E(X)^2<\infty$ *and* $g_m(0)\neq 0$ *we have*

$$A.R.E.(\hat{\Delta}_m(x, y)|\bar{y}-\bar{x})\geqq 0.864\,\sigma_f^2/m\sigma_m^2. \tag{3.4}$$

PROOF. We shall denote by $q_m(x)$ the p.d.f. of the statistics $T=h(X_1, \cdots, X_m)$. Then we have $g_m(0)=\int q_m^2(x)dx$. Moreover it has been shown by Hodges-Lehmann [2] that $12\,\sigma_m^2\left(\int q_m^2(x)dx\right)^2\geqq 0.864$, thus it follows

$$\sigma_f^2 g_m^2(0)\geqq 0.864\,\sigma_f^2/12\,\sigma_m^2. \tag{3.5}$$

Further from (3.2) and definition of $\zeta_{d_1d_2}$ we have

$$\zeta_{m0}=\zeta_{0m}=\mathrm{Cov}\{\varphi[h(X_1, \cdots, X_m); h(Y_1, \cdots, Y_m)], \varphi[h(X_1, \cdots, X_m; h(Y_1', \cdots, Y_m')]\}$$

$$=P[h(X_1, \cdots, X_m)<h(Y_1, \cdots, Y_m), h(X_1, \cdots, X_m)<h(Y_1', \cdots, Y_m')]$$

$$-\{P[h(X_1, \cdots, X_m)<h(Y_1, \cdots, Y_m)]\}^2=1/3-1/4=1/12.$$

Thus we have from (2.6) that

$$\gamma_m^2 \leqq \frac{m}{\rho}\zeta_{m0}+\frac{m}{1-\rho}\zeta_{0m}=\frac{m}{12\,\rho(1-\rho)}\,. \tag{3.6}$$

Thus substituting (3.5) and (3.6) to (3.3), we have the desired result. We shall in the sequel consider the estimator $\hat{\Delta}_{m1}$ given in the example 3.1.

COROLLARY. *Under the conditions of Theorem* 3.2 *we have*

$$A.\,R.\,E.\,(\hat{\Delta}_{m,1}|\bar{y}-\bar{x}) \geqq 0.864 \qquad \textit{for any } m \geqq 1\,.$$

PROOF. Since $h(x_1, \cdots, x_m)=(x_1+\cdots+x_m)/m$, we have $\sigma_m^2=\sigma_f^2/m$. Thus from (3.4) we have the result.

THEOREM 3.3. *Let the underlying distribution* $F(x)$ *be a standard normal c. d. f.. Then we have for* $m=1, 2, \cdots$

$$3/\pi \leqq A.\,R.\,E.\,(\hat{\Delta}_{m,1}|\bar{y}-\bar{x}) < A.\,R.\,E.\,(\hat{\Delta}_{m+1,1}|\bar{y}-\bar{x}) < 1\,.$$

PROOF. When $F(x)$ is a standard normal c. d. f. we have $\sigma_f^2=1$, $g_m(0)=(m/4\pi)^{\frac{1}{2}}$ and $\zeta_{10}=\zeta_{01}=1/4-(1/2\pi)\cos^{-1}(1/2m)$. Thus it follows from (2.2) and (3.3) that

$$A.\,R.\,E.\,(\hat{\Delta}_{m,1}|\bar{y}-\bar{x})=1/12m(\pi-2\cos^{-1}(1/2\,m))\,. \tag{3.7}$$

