ON DECOMPOSABILITY OF PROBABIKITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

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ON DECOMPOSABILITY OF PROBABILITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

By

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§ 1. Summary.

In this paper we consider probability measures on a complete separable metric space $T$ (or on a topological subspace $T$ of a complete separable metric space $T^*$) with which a compact metric group $G$ of homeomorphisms acting on $T$ is associated.

Let $\mu$ be an arbitrary probability measure on $T$ invariant under every $g \in G$. Then it is shown that for an arbitrary small $\varepsilon > 0$, there always exists a compact set $A_0 \subseteq T$ and a probability measure $\mu_0$ on $T$ invariant under every $g \in G$ such that

(i) $G \cdot A_0 = A_0$,
(ii) The support of $\mu_0$ is a closed subset of $A_0$,
(iii) For every Borel set $A$, $|\mu(A) - \mu_0(A)| < \varepsilon$.
(iv) Let $V$ be a Borel set in $A_0$ such that $G \cdot V = A_0$ and $Gv_1 \cap Gv_2 = \emptyset$ if $v_1 \neq v_2$, $(v_1, v_2 \in V)$, whose existence is shown in lemma 3. Then the probability measure $\mu_0$ is decomposed into a direct product measure of the normalized Haar measure of $G$ and a probability measure on $V$.

§ 2. Decomposition of $G$-invariant probability measures.

Let $T$ be a complete separable metric space or a topological subspace of a complete separable metric space $T^*$. Let $G$ be a compact metric group of homeomorphisms acting on $T$ such that $G \cdot T = T$ and the mapping $(g, t) \rightarrow gt$ from $G \times T$ into $T$ is continuous.

For any metric space $X$ we shall denote by $\mathcal{B}_X$ the $\sigma$-field of Borel subsets of $X$.

Let $\mu$ be a probability measure defined on $\mathcal{B}_T$ and invariant under every $g \in G$, that is,

(1) $\mu(T) = 1$.
(2) For each $g \in G$, $\mu(gA) = \mu(A)$, $A \in \mathcal{B}_T$.

Lemma 1. Let $g$ and $t$ be any elements of $G$ and $T$ respectively but be fixed. Let $W$ be an arbitrary neighbourhood of $gt \in T$. Then we can always find a neighbourhood $U$ of $t$ such that $W = gU$.
Proof. \( g: t \mapsto gt \) is a homeomorphism from \( T \) onto itself and \( g^{-1}(W) = g^{-1} \cdot W \) is an inverse image of the open set \( W \) under the mapping \( g \). Hence, \( g^{-1} \cdot W \) is open.

Since \( t = g^{-1} \cdot gt \subseteq g^{-1}W \), \( g^{-1}W \) is a neighbourhood of \( t \). Thus, by writing \( U = g^{-1} \cdot W \), we see that \( W = gU \).

Lemma 2. Let \( A \) be the support of the probability measure \( \mu \).

Then \( G \cdot A = A \).

Proof. \( A \) is a set of all elements \( t \in T \) such that for any neighbourhood \( U \) of \( t \), \( \mu(U) > 0 \). (See [2], page 28).

Let \( W \) be an arbitrary neighbourhood of \( gt \in T \), where \( g \) and \( t \) are any fixed elements of \( G \) and \( A \) respectively.

Then from lemma 1, we can choose a neighbourhood \( U \) of \( t \) such that \( W = gU \). Thus, we see that \( \mu(W) = \mu(gU) = \mu(U) > 0 \). This implies that \( gt \in A \) for all \( g \in G \) and hence \( G \cdot A = A \).

Since \( A \subseteq G \cdot A \), we have proved that \( G \cdot A = A \).

Lemma 3. Let \( \Gamma \subseteq T \) be a compact set invariant under every \( g \in G \).

Then there is a Borel set \( V \subseteq T \) such that

\[
\begin{align*}
\text{(3)} & \quad \Gamma = G \cdot V, \\
\text{(4)} & \quad Gv_1 \cap Gv_2 = \emptyset \quad \text{if} \quad v_1 \neq v_2, \quad (v_1, v_2 \in V).
\end{align*}
\]

Proof. For any two points \( t_1, t_2 \in \Gamma \), we shall say that \( t_1 \sim t_2 \) if there exists \( g \in G \) such that \( gt_1 = t_2 \). "\( \sim \)" is an equivalence relation. Let \( M \) be the space of all such equivalence classes. Let \([t]\) denote the equivalence class containing \( t \). Then, since the mapping \( t \mapsto [t] \) from \( \Gamma \) into \( M \) is continuous under the quotient topology and is onto, we see that \( M \) is a continuous image of the compact set \( \Gamma \) under this mapping and hence \( M \) is a compact metric space.

