# UNBIASED SLIPPAGE TESTS-(II)

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# UNBIASED SLIPPAGE TESTS—(II)

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## § 1. Introduction.

The main purpose of this paper is to note that for the validity of Theorem 3 in [4], the assumption A.3 can be replaced by (iv) of Lemma there. By looking over the proof of Theorem 3, it is clear that (iv) of Lemma plays the crucial importance, and A.3 is needed merely to guarantee (iv) of Lemma [4].

For the sake of clearness we restate Theorems 3 and 4, Lemma [4] and some useful definitions [3]. Consider a slippage problem of an exponential family, involing parameters $i$, $\Delta$ and $\theta_i$, where $i=0, 1, \cdots, m$, $\Delta$ is a real-valued parameter with $\Delta=0$ when $i=0$, and the remaining parameter $\theta$ is an unspecified nuisance parameter. Let $X$ be a random vector distributed according to one of the following densities

$$\text{(1)} \qquad \begin{cases} p_0(x\,;0,\theta)=C(0,\theta)\exp[\theta U(x)], \\ p_i(x\,;\Delta,\theta)=C(\Delta,\theta)\exp[\Delta T_i(x)+\theta U(x)] \qquad (i=1,\cdots,m) \end{cases}$$

with respect to a $\sigma$-additive measure $\mu$, where $-\infty<\Delta<\infty$. We assume there is a finite group of measurable transformations, $G=\{g\}$, on the sample space $\mathfrak{X}$ of $X$ such that $G$ is homomorphics to either the permutation group, $\Pi=\{\pi_g\}$, on $(1, 2, \cdots, m)$ or its transitive subgroup, and in addition, $G$ is homomorphic to a group, $H=\{h_g\}$, on $\Delta=\{\Delta\}$ consisting of only two elements $+1$ and $-1$ under multiplication. Furthermore the following assumptions are needed.

A.1 $$\Delta T_i(x)=(h_g\Delta)T_{\pi_g i}(gx),$$

A.2 $$U(x)=U(gx),$$

A.3 $$C(\Delta,\theta)=C(h_g\Delta,\theta)$$

for all $g\in G$ and $h_g$ is corresponding to each $g$.

A.4 $$U \text{ is complete for } p_0(x\,;0,\theta).$$

**Lemma 4.** *For all* $g\in G$

(i) *the distribution of* $X$ *satisfies*

$$P_i^X(A\,;\Delta,\theta)=P_{\pi_g i}^X(gA\,;h_g\Delta,\theta),$$

(ii) *the marginal distribution of* $U$ *satisfies*

$$P_i^U(B\,;\Delta,\theta)=P_{\pi_g i}^U(B\,;h_g\Delta,\theta),$$

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(iii) *the conditional distribution of X given* $U=u$ *satisfies*

$$P_i^{X/u}(A/u\,;\,\Delta,\theta)=P_{\pi_g i}^{X/u}(gA/u\,;\,h_g\Delta,\theta)\,,$$

(iv) *the density of the conditional distribution of X given* $U=u$ *is*

$$dP_i^{X/u}(x/u\,;\,\Delta)=C(\Delta\,;\,u)\exp[\Delta T_i(x)]d\nu(x\,;\,u)\,,$$

*where*

$$\text{(a)}\qquad \nu(A\,;\,u)=\nu(gA\,;\,u)$$

$$\text{(b)}\qquad C(\Delta\,;\,u)=C(h_g\Delta\,;\,u)$$

*for all* $g\in G$.

We considered a problem of testing the null hypothesis $H_0$ against the alternative $H_i$ $(i=1,\cdots,m)$:

$$\text{(2)}\qquad \begin{aligned} &H_0: X \text{ has the density } p_0(x\,;\,0,\theta)\\ &H_i: X \text{ has the density } p_i(x\,;\,\Delta,\theta)\quad (i=1,\cdots,m)\,,\end{aligned}$$

where $\Delta\neq 0$, but unspecified.

**Definitions** [3].

1. *A decision function* $\varphi(x)=(\varphi_0(x),\varphi_1(x),\cdots,\varphi_m(x))$ *is of size* $\alpha$ *for* $H_0$ *if* $E_0\varphi_0(X)\geqq 1-\alpha$.