Now we shall consider the function

$$a(x)=x[\pi-2\cos^{-1}(1/12\,x)] \qquad \text{for } x\geqq 1\,.$$

Since $d^2a(x)/d^2x=[4x^2(x^2-1/4)(1-1/4x^2)^{\frac{1}{2}}]^{-1}>0$ for $x\geqq 1$, then $da(x)/dx$ is a strictly increasing function of $x\geqq 1$. But since $da(x)/dx\rightarrow 0$ as $x\rightarrow\infty$, then $da(x)/dx<0$ for $x\geqq 1$. Thus $a(x)$ is a strictly decreasing function of $x\geqq 1$. Thus we have $m[\pi-2\cos^{-1}(1/2\,\mathrm{m})>(\mathrm{m}+1)[\pi-2\cos^{-1}(1/2(m+1)]$ for any integer $m\geqq 1$. Thus it follows from (3.7) that $A.\,R.\,E.\,(\hat{\Delta}_{m,1}|\bar{y}-\bar{x})<A.\,R.\,E.\,(\hat{\Delta}_{m+1,1}|\bar{y}-\bar{x})$. Further it is clear that $A.\,R.\,E.\,(\hat{\Delta}_{1,1}|\bar{y}-\bar{x})=3/\pi$ and $A.\,R.\,E.\,(\hat{\Delta}_{m,1}|\bar{y}-\bar{x})\rightarrow 1$ as $m\rightarrow\infty$. Thus the proof of the theorem is complete.

Now we shall give the numerical values of $A.\,R.\,E.\,(\hat{\Delta}_{m,1}|\bar{y}-\bar{x})$ for some specific distributions.

(1) Suppose that F be a standard normal c. d. f., then numerical values of (3.7) is obtained either by using the table [8], or by the six point interporation, and given in Table 3.1.

(2) Suppose that F be a chi-square c. d. f., $F(x)=\int_0^x \chi_\nu^2(t)dt$, $\chi_\nu^2(t)=(2\Gamma(\nu/2))^{-1}(t/2)^{\nu/2-1}e^{-t/2}$, then we have

$$\sigma_f^2=2\nu\,, \tag{3.8}$$

$$g_m(0)=m\Gamma(2a-1)/2^{2a}\Gamma^2(2a)\,, \tag{3.9}$$

$$\gamma_m^2=m^2(\lambda-1/4)/\rho(1-\rho)\,, \tag{3.10}$$

where $a=m\nu/2$

$$\lambda=A\sum_{i=0}^{a-1}\sum_{j=0}^{i}\sum_{k=0}^{a-1}\sum_{l=0}^{k}B(i,j,k,l)\;C(i,j,k,l)$$

$$A=[(a-\nu/2)]^{-2}[\Gamma(\nu/2]^{-1}\,,$$

$$B(i,j,k,l)=2^{i+k+\nu+2-4a}/3^{j+l+\nu/2}\,,$$

$$C(i, j, k, l)=\frac{\Gamma(2a-1-i-\nu/2)\Gamma(2a-1-k-\nu/2)\Gamma(j+l+\nu/2)}{\Gamma(a-i)\cdot\Gamma(j+1)\cdot\Gamma(a-k)\cdot\Gamma(l+1)}.$$

Substituting (3.8), (3.9) and (3.10) to (3.3), we can obtain numerical values of $A.R.E.(\hat{\Delta}_{m,1}|\bar{y}-\bar{x})$, which are calculated by using an electric computer, NEAC 2200-500, of Osaka University and given in Table 3.1.

(3) Suppose that F be a contaminated normal c. d. f., ; $F(x)=pN(x)+qN(x-\theta)$, $q=1-p$, where $N(x)$ is a standard normal c. d. f., then we have

(3.11) $$A.R.E.(\hat{\Delta}_{1,1}|\bar{y}-\bar{x})=(3/\pi)(1+pq\theta^2)[1+2pq(e^{-\theta^2/4}-1)]^2,$$