Thus, by a theorem of Federer and Morse (see [2], page 23, Theorem 4.2), it follows that there exists a Borel set \( V \subseteq \Gamma \) satisfying (3) and (4).

Now, we have the following:

Lemma 4. Let \( \Gamma \subseteq T \) be the set considered in lemma 3 and suppose for any \( g \in G \), \( g \neq e \), there is no fixed point in \( T \).

Then the mapping

\[
\xi: (g, v) \mapsto gv
\]

is a Borel isomorphism between \( G \times V \) and \( \Gamma \).

Proof. From lemma 3, we see that the mapping \( \xi: G \times V \to \Gamma \) is onto. Since there is no fixed point in \( \Gamma \) for any \( g \in G \), \( g \neq e \), \( \xi \) is one-one. From our assumption, it is clear that the mapping \( \xi \) is continuous.

In particular, \( \xi \) is measurable. Hence, by a theorem of Kuratowski (See [2], page 21, theorem 3.9), \( \xi^{-1} \) is measurable.

Thus, we have proved lemma 4.

Now we have the following theorem.

Theorem 1. Let the support \( A \) of \( \mu \) be compact. Let \( V \subseteq T \) be a Borel set such that \( G \cdot V = A \) and \( Gv_1 \cap Gv_2 = \emptyset \) if \( v_1 \neq v_2 \), \((v_1, v_2 \in V)\) and the mapping \( \xi: (g, v) \mapsto gv \) be a Borel isomorphism between \( G \times V \) and \( A \).
Let $C$ be a Borel set in $T$ and $A$ and $B$ Borel subsets in $G$ and $V$ respectively such that

$$\xi(A \times B) = C \cap A \in \mathcal{B}_A.$$ 

Then,

$$\mu(C) = \nu(A) \cdot \rho(B),$$

where $\nu$ is the normalized Haar measure on $G$ and $\rho$ a probability measure on $\mathcal{B}_V$.

**Proof.** Let us notice that $A$ is a compact set in $T$ and hence $A \in \mathcal{B}_T$. Therefore $\mathcal{B}_A$ is the $\sigma$-field of subsets of $A$ of the form $A \cap A$, $A \in \mathcal{B}_T$, that is, $\mathcal{B}_A = \{ \tilde{A} = A \cap A \mid A \in \mathcal{B}_T \}$.

Now let us define a set function $\tilde{\mu}$ on $\mathcal{B}_A$ in such a way that:

For each $\tilde{A} = A \cap A \in \mathcal{B}_A$, $A \in \mathcal{B}_T$,

$$\tilde{\mu}(\tilde{A}) = \mu(A).$$

It is easy to see that $\tilde{\mu}$ is a probability measure on $\mathcal{B}_A$. Indeed, for any $\tilde{A} \in \mathcal{B}_A$, $0 \leq \tilde{\mu}(\tilde{A}) = \mu(A) \leq 1$ and $\tilde{\mu}(A) = \mu(A) = 1$.

Let $\{ \tilde{A}_v = A_v \cap A_v, A_v \in \mathcal{B}_T, v = 1, 2, \ldots \}$ be a sequence of disjoint sets in $\mathcal{B}_A$. However $\{A_v; v = 1, 2, \ldots, A_v \in \mathcal{B}_T \}$ are not necessarily disjoint.

Let us notice that $\bigcup_{v=1}^{\infty} A_v = \bigcup_{v=1}^{\infty} (A_v \cap A) \cup \bigcup_{v=1}^{\infty} (A_v \cap A^c)$, and $\bigcup_{v=1}^{\infty} (A_v \cap A)$ and $\bigcup_{v=1}^{\infty} (A_v \cap A^c)$ are disjoint. Since $\bigcup_{v=1}^{\infty} \tilde{A}_v = \bigcup_{v=1}^{\infty} A_v \cap A$, we have

$$\tilde{\mu}\left(\bigcup_{v=1}^{\infty} \tilde{A}_v\right) = \mu\left(\bigcup_{v=1}^{\infty} A_v\right)$$

$$= \mu\left(\bigcup_{v=1}^{\infty} (A_v \cap A)\right) + \mu\left(\bigcup_{v=1}^{\infty} (A_v \cap A^c)\right).$$

