# ON DECOMPOSABILITY OF PROBABIKITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

Nagai, Takeaki
Faculty of Science, Kagoshima University

https://doi.org/10.5109/13036

出版情報：統計数理研究. 13 (3/4), pp.57-63, 1969-03. Research Association of Statistical Sciences
バージョン：
権利関係：

# ON DECOMPOSABILITY OF PROBABILITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

By

**Takeaki NAGAI***

(Received Jan. 5, 1969)

## §1. Summary.

In this paper we consider probability measures on a complete separable metric space $T$ (or on a topological subspace $T$ of a complete separable metric space $T^*$) with which a compact metric group $G$ of homeomorphisms acting on $T$ is associated.

Let $\mu$ be an arbitrary probability measure on $T$ invariant under every $g\in G$. Then it is shown that for an arbitrary small $\varepsilon>0$, there always exists a compact set $\Lambda_0\subset T$ and a probability measure $\mu_0$ on $T$ invariant under every $g\in G$ such that

(i) $G\cdot\Lambda_0=\Lambda_0$.

(ii) The support of $\mu_0$ is a closed subset of $\Lambda_0$.

(iii) For every Borel set $A$, $|\mu(A)-\mu_0(A)|<\varepsilon$.

(iv) Let $V$ be a Borel set in $\Lambda_0$ such that $G\cdot V=\Lambda_0$ and $Gv_1\cap Gv_2=\phi$ if $v_1\neq v_2$, $(v_1, v_2\in V)$, whose existence is shown in lemma 3. Then the probability measure $\mu_0$ is decomposed into a direct product measure of the normalized Haar measure of $G$ and a probability measure on $V$.

## §2. Decomposition of $G$-invariant probability measures.

Let $T$ be a complete separable metric space or a topological subspace of a complete separable metric space $T^*$. Let $G$ be a compact metric group of homeomorphisms acting on $T$ such that $G\cdot T=T$ and the mapping $(g, t)\rightarrow gt$ from $G\times T$ into $T$ is continuous.

For any metric space $X$ we shall denote by $\mathfrak{B}_X$ the $\sigma$-field of Borel subsets of $X$.

Let $\mu$ be a probability measure defined on $\mathfrak{B}_T$ and invariant under every $g\in G$, that is,

(1) $$\mu(T)=1.$$

(2) $$\text{For each } g\in G,\ \mu(gA)=\mu(A),\ A\in\mathfrak{B}_T.$$

**Lemma 1.** *Let $g$ and $t$ be any elements of $G$ and $T$ respectively but be fixed. Let $W$ be an arbitrary neighbourhood of $gt\in T$. Then we can always find a neighbourhood $U$ of $t$ such that $W=gU$.*

* Faculty of Science, Kagoshima University, Kagoshima.

**Proof.** $g: t \to gt$ is a homeomorphism from $T$ onto itself and $g^{-1}(W) = g^{-1} \cdot W$ is an inverse image of the open set $W$ under the mapping $g$. Hence, $g^{-1} \cdot W$ is open.

Since $t = g^{-1} \cdot gt \in g^{-1}W$, $g^{-1}W$ is a neighbourhood of $t$. Thus, by writing $U = g^{-1} \cdot W$, we see that $W = gU$.

**Lemma 2.** *Let $\Lambda$ be the support of the probability measure $\mu$.*

*Then $G \cdot \Lambda = \Lambda$.*

**Proof.** $\Lambda$ is a set of all elements $t \in T$ such that for any neighbourhood $U$ of $t$, $\mu(U) > 0$. (See [2], page 28).

Let $W$ be an arbitrary neighbourhood of $gt \in T$, where $g$ and $t$ are any fixed elements of $G$ and $\Lambda$ respectively.

Then from lemma 1, we can choose a neighbourhood $U$ of $t$ such that $W = gU$. Thus, we see that $\mu(W) = \mu(gU) = \mu(U) > 0$. This implies that $gt \in \Lambda$ for all $g \in G$ and hence $G \cdot \Lambda \subset \Lambda$.

Since $\Lambda \subset G \cdot \Lambda$, we have proved that $G \cdot \Lambda = \Lambda$.