(3.12) $$A.R.E.(\hat{\Delta}_{2,1}|\bar{y}-\bar{x})=\sigma_f^2 g_2^2(0)/\rho(1-\rho)\gamma_2^2,$$

where

$$\sigma_f^2=1+pq\theta^2,$$

$$g_2(0)=[p^4+4p^2q^4+q^4+4(p^3q+pq^3)e^{-\theta^2/8}+2p^2q^2e^{-\theta^2/2}]/\sqrt{2\pi},$$

$$\gamma_2^2=(4\lambda^*-1)/\rho(1-\rho),$$

$$\lambda^*=\sum_{i=-2}^{2}\sum_{j=1}^{2}A_{i,j}F(i,j,1/4)$$

$$A_{-2,-2}=qB_3^2,\quad A_{-2,-1}=2qB_1B_3,\quad A_{-2,0}=2qB_2B_3,$$

$$A_{-2,1}=2qB_3B_4,\quad A_{-2,2}=0,\quad A_{-1,-1}=pB_3^2+qB_1^2,$$

$$A_{-1,0}=2pB_1B_3+2qB_1B_2,\quad A_{-1,1}=2pB_2B_3+2qB_1B_4,\quad A_{-1,2}=2pB_3B_4,$$

$$A_{0,0}=pB_1^2+qB_2^2,\quad A_{0,1}=2pB_1B_2+2qB_2B_4,\quad A_{0,2}=2pB_1B_4,$$

$$A_{1,1}=pB_2^2+qB_4^2,\quad A_{1,2}=2pB_2B_4,\quad A_{2,2}=pB_4^2,$$

$$B_1=p(p^2+2q^2),\quad B_2=q(q^2+2p^2),\quad B_3=p^2q,\quad B_4=pq^2,$$

$$F(a,b,r)=\int_a^\infty\int_b^\infty\int\frac{1}{2\pi\sqrt{1-r^2}}\exp\left[-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)\right]dxdy.$$

Numerical values of (3.11) and (3.12) are given in Table 3.2, which were compued by means of NEAC 2200-500 of Osaka University either by using the table [8] utilising the identities

$$F(-a,-b,r)=F(a,b,r)+(\alpha(a)+\alpha(b))/2,$$

$$F(-a,b,r)=-F(a,b,-r)+(1-\alpha(b))/2,$$

where $\alpha(x)=\int_{-x}^{x}1/\sqrt{2\pi}\exp(-x^2/2)dx$, or by the six point interpolation.

Note that the U statistic corresponding to $\hat{\Delta}_{m,1}$, $m=1$, i. e. Wilcoxon two-sample statistic, is " distribution-free ", but the estimators $\hat{\Delta}_{m,1}$, $m\geqq 2$ are based on " non-distribution-free " U statistics.

Corollary, Theorem 3.3 and the inspection of the Table 3.1 and 3.2 lead one to the following conclusions ;

(i) $A.R.E.(\hat{\Delta}_{m,1}|\bar{y}-\bar{x})\geqq 0.864$ for all $m=1,2,\cdots$ and for any underlying distributions which satisfy the regularity conditions.

(ii) $A.R.E.(\hat{\Delta}_{m,1}|\bar{y}-\bar{x})$ increases monotonically from 0.955 to 1 as m increase from 1 to infinitive for a normal underlying distribution.

(iii) $\hat{\Delta}_{m,1}$ $m \geqq 2$ are more robust than $\hat{\Delta}_{m,1}$ $m=1$ for distributions nearest to the normal such as contaminated normal distributions $F(x)=pN(x)+qN(x-\theta)$ $0 \leqq q \leqq 0.2$ $0 \leqq \theta \leqq 1$.

(iv) For distributions not nearest to the normal distribution such as contaminated normal distributions $F(x)=pN(x)+qN(x-\theta)$, $0 \leqq q \leqq 0.2$, $\theta > 1$ and chi-square distributions, the robustness of $\hat{\Delta}_{m,1}$ $m \geqq 2$ decreases as m increases, though we have for latter distribution $A.R.E.(\hat{\Delta}_{m,1} | \bar{y}-\bar{x}) > 1$ for all $m=1, 2, \cdots$.

TABLE 3.1. Values of $A.R.E.$ $(\hat{\Delta}_{m,1} | \bar{y}-\bar{x})$ for the normal and the chi-square distributions with $\nu=1, 2, 3, 4, 5$ degree of freedom.