From the fact that $\mu\left(\bigcup_{v=1}^{\infty} (A_v \cap A^c)\right) \leq \mu(A^c) = 0$, it follows that

$$\tilde{\mu}\left(\bigcup_{v=1}^{\infty} \tilde{A}_v\right) = \mu\left(\bigcup_{v=1}^{\infty} (A_v \cap A)\right)$$

$$= \sum_{v=1}^{\infty} \mu(A_v \cap A)$$

$$= \sum_{v=1}^{\infty} \tilde{\mu}(\tilde{A}_v).$$

Thus, we have proved that $\tilde{\mu}$ is a probability measure on $\mathcal{B}_A$.

By $\mu^*$ let us denote the probability measure on $\mathcal{B}_{G \times V}$ induced by the mapping $\xi^{-1}$ from $A$ onto $G \times V$, that is, $\mu^*$ is the probability measure such that for any $A \times B \in \mathcal{B}_{G \times V}$, $(A \in \mathcal{B}_G, B \in \mathcal{B}_V)$,

$$\mu^*(A \times B) = \tilde{\mu}(\xi(A \times B)).$$

Since both $G \times V$ and $G$ are complete separable metric spaces, these are automatically separable standard Borel spaces (see [2], page 133).

Let us consider the mapping $\pi : (g, v) \to g$ from $G \times V$ onto $G$. Then it is clear that $\pi$ is measurable, since for any $E \in \mathcal{B}_G$, $\pi^{-1}(E) = E \times V \in \mathcal{B}_{G \times V}$.

Thus, it follows that there exists a regular conditional probability distribution
of $\mu^*$ given $\pi$, which we shall denote by $\bar{m}_g(A \times B)$, $g \in G$, $A \times B \in \mathfrak{B}_{G \times \nu}$. This satisfies the following conditions:

(i) For each $g \in G$, $\bar{m}_g(\cdot)$ is a probability measure on $\mathfrak{B}_{G \times \nu}$.

(ii) For each $A \times B \in \mathfrak{B}_{G \times \nu}$, the mapping $g \rightarrow \bar{m}_g(A \times B)$ is $\mathfrak{B}_G$-measurable and

$\mu^*(A \times B) = \int_G \bar{m}_g(A \times B) dg$,

where $\nu(E) = \mu^*(\pi^{-1}(E)) = \mu^*(E \times V)$, $E \in \mathfrak{B}_G$. (See [2], page 146).

It is obvious that $\nu$ is a probability measure on $\mathfrak{B}_G$.

In particular, we have for any $E \in \mathfrak{B}_G$,

$\mu^*(A \times B \cap \pi^{-1}(E)) = \mu^*((A \cap E) \times B)
= \int_G \bar{m}_g(A \times B) dg, \quad A \times B \in \mathfrak{B}_{G \times \nu}$.

Let us write

$m_g(B) = \bar{m}_g(G \times B), \quad B \in \mathfrak{B}_\nu$.

Then we have by putting $A = G$ in (6),

$\mu^*(E \times B) = \int_E m_g(B) dg$.

for any $E \in \mathfrak{B}_G$ and any $B \in \mathfrak{B}_\nu$, where $m_g(\cdot)$ is a probability measure on $\mathfrak{B}_\nu$.

Let $A \in \mathfrak{B}_G$ and $B \in \mathfrak{B}_\nu$ and $C = \{ \eta \in T | \eta = g', g' \in A, t \in B \}$. Then for any $g \in G$,

$\mu^*(gA \times B) = \mu^*(\pi(gA \times B)) = \rho(gC) = \rho(C)
= \rho(\pi(A \times B)) = \mu^*(A \times B)$.

Hence it follows by putting particularly $B = V$ in (7) that

$\nu(gA) = \mu^*(gA \times V) = \mu^*(A \times V) = \nu(A)$ for any $A \in \mathfrak{B}_G$.

This implies that $\nu$ is the normalized Haar measure on $G$.

Thus, we may write

$\mu^*(A \times B) = \int_A m_g(B) dg$.