**Lemma 3.** *Let $\Gamma \subset T$ be a compact set invariant under every $g \in G$.*

*Then there is a Borel set $V \subset T$ such that*

$$\Gamma = G \cdot V, \tag{3}$$

$$Gv_1 \cap Gv_2 = \phi \quad \text{if} \quad v_1 \neq v_2, \qquad (v_1, v_2 \in V). \tag{4}$$

**Proof.** For any two points $t_1, t_2 \in \Gamma$, we shall say that $t_1 \sim t_2$ if there exists $g \in G$ such that $gt_1 = t_2$. " $\sim$ " is an equivalence relation. Let $M$ be the space of all such equivalence classes. Let $[t]$ denote the equivalence class containing $t$. Then, since the mapping $t \to [t]$ from $\Gamma$ into $M$ is continuous under the quotient topology and is onto, we see that $M$ is a continuous image of the compact set $\Gamma$ under this mapping and hence $M$ is a compact metric space.

Thus, by a theorem of Federer and Morse (see [2], page 23, Theorem 4.2), it follows that there exists a Borel set $V \subset \Gamma$ satisfying (3) and (4).

Now, we have the following:

**Lemma 4.** *Let $\Gamma \subset T$ be the set considered in lemma 3 and suppose for any $g \in G$, $g \neq e$, there is no fixed point in $\Gamma$.*

*Then the mapping*

$$\xi : (g, v) \to gv \tag{5}$$

*is a Borel isomorphism between $G \times V$ and $\Gamma$.*

**Proof.** From lemma 3, we see that the mapping $\xi : G \times V \to \Gamma$ is onto. Since there is no fixed point in $\Gamma$ for any $g \in G$, $g \neq e$, $\xi$ is one-one. From our assumption, it is clear that the mapping $\xi$ is continuous.

In particular, $\xi$ is measurable. Hence, by a theorem of Kuratowski (See [2], page 21, theorem 3.9), $\xi^{-1}$ is measurable.

Thus, we have proved lemma 4.

Now we have the following theorem.

**Theorem 1.** *Let the support $\Lambda$ of $\mu$ be compact. Let $V \subset T$ be a Borel set such that $G \cdot V = \Lambda$ and $Gv_1 \cap Gv_2 = \phi$ if $v_1 \neq v_2$, $(v_1, v_2 \in V)$ and the mapping $\xi : (g, v) \to gv$ be a Borel isomorphism between $G \times V$ and $\Lambda$.*

*Let $C$ be a Borel set in $T$ and $A$ and $B$ Borel subsets in $G$ and $V$ respectively such that*

$$\xi(A\times B)=C\cap\Lambda\in\mathfrak{B}_\Lambda .$$

*Then,*

$$\mu(C)=\nu(A)\cdot\rho(B),$$

*where $\nu$ is the normalized Haar measure on $G$ and $\rho$ a probability measure on $\mathfrak{B}_V$.*

**Proof.** Let us notice that $\Lambda$ is a compact set in $T$ and hence $\Lambda\in\mathfrak{B}_T$. Therefore $\mathfrak{B}_\Lambda$ is the $\sigma$-field of subsets of $\Lambda$ of the form $A\cap\Lambda$, $A\in\mathfrak{B}_T$, that is, $\mathfrak{B}_\Lambda=\{\tilde{A}=A\cap\Lambda \mid A\in\mathfrak{B}_T\}$.

Now let us define a set function $\tilde{\mu}$ on $\mathfrak{B}_\Lambda$ in such a way that:

For each $\tilde{A}=A\cap\Lambda\in\mathfrak{B}_\Lambda$, $A\in\mathfrak{B}_T$,

$$\tilde{\mu}(\tilde{A})=\mu(A).$$

It is easy to see that $\tilde{\mu}$ is a probability measure on $\mathfrak{B}_\Lambda$. Indeed, for any $\tilde{A}\in\mathfrak{B}_\Lambda$, $0\leqq\tilde{\mu}(\tilde{A})=\mu(A)\leqq 1$ and $\tilde{\mu}(\Lambda)=\mu(\Lambda)=1$.