F \ m		1	2	4	6
Normal		0.9549	0.9894	0.9974	0.9989
χ^2_ν	$\nu=1$	∞	3.2341	1.6337	1.3739
	2	3.0000	1.5882	1.2543	1.1651
	3	1.8238	1.3311	1.1586	1.1060
	4	1.5000	1.2281	1.1151	1.0780
	5	1.3510	1.1728	1.0900	1.0615

TABLE 3.2. Values of $A.R.E.$ $(\hat{\Delta}_{m,1} | \bar{y}-\bar{x})$, $m=1, 2$, for the contaminated normal distribution $F(x)=pN(x)+qN(x-\theta)$, $q=1-p$.

m	q \ θ	0.4	0.8	1.0	2.0	3.0
1	0.0	0.9549	0.9549	0.9549	0.9549	0.9549
	0.1	0.9551	0.9569	0.9596	1.0220	1.2166
	0.2	0.9550	0.9555	0.9565	0.9966	1.1869
2	0.0	0.9894	0.9894	0.9894	0.9894	0.9894
	0.1	0.9895	0.9905	0.9920	1.0217	1.1173
	0.2	0.9894	0.9896	0.9900	0.9996	1.0522

4. Estimator for a dispersion parameter.

Let X_i, $i=1, 2, \cdots, n_1$ and Y_j, $j=1, 2, \cdots, n_2$ be independent random variables with c.d.f. $F^{(1)}(t)=F(t/\sigma_1)$ and $F^{(2)}(t)=F(t/\sigma_2)$ respectively. We shall consider in this section the robustness of the estimator for a dispersion parameter $c=\sigma_2/\sigma_1$.

Let $h(x_1, x_2, \cdots, x_m)$ be any m-variate symmetric real valued function such that

$$h(ax_1, \cdots, ax_m)=ah(x_1, \cdots, x_m) \quad \text{for any real } a>0\,. \tag{4.1}$$

We shall consider the U-statistics which satisfy (2.1), (3.2) and (4.1). These U statistics are seen to be "distribution-free" for $m=1$ but "non-distribution-free" for $m \geqq 2$. Following the same manner as (1.1) we can obtain an estimator $\hat{c}$ of c as follows.

(4.2) $$\hat{c}=(c^*+c^{**})/2$$

where

$$c^*=\sup\{a,\ U(x, y/a)>\mu\}$$

$$c^{**}=\inf\{a\ ;\ U(x, y/a)<\mu\}\ .$$

Since $\mu=E_0U(X, Y)=1/2$, it follows from (4.2) that the concrete form of the estimator $\hat{c}_m(x, y)$ of c derived from above U statistics is easily obtained as a median of $h(y_{j_1}, y_{j_2}, \cdots, y_{j_m})/h(x_{i_1}, x_{i_2}, \cdots, x_{i_m})$, $\binom{n_1}{m}\binom{n_2}{m}$ in number, where $x_{i_1}, x_{i_2}, \cdots, x_{i_m}$ $l\leqq i_1<i_2<\cdots<i_m\leqq n_1$ and $y_{j_1}, y_{j_2}, \cdots, y_{j_m}$ $l\leqq j_1<j_2<\cdots<j_m\leqq n_2$ are drawn from $x_1, x_2, \cdots, x_{n_1}$ and $y_1, y_2, \cdots, y_{n_2}$ respectively. We shall denote it by

$$\hat{c}_m(x, y)=\text{med}\left[h(y_{j_1}, y_{j_2}, \cdots, y_{jm})/h(x_{i_1}, x_{i_2}, \cdots, x_{im})\right].$$

Example 4.1. Suppose that $h(x_1, x_2, \cdots, x_m)=(x_1^2+x_2^2+\cdots+x_m^2)^{\frac{1}{2}}$, then it satisfies (4.1) and the estimator turns out to be

$$\hat{c}_{m,1}(x, y)=\text{med}\left[(y_{j_1}^2+y_{j_2}^2+\cdots+y_{jm}^2)/(x_{i_1}^2+x_{i_2}^2+\cdots+x_{im}^2)\right]^{\frac{1}{2}}.$$