Now, for any $A \in \mathfrak{B}_G$, $B \in \mathfrak{B}_\nu$ and $h \in G$,

$\mu^*(A \times B) = \int_A m_g(B) dg
= \mu^*(hA \times B)
= \int_{hA} m_g(B) dg
= \int_A m_{h^{-1}g}(B) dg$.

This implies that for any $B \in \mathfrak{B}_\nu$ and any $g, g' \in G$,

$m_g(B) = m_g(B), \quad B \in \mathfrak{B}_\nu$.

in other words, $m_g(B)$ is a probability measure on $\mathfrak{B}_\nu$ independent of $g \in G$.

Let us write

$\rho(B) = m_g(B), \quad B \in \mathfrak{B}_\nu$. 
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Then we have for any \( C \in \mathcal{B}_T \) such that \( \xi(A \times B) = C \cap A \), \( A \times B \in \mathcal{B}_G \times \mathcal{B}_T \),

\[
\mu(C) = \hat{\mu}(C \cap A) \\
= \hat{\mu}(\xi(A \times B)) \\
= \mu^*(A \times B) \\
= \int_A \rho(B) \, dg \\
= \nu(A) \cdot \rho(B).
\]

Thus, we have proved theorem 1.

Now, let us consider the case where the support of the probability measure \( \mu \) is not necessarily compact in \( T \).

**Lemma 5.** For any \( \varepsilon > 0 \), there exists a compact set \( A_0 \) in \( T \) such that

\[
(8) \quad G \cdot A_0 = A_0, \\
(9) \quad \mu(A_0) > 1 - \varepsilon/3.
\]

**Proof.** Every probability measure on a complete separable metric space or on a topological subspace which is a Borel subset of such a metric space is tight (See [2], page 29). Hence, for any small \( \varepsilon > 0 \), there is a compact set \( C \subset T \) such that

\[
\mu(C) > 1 - \varepsilon/3.
\]

From a theorem of Tychonoff, it follows that \( G \times C \) is compact, since it is a topological product space of a compact metric group \( G \) and a compact metric space \( C \). From our assumption, the mapping

\[
\gamma : (g, t) \rightarrow gt
\]

from \( G \times T \) into \( T \) is continuous and \( G \cdot C = \gamma(G \times C) \) is a continuous image of a compact set \( G \times C \). Hence \( G \cdot C \) is compact.

Let us write \( A_0 = G \cdot C \). Then \( A_0 \) is invariant under every \( g \in G \) and \( \mu(A_0) \geq \mu(C) > 1 - \varepsilon/3 \).

Let us consider a probability measure \( \mu_0 \) on \( \mathcal{B}_T \) defined by

\[
(10) \quad \mu_0(A) = \frac{\mu(A \cap A_0)}{\mu(A_0)}, \quad A \in \mathcal{B}_T.
\]

Then, since \( \mu_0(A_0) = 1 \), the support of \( \mu_0 \) is compact.

Since \( G \cdot A_0 = A_0 \), we see that for any \( g \in G \) and \( A \in \mathcal{B}_T \), \( gA \cap A_0 = g(A \cap A_0) \).

Thus, it is clear that for any \( g \in G \), \( \mu_0(gA) = \mu_0(A) \), \( A \in \mathcal{B}_T \).

Now, we have the following theorem:

**Theorem 2.** For an arbitrary small \( \varepsilon > 0 \), there exists a compact set \( A_0 \subset T \) and a probability measure \( \mu_0 \) on \( \mathcal{B}_T \), such that

\[
(11) \quad \text{The support of } \mu_0 \text{ is a closed subset of } A_0, \\
(12) \quad \text{For any } g \in G \text{ and } A \in \mathcal{B}_T, \quad \mu_0(gA) = \mu_0(A),
\]

and

\[
(13) \quad \text{For any } A \in \mathcal{B}_T, \quad |\mu(A) - \mu_0(A)| < \varepsilon.
\]

**Proof.** We have already shown that the existence of the compact set \( A_0 \) and the
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probability measure \( \mu_0 \) satisfying (11) and (12).

Now, we shall prove (13).

Since for any \( A \in \mathcal{B} \),

\[
\mu(A) = \mu(A \cap A_0) + \mu(A \cap A_0^c),
\]

we have the following inequality:

\[
|\mu(A) - \mu_0(A)|
= \left| \mu(A \cap A_0) + \mu(A \cap A_0^c) - \frac{\mu(A \cap A_0^c)}{\mu(A_0)} \right|
\leq \mu(A \cap A_0) + \mu(A \cap A_0^c) \cdot (1 - \mu(A_0)/\mu(A_0))
\leq \mu(A_0) \cdot (1 + 1/\mu(A_0)).
\]

As long as \( \varepsilon \) is not greater than 1, \( 1 + 1/\mu(A_0) \) is less than 5/2.