Let $\{\tilde{A}_\nu=A_\nu\cap\Lambda,\ A_\nu\in\mathfrak{B}_T,\ \nu=1,2,\cdots\}$ be a sequence of disjoint sets in $\mathfrak{B}_\Lambda$. However $\{A_\nu;\ \nu=1,2,\cdots,\ A_\nu\in\mathfrak{B}_T\}$ are not necessarily disjoint.

Let us notice that $\bigcup_{\nu=1}^{\infty}A_\nu=\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda)\cup\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda^c)$, and $\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda)$ and $\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda^c)$ are disjoint. Since $\bigcup_{\nu=1}^{\infty}\tilde{A}_\nu=\bigcup_{\nu=1}^{\infty}A_\nu\cap\Lambda$, we have

$$\tilde{\mu}\Big(\bigcup_{\nu=1}^{\infty}\tilde{A}_\nu\Big)=\mu\Big(\bigcup_{\nu=1}^{\infty}A_\nu\Big)$$

$$=\mu\Big(\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda)\Big)+\mu\Big(\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda^c)\Big).$$

From the fact that $\mu\Big(\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda^c)\Big)\leqq\mu(\Lambda^c)=0$, it follows that

$$\tilde{\mu}\Big(\bigcup_{\nu=1}^{\infty}\tilde{A}_\nu\Big)=\mu\Big(\bigcup_{\nu=1}^{\infty}(A_\nu\cap\Lambda)\Big)$$

$$=\sum_{\nu=1}^{\infty}\mu(A_\nu\cap\Lambda)$$

$$=\sum_{\nu=1}^{\infty}\tilde{\mu}(\tilde{A}_\nu).$$

Thus, we have proved that $\tilde{\mu}$ is a probability measure on $\mathfrak{B}_\Lambda$.

By $\mu^*$ let us denote the probability measure on $\mathfrak{B}_{G\times V}$ induced by the mapping $\xi^{-1}$ from $\Lambda$ onto $G\times V$, that is, $\mu^*$ is the probability measure such that for any $A\times B\in\mathfrak{B}_{G\times V}$, $(A\in\mathfrak{B}_G,\ B\in\mathfrak{B}_V)$,

$$\mu^*(A\times B)=\tilde{\mu}(\xi(A\times B)).$$

Since both $G\times V$ and $G$ are complete separable metric spaces, these are automatically separable standard Borel spaces (see [2], page 133).

Let us consider the mapping $\pi:(g,v)\to g$ from $G\times V$ onto $G$. Then it is clear that $\pi$ is measurable, since for any $E\in\mathfrak{B}_G$, $\pi^{-1}(E)=E\times V\in\mathfrak{B}_{G\times V}$.

Thus, it follows that there exists a regular conditional probability distribution of $\mu^*$ given $\pi$, which we shall denote by $\tilde{m}_g(A\times B)$, $g\in G$, $A\times B\in\mathfrak{B}_{G\times V}$. This satisfies the following conditions:

(i) For each $g\in G$, $\tilde{m}_g(\cdot)$ is a probability measure on $\mathfrak{B}_{G\times V}$.

(ii) For each $A\times B\in\mathfrak{B}_{G\times V}$, the mapping $g\to\tilde{m}_g(A\times B)$ is $\mathfrak{B}_G$-measurable and

(iii) $\mu^*(A\times B)=\int_G \tilde{m}_g(A\times B)d\nu(g)$,

where $\nu(E)=\mu^*(\pi^{-1}(E))=\mu^*(E\times V)$, $E\in\mathfrak{B}_G$. (See [2], page 146).

It is obvious that $\nu$ is a probability measure on $\mathfrak{B}_G$.

In particular, we have for any $E\in\mathfrak{B}_G$,

$$\mu^*(A\times B\cap\pi^{-1}(E))=\mu^*((A\cap E)\times B) \tag{6}$$

$$=\int_E \tilde{m}_g(A\times B)d\nu(g)\,, \qquad A\times B\in\mathfrak{B}_{G\times V}\,.$$

Let us write

$$m_g(B)=\tilde{m}_g(G\times B)\,, \qquad B\in\mathfrak{B}_V\,.$$

Then we have by putting $A=G$ in (6),

$$\mu^*(E\times B)=\int_E m_g(B)d\nu(g)\,, \tag{7}$$

for any $E\in\mathfrak{B}_G$ and any $B\in\mathfrak{B}_V$, where $m_g(\cdot)$ is a probability measure on $\mathfrak{B}_V$.