Example 4.2. Suppose that $h(x_1, x_2, \cdots, x_m)=\text{med}\,(|x_1|, |x_2|, \cdots, |x_m|)$. Then it satisfies (4.1) and the corresponding estimator is

$$\hat{c}_{m,2}(x, y)=\text{med}\,\{\text{med}\,(|y_{j_1}|, \cdots, |y_{jm}|)/\text{med}\,(|x_{i_1}|, \cdots, |x_{im}|)\}\ .$$

Let the p. d. f. of the satistic $T=h(X_1, X_2, \cdots, X_m)/h(Y_1, Y_2, \cdots, Y_m)$ with X_i, Y_j $i, j=1, 2, \cdots, m$ being independent and identically distributed as $F(x)$ be $p_m(x)$. Then we have

THEOREM 4.1. *Suppose that $p_m(1)\neq 0$. Then $N^{\frac{1}{2}}(\hat{c}_m(x, y)-c)$ has a limiting normal distribution with mean* 0 *and variance $c^2\gamma_m^2/p_m^2(1)$, where γ_m^2 is given in Theorem* 3.1.

PROOF. We shall omit the proof, since it is obtained by an analogous manner as that of Theorem 3.1.

Let the maximum likelihood estimator of c with $F(x)$ standard normal c. d. f. be $\hat{c}_{M.L.}$ Then we have

$$\hat{c}_{M.L.}=(n_1\sum_{j=1}^{n_2}y_j^2/n_2\sum_{i=1}^{n_1}x_i^2)^{1/2},$$

which is distributed according as asymptotically normal distribution with mean 0 and variance

$$\kappa^2=c^2(\mu_4-\mu_2)/4\rho(1-\rho)\mu_2^2, \qquad \text{where } \mu_i=\int x^i f(x)dx\,.$$

Thus the asymptotic efficiency of $\hat{c}_m(x, y)$ relative to $\hat{c}_{M.L.}$ is given as follows.

(4.3) $$A.\,R.\,E.\,(\hat{c}_m(x, y)\,|\,\hat{c}_{M.L.})=\kappa^2p_m^2(1)/c^2\gamma_m^2\,.$$

Now we shall give the numerical values of (4.3) for the estimator $\hat{c}_{m,1}$ given in the example 4.1 for specific distributions.

Let the c. d. f. of random variable X be F and suppose that X^2 is distributed as chi-square distribution with ν degree of freedom so F turns to a standered normal

c. d. f. when $\nu=1$. Then we have

$$\kappa^2 = c^2(2\nu\rho(1-\rho))^{-1},$$

$$p_m(1) = \Gamma(m\nu)[2^{m\nu-1}\Gamma^2(m\nu/2)]^{-1},$$

and γ_m^2 is given in (3.10). Substituting above equalities to (4.3) we obtain numercal values of $A.R.E.(\hat{c}_{m,1}|c_{M.L.})$, which are given in Table 4.1.

Note that the U statistic corresponding to the estimator $\hat{c}_{m,1}$ $m=1$, which is seen to be equivalent to Sukhatme's T statistic [12] for symmetric distribution functions, is "distribution-free", while the estimators $\hat{c}_{m,1}$ $m=2$ are based on "non-distribution-free" U statistics.

Inspection of Table 4.1 leads one to the following conclusions;

(i) Asymptotic relative efficiency of $\hat{c}_{m,1}$ $m=1$ relative to the maximum likelihood estimator $\hat{c}_{M,L}$ is only 0.61 for the underlying normal distribution, thus the efficiency loss is large for one to use ordinary nonparametric estimator $\hat{c}_{m,1}$ $m=1$ instead of $\hat{c}_{M,L}$ when the underlying distribution has large possibility to be a normal.

(ii) $A.R.E.(\hat{c}_{m,1}|\hat{c}_{M,L})$, $m=2$ is quite high for the underlying normal distribution, for example $A.R.E.(\hat{c}_{m,1}|\hat{c}_{M,L})=0.95$, $m=6$, thus we can improve $A.R.E.(\hat{c}_m|\hat{c}_{M,L})$ by extending our consideration to the estimators $\hat{c}_{m,1}$ $m\geqq 2$ based on "non-distribution-free" statistics.