*It is of exact size* $\alpha$ *for* $H_0$ *if the equality holds.*

2. *A decision function* $\varphi(x)=(\varphi_0(x),\varphi_1(x),\cdots,\varphi_m(x))$ *is unbiased of size* $\alpha$ *if*

$$E_0\varphi_1(X)=\cdots=E_0\varphi_m(X)\,,\qquad \sum_{i=1}^{m}E_0\varphi_i(X)\leqq\alpha\quad \textit{and}\quad \sum_{i=1}^{m}E_i\varphi_i(X)\geqq\alpha\,.$$

3. *A decision function* $\varphi(x)=(\varphi_0(x),\varphi_1(x),\cdots,\varphi_m(x))$ *is symmetric in power for* $H_i$ *if*

$$\text{(3)}\qquad E_1\varphi_1(X)=\cdots=E_m\varphi_m(X)\,.$$

*The common value of* (3) *is called the power of* $\varphi$ *for* $H_i$ $(i=1,\cdots,m)$. $\varphi(x)$ *is called the most powerful symmetric of size* $\alpha$ *(MPS of size* $\alpha$*) for* $H_i$, *if it maximizes each term of* (3) *subject to size* $\alpha$ *and* (3 .

We have the following theorems.

**Theorem 3** [4]. *A decision function defined by*

$$\varphi_0(x,u)=1,\quad \xi(u),\quad 0\qquad \textit{when}\qquad \max_i|T_i(x)|<,\quad =,\quad >k(u)$$

$$\varphi_j(x,u)=\frac{1-\varphi_0(x,u)}{J(x,u)},\quad 0\qquad \textit{when}\qquad |T_j(x)|=,\quad <\max_i T_i(x)$$

$(i=1,2,\cdots,m)$ *and, the functions* $\xi(u)$ *and* $k(u)$ *are determined by*

$$\int_{\mathfrak{X}}\varphi_0(x,u)p_0(x/u\,;\,0)d\nu(x\,;\,u)=1-\alpha\qquad \textit{for all}\quad u$$

*consititutes an unbiased uniformly MPS of size* $\alpha$ *test for testing* (2) *when the probability density is given by* (1) *and when the assumptions above hold.*

**Theorem 4** [4]. *Assuming that the conditions in Theorem 3 hold. Suppose that there exists statistics* $V_i=H(U,T_i)$ $(i=1,\cdots,m)$ *which are independent of* $U$ *when*

$\varDelta=0$, *and* $\max_i |v_i|=|v_j|$ *if and only if* $\max_i |t_i|=|t_j|$ *for each fixed* $u$. *Let* $V=(V_1, \cdots, V_m)$ *then an unbiased uniformly MPS of size-*$\alpha$ *test for testing* (2) *is given by*

$$\varphi_0(v)=1,\ \xi,\ 0 \qquad \textit{when} \quad \max_i |v_i| <,\ =,\ > k$$

$$\varphi_j(v)=\frac{1-\varphi_0(v)}{J(v)},\ 0 \quad \textit{when} \quad |v_j|=,\ < \max_j |v_j|,$$

*where* $\xi$ *and* $k$ *are no longer dependent on* $u$ *and depending on the size condition.*

## § 2. Main Theorem and Examples.

For the sake of completeness, we shall state above remark as follows. Let $X$ be a random vector distributed according to a density (1), and hence the conditional density of $X$ given $U=u$ with respect to a probability measure $\nu(A/u)$, $A\in\mathfrak{A}$ ($\mathfrak{A}$ is a $\sigma$-field of subsets of the sample space $\mathfrak{X}$), again belongs to the exponential family distributions of the form, say,

$$p_i^{X/u}(x/u\,;\,\varDelta)=C(\varDelta/u)\exp[\varDelta S_i(x)]\,.$$

All the assumptions in section 1 remain validly with the exception of A.3. For the sake of clearness we revise from A.1 till A.4 as follows.

