ON DECOMPOSABILITY OF PROBABILITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

Nagai, Takeaki
Faculty of Science, Kagoshima University

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ON DECOMPOSABILITY OF PROBABILITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

By

Takeaki NAGAI*

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§ 1. Summary.

In this paper we consider probability measures on a complete separable metric space T (or on a topological subspace T of a complete separable metric space T^*) with which a compact metric group G of homeomorphisms acting on T is associated.

Let μ be an arbitrary probability measure on T invariant under every $g \in G$. Then it is shown that for an arbitrary small $\varepsilon > 0$, there always exists a compact set $A_0 \subset T$ and a probability measure μ_0 on T invariant under every $g \in G$ such that

- (i) $G \cdot A_0 = A_0$.
- (ii) The support of μ_0 is a closed subset of A_0 .
- (iii) For every Borel set A , $|\mu(A) - \mu_0(A)| < \varepsilon$.
- (iv) Let V be a Borel set in A_0 such that $G \cdot V = A_0$ and $Gv_1 \cap Gv_2 = \emptyset$ if $v_1 \neq v_2$, ($v_1, v_2 \in V$), whose existence is shown in lemma 3. Then the probability measure μ_0 is decomposed into a direct product measure of the normalized Haar measure of G and a probability measure on V .

§ 2. Decomposition of G -invariant probability measures.

Let T be a complete separable metric space or a topological subspace of a complete separable metric space T^* . Let G be a compact metric group of homeomorphisms acting on T such that $G \cdot T = T$ and the mapping $(g, t) \rightarrow gt$ from $G \times T$ into T is continuous.

For any metric space X we shall denote by $\mathfrak{B}_X$ the σ -field of Borel subsets of X .

Let μ be a probability measure defined on $\mathfrak{B}_T$ and invariant under every $g \in G$, that is,

- (1) $\mu(T) = 1$.
- (2) For each $g \in G$, $\mu(gA) = \mu(A)$, $A \in \mathfrak{B}_T$.

Lemma 1. *Let g and t be any elements of G and T respectively but be fixed. Let W be an arbitrary neighbourhood of $gt \in T$. Then we can always find a neighbourhood U of t such that $W = gU$.*

* Faculty of Science, Kagoshima University, Kagoshima.

Proof. $g: t \rightarrow gt$ is a homeomorphism from T onto itself and $g^{-1}(W) = g^{-1} \cdot W$ is an inverse image of the open set W under the mapping g . Hence, $g^{-1} \cdot W$ is open.

Since $t = g^{-1} \cdot gt \in g^{-1}W$, $g^{-1}W$ is a neighbourhood of t . Thus, by writing $U = g^{-1} \cdot W$, we see that $W = gU$.

Lemma 2. *Let A be the support of the probability measure μ .
Then $G \cdot A = A$.*

Proof. A is a set of all elements $t \in T$ such that for any neighbourhood U of t , $\mu(U) > 0$. (See [2], page 28).

Let W be an arbitrary neighbourhood of $gt \in T$, where g and t are any fixed elements of G and A respectively.

Then from lemma 1, we can choose a neighbourhood U of t such that $W = gU$. Thus, we see that $\mu(W) = \mu(gU) = \mu(U) > 0$. This implies that $gt \in A$ for all $g \in G$ and hence $G \cdot A \subset A$.

Since $A \subset G \cdot A$, we have proved that $G \cdot A = A$.

Lemma 3. *Let $\Gamma \subset T$ be a compact set invariant under every $g \in G$.
Then there is a Borel set $V \subset T$ such that*

$$(3) \quad \Gamma = G \cdot V,$$

$$(4) \quad Gv_1 \cap Gv_2 = \phi \quad \text{if} \quad v_1 \neq v_2, \quad (v_1, v_2 \in V).$$

Proof. For any two points $t_1, t_2 \in \Gamma$, we shall say that $t_1 \sim t_2$ if there exists $g \in G$ such that $gt_1 = t_2$. " $\sim$ " is an equivalence relation. Let M be the space of all such equivalence classes. Let $[t]$ denote the equivalence class containing t . Then, since the mapping $t \rightarrow [t]$ from Γ into M is continuous under the quotient topology and is onto, we see that M is a continuous image of the compact set Γ under this mapping and hence M is a compact metric space.