(iii) $A.R.E.(\hat{c}_{m,1}|\hat{c}_{M,L})$ for chi-square distribution with ν degree of freedom increases as ν increases from 1 to 5 for each m, which shows the robustness of $\hat{c}_{m,1}$ for each m.

Of course there exists family of distributions which make $A.R.E.(\hat{c}_{m,1}|\hat{c}_{M,L})$ larger than 1, for example $A.R.E.(\hat{c}_{1,1}|\hat{c}_{M,L})>1$ for distributions which have the p. d. f.

$$f(x) = \frac{1}{\Gamma((3+\alpha)/2)} \exp\left(-\left|\frac{x}{2}\right|^{\alpha/(1+\alpha)}\right), \qquad \alpha \geqq 1.5$$

and $A.R.E.(\hat{c}_{1,1}|\hat{c}_{M,L})\to\infty$ as $\alpha\to\infty$.

TABLE 4.1. Values of $A.R.E.$ $(\hat{c}_{m,1}|\hat{c}_{M.L})$, $m=1, 2, 4, 6$, when X^2 is distributed as chi-square distribution with ν degree of freedom.

χ^2_ν \ m	1	2	4	6
$\nu=1$	0. 6079	0. 8080	0. 9190	0. 9541
2	0. 7500	0. 8933	0. 9603	0. 9790
3	0. 8106	0. 9244	0. 9736	0. 9865
4	0. 8438	0. 9403	0. 9800	0. 9900
5	0. 8488	0. 9500	0. 9837	0. 9919

5. Acknowledgements.

The author wishes to express his hearty thanks to Professor A. Kudô of Kyushu University for his helpful guidance and kind encouragements during the course of the entire work. He also wishes to express his hearty thanks to Professor R. Tamura of Kyusnu Institute of Design for his criticisms and variable discussions on the topics.

References

[1] FINE, TERRENCE (1966). *On the Hodges and Lehmann shift estimator in the two sample problem.* Ann. Math. Statist. **37**, 1814–1818.

[2] HODGES, J.L. Jr. and LEHMANN, E.L. (1956). *The efficiency of some nonparametric competitors of the t-test.* Ann. Math. Statist., **27**, 324–355.

[3] HODGES, J.L. Jr. and LEHMANN, E.L. (1963). *Estimates of localion based on rank test.* Ann. Math. Statist. **34**, 599–611.

[4] HOEFFDING, W. (1948). *A class of statistics with asymptotically normal distribution.* Ann. Math. Statist. **19**, 293–325.

[5] HØYLAND, A. (1964). *Robustness of the Hodges—Lehmann estimates for shift.* Ann. Math. Statist. **36**, 174–197.

[6] LEHMANN, E.L. (1963). *Robust estimation in analysis of variance.* Ann. Math. Statist. **34**, 957–966.

[7] LEHMANN, E.L. (1964). *Asymptotically nonparametric inference in linear models with one observation per cell.* Ann. Math. Statist. **35**, 726–734.

[8] NATIONAL BUREAU OF STANDARDS (1959). *Table of the Bivariate Normal Distribution Function and Related Functions.* National Bureau of Standards Applied Mathmatics Series 50.

[9] RAMACHANDRAMURTY, P.V. (1966). *On some nonparametric estimates for shift in the Behrens-Fisher situation.* Ann. Math. Statist. **37**, 593–610.

[10] SPJØTVOLL (1968). *A note on robust estimation in analysis of variance.* Ann. Math. Statist. **39**, 1486–1492.

[11] SUGIURA, N. (1965). *Multisample and multivariate nonparametric teses based on U statistics and their asymptotic efficiencies.* Osaka J. Math. **2**, 385–426.

[12] SUKHATME, B.V. (1957). *On certain two-sample nonparametric tests for variance.* Ann. Math. Statist. **28**, 188–194.