A′1 $$S_i(x)=(h_g\varDelta)S_{\pi_g i}(gx)$$

A′2 $$U(x)=U(gx)$$

A′3 $$\nu(A/u)=\nu(gA/u)\,,\qquad C(\varDelta/u)=C(h_g\varDelta/u)$$

for all $g\in G$ and $h_g$ is corresponding to each $g$.

A′4 $$U \text{ is complete for } p_0(x\,;\,0,\theta)\,.$$

**Main Theorem.** *Consider a vector-valued decision function* $\varphi(x, u)$ *given by:*

$$\varphi_0(x, u)=1,\ \xi(u),\ 0 \qquad \textit{when} \quad \max_i |S_i(x)| <,\ =,\ > k(u)$$

$$\varphi_j(x, u)=\frac{1-\varphi_0(x, u)}{J(x, u)},\ 0 \quad \textit{when} \quad |S_j(x)|=,\ < \max_j |S_i(x)|,$$

*and*

$$\int_{\mathfrak{X}}\varphi_0(x, u)p_0(x\,|\,u\,;\,0)d\nu(x\,|\,u)=1-\alpha \qquad \textit{for all } u\,,$$

*constitutes an unbiased uniformly MPS of size* $\alpha$ *test for testing* (2) *when the probability desity is given by* (1) *and the assumptions above hold.* $J(x, u)$ *is the number of times that* $|S_i(x)|$ *attains the maximum.*

For the sake of demonstration, we consider the following examples.

**Example 1.** Let $X=(X_1, \cdots, X_m)$ be a random vector variable distributed according to the normal distribution with a vector mean $(\theta_1, \cdots, \theta_m)$ and a variance convariance matrix $\sigma^2 I$, where $\sigma^2$ is known. Without loss of generality we put $\sigma^2=1$.

We shall consider the problem of testing the hypothesis $H_0$ against the alternative $H_i$ $(i=1, \cdots, m)$:

$$H_0:\ \theta_1 = \cdots = \theta_m = \theta$$
$$H_i:\ \theta_1 = \cdots = \theta_{i-1} = \theta_i - \Delta = \theta_{i+1} = \cdots = \theta_m = \theta$$

where $\Delta \neq 0$, $-\infty < \theta < \infty$ and both are unspecified.

The probability density of $X$ with respect to the Lebesgue measure $\mu$ is

$$p_0(x\,;\,0,\,\theta) = C(0,\,\theta)\exp[m\theta\bar{x}]$$
$$p_j(x\,;\,\Delta,\,\theta) = C(\Delta,\,\theta)\exp[\Delta x_j + m\theta\bar{x}] \qquad (j=1,\cdots,m)$$

where

$$C(\Delta,\,\theta) = (1/\sqrt{2\pi})^m \exp[-\Delta^2/2 - \Delta\theta - m\theta^2/2]$$

and

$$\bar{x} = \frac{1}{m}\sum_{i=1}^m x_i\,.$$

We have therefore

$$T_j(x) = x_j \quad \text{and} \quad U(x) = \sum_{i=1}^m x_i\,.$$

Clearly A′4 is satisfied. There exists a measurable transformation group $G=\{g\}$ of order $2(m\,!)$, namely the permutation of $(x_1,\cdots,x_m)$ combined with the transformation of $(x_1,\cdots,x_m)$ to $(2\bar{x}-x_1,\cdots,2\bar{x}-x_m)$, satisfies the assumption A′2.

In order to establish A′1 and A′3, we have to work out the conditional distribution of $X_1,\cdots,X_m$ given $\bar{X}=\bar{x}$. It is of course, a singular normal distribution and has no density with respect to the ordinary Lebesgue measure over $m$ dimensional Euclidean space, but it has a density with respect to the Lebesgue measure $\nu$ over $m-1$ dimensional hypersurface where $\sum x_i$ is fixed at $m\bar{x}$. For the sake of convenience, putting $y_i = x_i - \bar{x}$ $(i=1,\cdots,m-1)$, we start with the joint density of $(y_1, y_2,\cdots,y_{m-1},\bar{x})$, which is justified by the one to one correspondence to $(x_1,\cdots,x_m)$. The transformation group now reduced to the group $G'=\{g'\}$, which consists of $2(m\,!)$ transformations from $(y_1,\cdots,y_{m-1},\bar{x})$ to $(y_{\pi_g^{-1}1},\cdots,y_{\pi_g^{-1}(m-1)},\bar{x})$ combined with the one to $(-y_1,\cdots,-y_{m-1},\bar{x})$, where $y_m$ here is to be understood as $-(y_1+\cdots+y_{m-1})$. The conditional density of $(y_1,\cdots,y_{m-1})$ is given by