Thus, by a theorem of Federer and Morse (see [2], page 23, Theorem 4.2), it follows that there exists a Borel set $V \subset \Gamma$ satisfying (3) and (4).

Now, we have the following:

Lemma 4. *Let $\Gamma \subset T$ be the set considered in lemma 3 and suppose for any $g \in G$, $g \neq e$, there is no fixed point in Γ .*

Then the mapping

$$(5) \quad \xi: (g, v) \rightarrow gv$$

is a Borel isomorphism between $G \times V$ and Γ .

Proof. From lemma 3, we see that the mapping $\xi: G \times V \rightarrow \Gamma$ is onto. Since there is no fixed point in Γ for any $g \in G$, $g \neq e$, ξ is one-one. From our assumption, it is clear that the mapping ξ is continuous.

In particular, ξ is measurable. Hence, by a theorem of Kuratowski (See [2], page 21, theorem 3.9), ξ^{-1} is measurable.

Thus, we have proved lemma 4.

Now we have the following theorem.

Theorem 1. *Let the support A of μ be compact. Let $V \subset T$ be a Borel set such that $G \cdot V = A$ and $Gv_1 \cap Gv_2 = \phi$ if $v_1 \neq v_2$, $(v_1, v_2 \in V)$ and the mapping $\xi: (g, v) \rightarrow gv$ be a Borel isomorphism between $G \times V$ and A .*

Let C be a Borel set in T and A and B Borel subsets in G and V respectively such that

$$\xi(A \times B) = C \cap A \in \mathfrak{B}_A.$$

Then,

$$\mu(C) = \nu(A) \cdot \rho(B),$$

where ν is the normalized Haar measure on G and ρ a probability measure on $\mathfrak{B}_V$.

Proof. Let us notice that A is a compact set in T and hence $A \in \mathfrak{B}_T$. Therefore $\mathfrak{B}_A$ is the σ -field of subsets of A of the form $A \cap A$, $A \in \mathfrak{B}_T$, that is, $\mathfrak{B}_A = \{\tilde{A} = A \cap A \mid A \in \mathfrak{B}_T\}$.

Now let us define a set function $\tilde{\mu}$ on $\mathfrak{B}_A$ in such a way that:

For each $\tilde{A} = A \cap A \in \mathfrak{B}_A$, $A \in \mathfrak{B}_T$,

$$\tilde{\mu}(\tilde{A}) = \mu(A).$$

It is easy to see that $\tilde{\mu}$ is a probability measure on $\mathfrak{B}_A$. Indeed, for any $\tilde{A} \in \mathfrak{B}_A$, $0 \leq \tilde{\mu}(\tilde{A}) = \mu(A) \leq 1$ and $\tilde{\mu}(A) = \mu(A) = 1$.

Let $\{\tilde{A}_\nu = A_\nu \cap A, A_\nu \in \mathfrak{B}_T, \nu = 1, 2, \dots\}$ be a sequence of disjoint sets in $\mathfrak{B}_A$. However $\{A_\nu; \nu = 1, 2, \dots, A_\nu \in \mathfrak{B}_T\}$ are not necessarily disjoint.

Let us notice that $\bigcup_{\nu=1}^{\infty} A_\nu = \bigcup_{\nu=1}^{\infty} (A_\nu \cap A) \cup \bigcup_{\nu=1}^{\infty} (A_\nu \cap A^c)$, and $\bigcup_{\nu=1}^{\infty} (A_\nu \cap A)$ and $\bigcup_{\nu=1}^{\infty} (A_\nu \cap A^c)$ are disjoint. Since $\bigcup_{\nu=1}^{\infty} \tilde{A}_\nu = \bigcup_{\nu=1}^{\infty} A_\nu \cap A$, we have

$$\begin{aligned} \tilde{\mu}\left(\bigcup_{\nu=1}^{\infty} \tilde{A}_\nu\right) &= \mu\left(\bigcup_{\nu=1}^{\infty} A_\nu\right) \\ &= \mu\left(\bigcup_{\nu=1}^{\infty} (A_\nu \cap A)\right) + \mu\left(\bigcup_{\nu=1}^{\infty} (A_\nu \cap A^c)\right). \end{aligned}$$

From the fact that $\mu\left(\bigcup_{\nu=1}^{\infty} (A_\nu \cap A^c)\right) \leq \mu(A^c) = 0$, it follows that

$$\begin{aligned} \tilde{\mu}\left(\bigcup_{\nu=1}^{\infty} \tilde{A}_\nu\right) &= \mu\left(\bigcup_{\nu=1}^{\infty} (A_\nu \cap A)\right) \\ &= \sum_{\nu=1}^{\infty} \mu(A_\nu \cap A) \\ &= \sum_{\nu=1}^{\infty} \tilde{\mu}(\tilde{A}_\nu). \end{aligned}$$

Thus, we have proved that $\tilde{\mu}$ is a probability measure on $\mathfrak{B}_A$.