Let $A\in\mathfrak{B}_G$ and $B\in\mathfrak{B}_V$ and $C=\{\eta\in T\mid\eta=g't,\ g'\in A,\ t\in B\}$. Then for any $g\in G$,

$$\mu^*(gA\times B)=\tilde{\mu}(\xi(gA\times B))=\tilde{\mu}(gC)=\tilde{\mu}(C)$$

$$=\tilde{\mu}(\xi(A\times B))=\mu^*(A\times B)\,.$$

Hence it follows by putting particularly $B=V$ in (7) that

$$\nu(gA)=\mu^*(gA\times V)=\mu^*(A\times V)=\nu(A) \qquad \text{for any} \quad A\in\mathfrak{B}_G\,.$$

This implies that $\nu$ is the normalized Haar measure on $G$.

Thus, we may write

$$\mu^*(A\times B)=\int_A m_g(B)dg\,.$$

Now, for any $A\in\mathfrak{B}_G$, $B\in\mathfrak{B}_V$ and $h\in G$,

$$\mu^*(A\times B)=\int_A m_g(B)dg$$

$$=\mu^*(hA\times B)$$

$$=\int_{hA} m_g(B)dg$$

$$=\int_A m_{h^{-1}g}(B)dg\,.$$

This implies that for any $B\in\mathfrak{B}_V$ and any $g, g'\in G$,

$$m_g(B)=m_g(B)\,,$$

in other words, $m_g(B)$ is a probability measure on $\mathfrak{B}_V$ independent of $g\in G$.

Let us write

$$\rho(B)=m_g(B)\,, \qquad B\in\mathfrak{B}_V\,.$$

Then we have for any $C\in\mathfrak{B}_T$ such that $\xi(A\times B)=C\cap\Lambda$, $A\times B\in\mathfrak{B}_{G\times V}$,

$$\begin{aligned}\mu(C)&=\tilde{\mu}(C\cap\Lambda)\\&=\tilde{\mu}(\xi(A\times B))\\&=\mu^*(A\times B)\\&=\int_A\rho(B)dg\\&=\nu(A)\cdot\rho(B).\end{aligned}$$

Thus, we have proved theorem 1.

Now, let us consider the case where the support of the probability measure $\mu$ is not necessarily compact in $T$.

**Lemma 5.** *For any $\varepsilon>0$, there exists a compact set $\Lambda_0$ in $T$ such that*

(8) $$G\cdot\Lambda_0=\Lambda_0.$$

(9) $$\mu(\Lambda_0)>1-\varepsilon/3.$$

**Proof.** Every probability measure on a complete separable metric space or on a topological subspace which is a Borel subset of such a metric space is tight (See [2], page 29). Hence, for any small $\varepsilon>0$, there is a compact set $C\subset T$ such that

$$\mu(C)>1-\varepsilon/3.$$

From a theorem of Tychonoff, it follows that $G\times C$ is compact, since it is a topological product space of a compact metric group $G$ and a compact metric space $C$. From our assumption, the mapping $\gamma:(g,t)\to gt$ from $G\times T$ into $T$ is continuous and $G\cdot C=\gamma(G\times C)$ is a continuous image of a compact set $G\times C$. Hence $G\cdot C$ is compact.

Let us write $\Lambda_0=G\cdot C$. Then $\Lambda_0$ is invariant under every $g\in G$ and $\mu(\Lambda_0)\geqq\mu(C)>1-\varepsilon/3$.

Let us consider a probability measure $\mu_0$ on $\mathfrak{B}_T$ defined by

(10) $$\mu_0(A)=\frac{\mu(A\cap\Lambda_0)}{\mu(\Lambda_0)},\qquad A\in\mathfrak{B}_T.$$

Then, since $\mu_0(\Lambda_0)=1$, the support of $\mu_0$ is compact.