$$f_j(y_1,\cdots,y_{m-1}\,|\,\bar{x}) = C\exp\Big[-\frac{(m-1)\Delta^2}{2m}\Big]\exp[\Delta y_j] \qquad (j=0,1,\cdots,m) \tag{4}$$

with respect to the $m-1$ dimensional Lebesgue measure $\nu(A\,|\,\bar{x})$ over the hypersurface. It can be seen that it satisfies A′1 and A′3 for the group $G'$. This becomes more clear by expressing (4) in terms of $(x_1,\cdots,x_m)$ which is gtven by

$$p_j^{X|\bar{x}}(x\,|\,\bar{x}\,;\,\Delta) = C(\Delta)\exp[\Delta(x_i-\bar{x})] \qquad (j=0,1,\cdots,m)$$

with respect to $\nu$, where

$$C(\Delta) = (1/\sqrt{2\pi})^m\cdot\exp\Big[-\frac{(m-1)\Delta^2}{2m}\Big].$$

By Main Theorem an unbiased uniformly MPS of size $\alpha$ test $\varphi$ exists and it is of the form:

$$
(5) \qquad
\begin{aligned}
&\varphi_0(x, \bar{x}) = 1,\ \xi(\bar{x}),\ 0 && \text{when} \quad \max_i |x_i - \bar{x}| <,\ =,\ > k(\bar{x}) \\
&\varphi_j(x, \bar{x}) = \frac{1 - \varphi_0(x, \bar{x})}{J(x, \bar{x})},\ 0 && \text{when} \quad |x_j - \bar{x}| =,\ < \max_i |x_i - \bar{x}|
\end{aligned}
$$

where $k(\bar{x})$ and $\xi(\bar{x})$ are determined by

$$E_0[\varphi_0(X, \bar{X}) | \bar{x}\,;\, 0] = 1 - \alpha \qquad \text{for all } \bar{x}\,.$$

Let

$$V_i \equiv H(U, S_i) = x_i - \bar{x} \qquad (i = 1, \cdots, m)\,,$$

we see that $V_i$ are independent of $U$ when $\Delta = 0$ and $\max |v_i| = |v_j|$ if and only if $\max |x_i - \bar{x}| = |x_j - \bar{x}|$ for each fixed $u \equiv \bar{x}$. By Theorem 4 [cf. Introduction] the form (5) of $\varphi$ can be written by

$$
\begin{aligned}
&\varphi_0(v) = 1,\ \xi,\ 0 && \text{when} \quad \max |v_i| <,\ =,\ > k \\
&\varphi_j(v) = \frac{1 - \varphi_0(v)}{J(v)},\ 0 && \text{when} \quad |v_j| =,\ < \max_i |v_i|
\end{aligned}
$$

where $\xi$ and $k$ are no longer dependent on $\bar{x}$ and depending only on the size condition.

**Example 2.** The case where $\sigma^2$ is unknown.

Suppose we have $m$ normal populations $N(\theta_i, \sigma^2)$ $(i = 1, \cdots, m)$, where $\sigma^2$ is common and unknown. We consider testing of the following $m+1$ hypotheses:

$$H_0:\ \theta_1 = \theta_2 = \cdots = \theta_m = \theta$$

against

$$H_i:\ \theta_1 = \theta_2 = \cdots = \theta_{i-1} = \theta_i - \Delta = \theta_{i+1} = \cdots = \theta_m = \theta \qquad (i = 1, \cdots, m)$$

from the sample of size $n$, $(\theta_i\,;\, x_{i1}, \cdots, x_{in})$ $(i = 1, \cdots, m)$, from each one of the $m$ populations, where $\Delta \neq 0$, $-\infty < \theta < \infty$, both are unspecified.