By μ^* let us denote the probability measure on $\mathfrak{B}_{G \times V}$ induced by the mapping ξ^{-1} from A onto $G \times V$, that is, μ^* is the probability measure such that for any $A \times B \in \mathfrak{B}_{G \times V}$, ($A \in \mathfrak{B}_G$, $B \in \mathfrak{B}_V$),

$$\mu^*(A \times B) = \tilde{\mu}(\xi(A \times B)).$$

Since both $G \times V$ and G are complete separable metric spaces, these are automatically separable standard Borel spaces (see [2], page 133).

Let us consider the mapping $\pi: (g, v) \rightarrow g$ from $G \times V$ onto G . Then it is clear that π is measurable, since for any $E \in \mathfrak{B}_G$, $\pi^{-1}(E) = E \times V \in \mathfrak{B}_{G \times V}$.

Thus, it follows that there exists a regular conditional probability distribution

of μ^* given π , which we shall denote by $\tilde{m}_g(A \times B)$, $g \in G$, $A \times B \in \mathfrak{B}_{G \times V}$. This satisfies the following conditions:

- (i) For each $g \in G$, $\tilde{m}_g(\cdot)$ is a probability measure on $\mathfrak{B}_{G \times V}$.
- (ii) For each $A \times B \in \mathfrak{B}_{G \times V}$, the mapping $g \rightarrow \tilde{m}_g(A \times B)$ is $\mathfrak{B}_G$ -measurable and
- (iii) $\mu^*(A \times B) = \int_G \tilde{m}_g(A \times B) d\nu(g)$,

where $\nu(E) = \mu^*(\pi^{-1}(E)) = \mu^*(E \times V)$, $E \in \mathfrak{B}_G$. (See [2], page 146).

It is obvious that ν is a probability measure on $\mathfrak{B}_G$.

In particular, we have for any $E \in \mathfrak{B}_G$,

$$(6) \quad \begin{aligned} \mu^*(A \times B \cap \pi^{-1}(E)) &= \mu^*((A \cap E) \times B) \\ &= \int_E \tilde{m}_g(A \times B) d\nu(g), \quad A \times B \in \mathfrak{B}_{G \times V}. \end{aligned}$$

Let us write

$$m_g(B) = \tilde{m}_g(G \times B), \quad B \in \mathfrak{B}_V.$$

Then we have by putting $A = G$ in (6),

$$(7) \quad \mu^*(E \times B) = \int_E m_g(B) d\nu(g),$$

for any $E \in \mathfrak{B}_G$ and any $B \in \mathfrak{B}_V$, where $m_g(\cdot)$ is a probability measure on $\mathfrak{B}_V$.

Let $A \in \mathfrak{B}_G$ and $B \in \mathfrak{B}_V$ and $C = \{\eta \in T \mid \eta = g't, g' \in A, t \in B\}$. Then for any $g \in G$,

$$\begin{aligned} \mu^*(gA \times B) &= \tilde{\mu}(\xi(gA \times B)) = \tilde{\mu}(gC) = \tilde{\mu}(C) \\ &= \tilde{\mu}(\xi(A \times B)) = \mu^*(A \times B). \end{aligned}$$

Hence it follows by putting particularly $B = V$ in (7) that

$$\nu(gA) = \mu^*(gA \times V) = \mu^*(A \times V) = \nu(A) \quad \text{for any } A \in \mathfrak{B}_G.$$

This implies that ν is the normalized Haar measure on G .