Thus, we have \( |\mu(A) - \mu_0(A)| < (5/6)\varepsilon < \varepsilon \), for any \( A \in \mathcal{B} \).

This completes the proof.

Thus, in generally, every probability measure on \( T \) invariant under every \( g \in G \)
can be approximated by a direct product probability measure of the normalized Haar
measure on \( G \) and a probability measure on a Borel set \( V \) satisfying the conditions
stated in lemma 3 as closely as possible.

**Example 1.** Let \( T = \mathbb{R}^2 = \{(\theta, r) \mid 0 \leq r < \infty, 0 \leq \theta < 2\pi \} \) and \( G = SO(2) \). Let us
consider a compact set \( \Gamma = \{(\theta, r) \mid 0 \leq r \leq \alpha, 0 \leq \theta \leq 2\pi \} \) and a set \( V = \{(0, r) \mid 0 \leq r \leq \alpha \} \),
where \( \alpha \) is a finite real number.

Then it is obvious that \( V \) satisfies the conditions (3) and (4) in lemma 3.

Let \( \mu \) be a probability measure on \( \mathcal{B}_T \) and \( \mu(\Gamma) = 1 \). If for any \( A \in \mathcal{B}_T \), \( \mu(A) \) is
proportional to the area of the set \( A \), then the probability measure \( \mu \) is invariant
under every \( g \in SO(2) \) and it is written in the view of theorem 1 as follows:

\[
\int_{A \in \mathcal{B}} d\mu(\theta, r) = \int_{A} \frac{1}{2\pi} d\theta \cdot \int_{B} \frac{2}{\alpha^2} r dr,
\]

where \( A \in \mathcal{B}_T \) and \( B \in \mathcal{B}_V \).

**Example 2.** Let \( L_3 = \{(t, x, y, z) \mid t^2 - (x^2 + y^2 + z^2) = c^2 \} \), where \( c \) is a real number
and \((t, x, y, z)'s\) are points in 4-dim. Euclidean space \( \mathbb{R}_c \). Let \( G \) be a subgroup of
the proper Lorentz group of the order (3, 1) such that its elements are of the following
form:

\[
g = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \quad h \in SO(3),
\]

where 0 in the above expression denotes 3×1-zero vector and also 1×3-zero vector.

Let \( \mu \) be a probability measure on \( L_3 \) invariant under every \( g \in G \). Let the
support of the measure \( \mu \) be such that

\[
\Gamma = \{(t, x, y, z) \in L_3 \mid 0 \leq x^2 + y^2 + z^2 \leq \alpha^2 \},
\]

where \( \alpha \) is a finite real number.

Let us consider a set

\[
V = \{ (\sqrt{c^2+r^2}, r, 0, 0) \mid 0 \leq r \leq \alpha \} \subset \Gamma.
\]
Then $V$ satisfies the conditions (3) and (4) in lemma 3.

Suppose that for any $C \in \mathcal{B}_T$, $\mu(C)$ is proportional to the area of $C$. Then it is written in the view of theorem 1 as follows:

$$\int_C d\mu(x, y, z) = \int_{A_1 \times A_2 \times B} d\mu(r, \theta, \varphi)$$

$$= \int_{A_1 \times A_2} \frac{1}{4\pi} \sin \theta d\theta d\varphi \cdot \int_B \frac{\beta r^2}{\sqrt{c^2 + r^2}} dr,$$

where $C$ is a set of all points $(t, x, y, z)$ in $\Gamma$ such that

$x = r \cdot \sin \theta \cdot \cos \varphi$, 
$y = r \cdot \sin \theta \cdot \sin \varphi$, 
$z = r \cdot \cos \theta$, 
$t = \sqrt{c^2 + r^2}$,

$(\theta, \varphi) \in A_1 \times A_2 \in \mathcal{A}_0$ and $r \in B \in \mathcal{B}_r$, and $\beta = \left[ \int_0^\infty \frac{r^2}{(c^2 + r^2)^{3/2}} dr \right]^{-1}$.

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References