The joint distribution density of $X = (X_{11}, \cdots, X_{1n}, X_{21}, \cdots, X_{2n}, \cdots, X_{m1}, \cdots, X_{mn})$ under $H_0$ and $H_i$ $(i = 1, \cdots, m)$ are given respectively by

$$
\begin{aligned}
p_0(x\,;\, 0, \theta, \sigma^2) &= C \exp\Big[-\frac{mn\theta^2}{2\sigma^2}\Big] \exp\Big[-\frac{1}{2\sigma^2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2 + \frac{\theta}{\sigma^2}\, mn\bar{\bar{x}}\Big] \\
p_i(x\,;\, \Delta, \theta, \sigma^2) &= C \exp\Big[-\frac{mn\theta^2}{2\sigma^2} - \frac{n\Delta}{2\sigma^2}(2\theta + \Delta)\Big] \\
&\quad \exp\Big[-\frac{1}{2\sigma^2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2 + \frac{\theta}{\sigma^2}\, mn\bar{\bar{x}} + \frac{\Delta}{\sigma^2}\, n\bar{x}_i\Big] \qquad (i = 1, 2, \cdots, m)
\end{aligned}
$$

with respect to the Lebesgue measure $\mu$, where

$$C = (1/2\pi\sigma^2)^{\frac{mn}{2}}\,, \qquad n\bar{x}_i = \sum_{j=1}^{m} x_{ij} \qquad (i = 1, \cdots, m)$$

and

$$mn\bar{\bar{x}} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\,.$$

We may put

$$T_i(x) = \bar{x}_i$$

and

$$U(x)=\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}^2,\ \bar{\bar{x}}\Big).$$

Thus the minimal sufficient statistic for the parameter space for all the hypotheses is $(\bar{x}_1, \cdots, \bar{x}_m, \sum\sum x_{ij}^2)$, whereas it is $(\bar{x}_i, \bar{\bar{x}}, \sum\sum x_{ij}^2)$ for $H_0$ and $H_i$, and $(\bar{\bar{x}}, \sum\sum x_{ij}^2)$ for $H_0$. It is clear that A′4 is satisfied.

There is again a measurable transformation group $G=\{g\}$ of the sample space $\mathfrak{X}$, consisting of $2(m!)$ elements, of the same form as in Example 1 where $x_i$ is conceived to be $(x_{i1}, \cdots, x_{in})$ $(i=1, \cdots, m)$ and $\bar{x}$ to be $\bar{\bar{x}}$ in the present example, satisfies the assumption A′2.

In order to establish A′1 and A′3, we have to work out the conditional density given $\bar{x}$ and $\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}-\bar{x})^2$. This conditional distribution is again a singular distribution. To do this we make transformation of variables, $y=Ax$ where $y'=(y_{11}, \cdots, y_{1n}, y_{21}, \cdots, y_{2n}, \cdots, y_{m1}, \cdots, y_{mn})$, $x'=(x_{11}, \cdots, x_{1n}, x_{21}, \cdots, x_{2n}, \cdots, x_{m1}, \cdots, x_{mn})$ and $A$ is an orthogonal matrix with $y_{11}=\sqrt{mn}\,\bar{\bar{x}}$ and

$$mns^2=\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}-\bar{\bar{x}})^2=\sum_{i=1}^{m}\sum_{j=1}^{n}y_{ij}^2-y_{11}^2.$$

The joint density turns out to be

$$p_i(y;\Delta,\theta,\sigma^2)=C\exp\Big[-\frac{mn\theta^2}{2\sigma^2}-\frac{n\Delta}{2\sigma^2}(2\theta+\Delta)\Big]$$
$$\exp\Big[-\frac{mn}{2\sigma^2}(s^2+\bar{\bar{x}})+\frac{\theta}{\sigma^2}mn\bar{\bar{x}}+\frac{n\Delta}{\sigma^2}\bar{x}_i(y)\Big]$$

where $\bar{x}_i(y)$ is an expression of $\bar{x}_i$ as a linear function of $y$. This is also equal to