Thus, we may write

$$\mu^*(A \times B) = \int_A m_g(B) dg.$$

Now, for any $A \in \mathfrak{B}_G$, $B \in \mathfrak{B}_V$ and $h \in G$,

$$\begin{aligned} \mu^*(A \times B) &= \int_A m_g(B) dg \\ &= \mu^*(hA \times B) \\ &= \int_{hA} m_g(B) dg \\ &= \int_A m_{h^{-1}g}(B) dg. \end{aligned}$$

This implies that for any $B \in \mathfrak{B}_V$ and any $g, g' \in G$,

$$m_g(B) = m_{g'}(B),$$

in other words, $m_g(B)$ is a probability measure on $\mathfrak{B}_V$ independent of $g \in G$.

Let us write

$$\rho(B) = m_g(B), \quad B \in \mathfrak{B}_V.$$

Then we have for any $C \in \mathfrak{B}_T$ such that $\xi(A \times B) = C \cap A$, $A \times B \in \mathfrak{B}_{G \times V}$,

$$\begin{aligned} \mu(C) &= \tilde{\mu}(C \cap A) \\ &= \tilde{\mu}(\xi(A \times B)) \\ &= \mu^*(A \times B) \\ &= \int_A \rho(B) dg \\ &= \nu(A) \cdot \rho(B). \end{aligned}$$

Thus, we have proved theorem 1.

Now, let us consider the case where the support of the probability measure μ is not necessarily compact in T .

Lemma 5. *For any $\varepsilon > 0$, there exists a compact set A_0 in T such that*

$$(8) \quad G \cdot A_0 = A_0.$$

$$(9) \quad \mu(A_0) > 1 - \varepsilon/3.$$

Proof. Every probability measure on a complete separable metric space or on a topological subspace which is a Borel subset of such a metric space is tight (See [2], page 29). Hence, for any small $\varepsilon > 0$, there is a compact set $C \subset T$ such that

$$\mu(C) > 1 - \varepsilon/3.$$

From a theorem of Tychonoff, it follows that $G \times C$ is compact, since it is a topological product space of a compact metric group G and a compact metric space C . From our assumption, the mapping $\gamma: (g, t) \rightarrow gt$ from $G \times T$ into T is continuous and $G \cdot C = \gamma(G \times C)$ is a continuous image of a compact set $G \times C$. Hence $G \cdot C$ is compact.

Let us write $A_0 = G \cdot C$. Then A_0 is invariant under every $g \in G$ and $\mu(A_0) \geq \mu(C) > 1 - \varepsilon/3$.

Let us consider a probability measure μ_0 on $\mathfrak{B}_T$ defined by

$$(10) \quad \mu_0(A) = \frac{\mu(A \cap A_0)}{\mu(A_0)}, \quad A \in \mathfrak{B}_T.$$

Then, since $\mu_0(A_0) = 1$, the support of μ_0 is compact.

Since $G \cdot A_0 = A_0$, we see that for any $g \in G$ and $A \in \mathfrak{B}_T$, $gA \cap A_0 = g(A \cap A_0)$. Thus, it is clear that for any $g \in G$, $\mu_0(gA) = \mu_0(A)$, $A \in \mathfrak{B}_T$.

Now, we have the following theorem:

Theorem 2. *For an arbitrary small $\varepsilon > 0$, there exists a compact set $A_0 \subset T$ and a probability measure μ_0 on $\mathfrak{B}_T$ such that*

$$(11) \quad \text{The support of } \mu_0 \text{ is a closed subset of } A_0.$$

$$(12) \quad \text{For any } g \in G \text{ and } A \in \mathfrak{B}_T,$$

$$\mu_0(gA) = \mu_0(A),$$

and

$$(13) \quad \text{For any } A \in \mathfrak{B}_T,$$

$$|\mu(A) - \mu_0(A)| < \varepsilon.$$

Proof. We have already shown that the existence of the compact set A_0 and the

probability measure μ_0 satisfying (11) and (12).

Now, we shall prove (13).

Since for any $A \in \mathfrak{B}_T$,

$$\mu(A) = \mu(A \cap A_0) + \mu(A \cap A_0^c),$$

we have the following inequality :

$$\begin{aligned} |\mu(A) - \mu_0(A)| &= \left| \mu(A \cap A_0^c) + \mu(A \cap A_0) - \frac{\mu(A \cap A_0)}{\mu(A_0)} \right| \\ &\leq \mu(A \cap A_0^c) + \mu(A \cap A_0) \cdot (1 - \mu(A_0)) / \mu(A_0) \\ &= \mu(A_0^c) \cdot (1 + 1/\mu(A_0)). \end{aligned}$$

As long as ε is not greater than 1, $1 + 1/\mu(A_0)$ is less than 5/2.