$$p_i(\bar{\bar{x}}, y_{12}, \cdots, y_{1n}, y_{21}, \cdots, y_{2n}, \cdots, y_{m1}, \cdots, y_{mn}: \Delta, \theta, \sigma^2)$$
$$=\Big(\frac{1}{\pi}\Big)^{\frac{mn-1}{2}}\Big(\frac{1}{2\sigma^2}\Big)^{\frac{mn-1}{2}}\exp\Big[-\frac{n\Delta^2(n-1)}{2\sigma^2 m}\Big]$$
$$\cdot\exp\Big[-\frac{mns^2}{2\sigma^2}+\frac{n\Delta}{\sigma^2}w(\bar{\bar{x}}, y_{12}, \cdots, y_{mn})\Big]f(\bar{\bar{x}})\qquad(i=0, 1, \cdots, m)$$

where $f(\bar{\bar{x}})$ is the probability density of $\bar{\bar{x}}$ and $w(\bar{\bar{x}}, y_{12}, \cdots, y_{mn})$ is again an expression of $\bar{x}_i-\bar{\bar{x}}$ in terms of $y$.

We again make the transformation

$$y_{ij}=\sqrt{mn}\,sz_{ij}\qquad(i=1, \cdots, m;\ j=1, \cdots, n;\ (i, j)\neq(1, 1)).$$

Then it satisfies

$$\sum{}' x_{ij}^2=1$$

where the summation is over $(i=1, \cdots, m, j=1, \cdots, n, (i, j)\neq(1, 1))$. The Jacobian of this transformation is

$$J=\pm\frac{(mn)^{\frac{mn-1}{2}}s^{mn-2}}{\sqrt{1-z_{12}^2-\cdots-z_{m(n-1)}^2}}.$$

The joint probability of $(z_{12}, z_{13}, \cdots, z_{mn}, \bar{\bar{x}}, mns^2)$ is therefore given by

$$p_i(z_{12}, \cdots, z_{m(n-1)}, \bar{\bar{x}}, mns^2; \Delta, \theta, \sigma^2)$$
$$=(mn)^{\frac{mn-3}{2}} s^{mn-3}\Big(\frac{1}{\pi}\Big)^{\frac{mn-1}{2}}\Big(\frac{1}{2\sigma^2}\Big)^{\frac{mn-1}{2}} \exp\Big[-\frac{n\Delta^2(m-1)}{2\sigma^2 m}\Big]$$
$$\cdot \exp\Big[-\frac{mns^2}{2\sigma^2}+\frac{n\Delta}{\sigma^2} r(z_{12}, \cdots, z_{m(n-1)}, \bar{\bar{x}}, mns^2)\Big]f(\bar{\bar{x}})$$
$$(i=0, 1, \cdots, m)$$

with respect to the Lebesgue measure, and $r(z_{12}, \cdots, z_{m(n-1)}, \bar{\bar{x}}, mns^2)$ can be expressed as $\bar{x}_i-\bar{\bar{x}}$ in terms of $x$.

It is known that $\frac{mns^2}{\sigma^2}$ is distributed according to the non-central chi-square distribution with degrees of freedom $mn-1$ and parameter of noncentrality $\lambda$ is $n\Delta^2(m-1)/2m\sigma^2$. Therefore the probability density of $mnS^2$ is given by

$$\frac{1}{2\sigma^2} e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\frac{\Big(\frac{mns^2}{2\sigma^2}\Big)^{\frac{mn-3}{2}+k}}{\Gamma\Big(\frac{mn-1}{2}+k\Big)} \exp\Big[-\frac{mns^2}{2\sigma^2}\Big].$$

Hence the conditional probability density of $(z_{12}, \cdots, z_{m(n-1)})$ given $\bar{\bar{x}}$ and $mns^2$ is given by