Thus, we have $|\mu(A) - \mu_0(A)| < (5/6)\varepsilon < \varepsilon$, for any $A \in \mathfrak{B}_T$.

This completes the proof.

Thus, in generally, every probability measure on T invariant under every $g \in G$ can be approximated by a direct product probability measure of the normalized Haar measure on G and a probability measure on a Borel set V satisfying the conditions stated in lemma 3 as closely as possible.

Example 1. Let $T = R_2 = \{(\theta, r) | 0 \leq r < \infty, 0 \leq \theta < 2\pi\}$ and $G = SO(2)$. Let us consider a compact set $\Gamma = \{(\theta, r) | 0 \leq r \leq \alpha, 0 \leq \theta \leq 2\pi\}$ and a set $V = \{(0, r) | 0 \leq r \leq \alpha\}$, where α is a finite real number.

Then it is obvious that V satisfies the conditions (3) and (4) in lemma 3.

Let μ be a probability measure on $\mathfrak{B}_T$ and $\mu(\Gamma) = 1$. If for any $A \in \mathfrak{B}_T$, $\mu(A)$ is proportional to the area of the set A , then the probability measure μ is invariant under every $g \in SO(2)$ and it is written in the view of theorem 1 as follows:

$$\int_{A \times B} d\mu(\theta, r) = \int_A \frac{1}{2\pi} d\theta \cdot \int_B \frac{2}{\alpha^2} r dr,$$

where $A \in \mathfrak{B}_G$ and $B \in \mathfrak{B}_V$.

Example 2. Let $L_3 = \{(t, x, y, z) | t^2 - (x^2 + y^2 + z^2) = c^2\}$, where c is a real number and (t, x, y, z) 's are points in 4-dim. Euclidean space R_4 . Let G be a subgroup of the proper Lorentz group of the order (3, 1) such that its elements are of the following form:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \quad h \in SO(3),$$

where 0 in the above expression denotes 3×1 -zero vector and also 1×3 -zero vector.

Let μ be a probability measure on L_3 invariant under every $g \in G$. Let the support of the measure μ be such that

$$\Gamma = \{(t, x, y, z) \in L_3 | 0 \leq x^2 + y^2 + z^2 \leq \alpha^2\},$$

where α is a finite real number.

Let us consider a set

$$V = \{(\sqrt{c^2 + r^2}, r, 0, 0) | 0 \leq r \leq \alpha\} \subset \Gamma.$$

Then V satisfies the conditions (3) and (4) in lemma 3.

Suppose that for any $C \in \mathfrak{B}_T$, $\mu(C)$ is proportional to the area of C . Then it is written in the view of theorem 1 as follows:

$$\begin{aligned} \int_C d\mu(x, y, z) &= \int_{A_1 \times A_2 \times B} d\mu(r, \theta, \varphi) \\ &= \int_{A_1 \times A_2} \frac{1}{4\pi} \sin \theta d\theta d\varphi \cdot \int_B \frac{\beta r^2}{\sqrt{c^2 + r^2}} dr, \end{aligned}$$

where C is a set of all points (t, x, y, z) in Γ such that

$$x = r \cdot \sin \theta \cdot \cos \varphi,$$

$$y = r \cdot \sin \theta \cdot \sin \varphi,$$

$$z = r \cdot \cos \theta,$$

$$t = \sqrt{c^2 + r^2},$$

$$(\theta, \varphi) \in A_1 \times A_2 \in \mathfrak{B}_G \text{ and } r \in B \in \mathfrak{B}_r, \text{ and } \beta = \left[\int_0^\alpha r^2 / (c^2 + r^2)^{1/2} dr \right]^{-1}.$$

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References

- [1] KELLEY, J. L.: "General Topology", Van Nostrand, N. Y., 1961.
- [2] PARTHASARATHY, K. R.: "Probability measures on metric spaces," Academic Press, N. Y., 1967.
- [3] PONTRJAGIN, L. S.: "Topological Groups," Princeton Univ. Press, Princeton, New Jersey, 1946.
- [4] PROHOROV, YU. V.: "Convergence of random processes and limit theorems in probability theory," Theor. Probability Appl. 1 (1956), 157-214.
- [5] SIMMONS, G. F.: "Introduction to Topology and Modern Analysis," McGraw-Hill, N. Y., 1963.