(6) $$p_i(z_{12}, \cdots, z_{m(n-1)}|\bar{\bar{x}}, mns^2; \Delta/\sigma^2)$$
$$=(\sqrt{\pi})^{1-mn}\Big[\sum_{k=0}^{\infty}\frac{\Big(\frac{\Delta}{\sigma^2}\Big)^{2k}(mns^2)^k}{k!\,2^{2k}\Gamma\Big(\frac{mn-1}{2}+k\Big)}\Big(\frac{mn-n}{m}\Big)^k\Big]^{-1}$$
$$\cdot \exp\Big[\frac{n\Delta}{\sigma^2} r(z_{12}, \cdots, z_{m(n-1)}, \bar{\bar{x}}, mns^2)\Big] \qquad (i=0, 1, \cdots, m)$$

with respect to $m-2$ dimensional Lebesgue measure $\nu(A|\bar{\bar{x}}, mns^2)$. We express (6) in terms of $x$ which is given by

$$p_i^{X|u}(x|\bar{\bar{x}}, mns^2; \Delta/\sigma^2)$$
$$=(\sqrt{\pi})^{1-mn}\Big[\sum_{k=0}^{\infty}\frac{\Big(\frac{\Delta}{\sigma^2}\Big)^{2k}(mns^2)^k}{k!\,2^{2k}\Gamma\Big(\frac{mn-1}{2}+k\Big)}\Big(\frac{mn-n}{m}\Big)^k\Big]^{-1}$$
$$\cdot \exp\Big[\frac{n\Delta}{\sigma^2}(\bar{x}_i-\bar{\bar{x}})\Big] \qquad (i=0, 1, \cdots, m)$$

with respect to $\nu$.

It can be seen that it satisfies A′1 and A′3. By Main Theorem, an unbiased uniformly MPS of size $\alpha$ test $\varphi$ exists and it is of the form:

(7)
$$\varphi_0(x, \bar{\bar{x}}, mns^2)=1,\ \xi(\bar{\bar{x}}, mns^2),\ 0 \qquad \text{when } \max_i|\bar{x}_i-\bar{\bar{x}}|<,\ =,\ >k(\bar{\bar{x}}, mns^2)$$
$$\varphi_j(x, \bar{\bar{x}}, mns^2)=\frac{1-\varphi_0(x, \bar{\bar{x}}, mns^2)}{J(x, \bar{\bar{x}}, mns^2)},\ 0 \qquad \text{when } |\bar{x}_j-\bar{\bar{x}}|=,\ <\max_i|\bar{x}_i-\bar{\bar{x}}|$$

where $J(x, \bar{x}, mns^2)$ is the number of times that $\max_i |\bar{x}_i - \bar{\bar{x}}|$ is attained. $k(\bar{\bar{x}}, mns^2)$ and $\xi(\bar{\bar{x}}, mns^2)$ are determined by

$$E_0[\varphi_0(X, \bar{\bar{X}}, mnS^2) | \bar{\bar{x}}, mns^2 ;\ 0] = 1-\alpha \qquad \text{for all } \bar{\bar{x}} \text{ and } mns^2 .$$

Let

$$V_i \equiv H(U, S_i) = (\bar{x}_i - \bar{\bar{x}}) / \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - \bar{\bar{x}})^2} \qquad (i = 1, 2, \cdots, m) .$$

We see that $V_i$ are independent of $U$ when $\Delta/\sigma^2 = 0$ and $\max |v_i| = |v_j|$ if and only if $\max |\bar{x}_i - \bar{\bar{x}}| = |\bar{x}_j - \bar{\bar{x}}|$ for each fixed $\bar{\bar{x}}$ and $mns^2$. By Theorem 4 [cf. Introduction] the form (6) of $\varphi$ can be written by

$$\varphi_0(v) = 1,\ \xi,\ 0 \qquad \text{when} \quad \max_i |v_i| <,\ =,\ > k$$

$$\varphi_j(v) = \frac{1-\varphi_0(v)}{J(v)},\ 0 \qquad \text{when} \quad |v_j| =,\ < \max_i |v_i|$$

where $\xi$ and $k$ are no longer dependent upon $\bar{\bar{x}}$ and $\sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - \bar{\bar{x}})^2$.

**Remark.** If we examine Example 2, it can be easily seen that $n$ need not be more than 1, it can be equal to 1. The procedure of this case was proposed in Kudô [2], as a test of an outlier, whose table has been computed, in part, by Grubbs [1].

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