Since $G\cdot\Lambda_0=\Lambda_0$, we see that for any $g\in G$ and $A\in\mathfrak{B}_T$, $gA\cap\Lambda_0=g(A\cap\Lambda_0)$. Thus, it is clear that for any $g\in G$, $\mu_0(gA)=\mu_0(A)$, $A\in\mathfrak{B}_T$.

Now, we have the following theorem:

**Theorem 2.** *For an arbitrary small $\varepsilon>0$, there exists a compact set $\Lambda_0\subset T$ and a probability measure $\mu_0$ on $\mathfrak{B}_T$ such that*

(11) *The support of $\mu_0$ is a closed subset of $\Lambda_0$.*

(12) *For any $g\in G$ and $A\in\mathfrak{B}_T$,*

$$\mu_0(gA)=\mu_0(A),$$

*and*

(13) *For any $A\in\mathfrak{B}_T$,*

$$|\mu(A)-\mu_0(A)|<\varepsilon.$$

**Proof.** We have already shown that the existence of the compact set $\Lambda_0$ and the probability measure $\mu_0$ satisfying (11) and (12).

Now, we shall prove (13).

Since for any $A \in \mathfrak{B}_T$,

$$\mu(A)=\mu(A \cap A_0)+\mu(A \cap A_0^c),$$

we have the following inequality:

$$\begin{aligned}
&|\mu(A)-\mu_0(A)| \\
&=\left|\mu(A \cap A_0^c)+\mu(A \cap A_0)-\frac{\mu(A \cap A_0)}{\mu(A_0)}\right| \\
&\leqq \mu(A \cap A_0^c)+\mu(A \cap A_0)\cdot(1-\mu(A_0))/\mu(A_0) \\
&=\mu(A_0^c)\cdot(1+1/\mu(A_0)).
\end{aligned}$$

As long as $\varepsilon$ is not greater than 1, $1+1/\mu(A_0)$ is less than 5/2.

Thus, we have $|\mu(A)-\mu_0(A)|<(5/6)\varepsilon<\varepsilon$, for any $A \in \mathfrak{B}_T$.

This completes the proof.

Thus, in generally, every probability measure on $T$ invariant under every $g \in G$ can be approximated by a direct product probability measure of the normalized Haar measure on $G$ and a probability measure on a Borel set $V$ satisfying the conditions stated in lemma 3 as closely as possible.

**Example 1.** Let $T=R_2=\{(\theta, r) \mid 0 \leqq r<\infty, 0 \leqq \theta<2\pi\}$ and $G=SO(2)$. Let us consider a compact set $\Gamma=\{(\theta, r) \mid 0 \leqq r \leqq \alpha, 0 \leqq \theta \leqq 2\pi\}$ and a set $V=\{(0, r) \mid 0 \leqq r \leqq \alpha\}$, where $\alpha$ is a finite real number.

Then it is obvious that $V$ satisfies the conditions (3) and (4) in lemma 3.

Let $\mu$ be a probability measure on $\mathfrak{B}_T$ and $\mu(\Gamma)=1$. If for any $A \in \mathfrak{B}_\Gamma$, $\mu(A)$ is proportional to the area of the set $A$, then the probability measure $\mu$ is invariant under every $g \in SO(2)$ and it is written in the view of theorem 1 as follows:

$$\int_{A \times B} d\mu(\theta, r)=\int_A \frac{1}{2\pi} d\theta \cdot \int_B \frac{2}{\alpha^2} r dr,$$

where $A \in \mathfrak{B}_G$ and $B \in \mathfrak{B}_V$.

**Example 2.** Let $L_3=\{(t, x, y, z) \mid t^2-(x^2+y^2+z^2)=c^2\}$, where $c$ is a real number and $(t, x, y, z)$'s are points in 4-dim. Euclidean space $R_4$. Let $G$ be a subgroup of the proper Lorentz group of the order (3, 1) such that its elements are of the following form:

$$g=\begin{pmatrix} 1, & 0 \\ 0, & h \end{pmatrix}, \qquad h \in SO(3),$$

where 0 in the above expression denotes $3 \times 1$-zero vector and also $1 \times 3$-zero vector.

Let $\mu$ be a probability measure on $L_3$ invariant under every $g \in G$. Let the support of the measure $\mu$ be such that

$$\Gamma=\{(t, x, y, z) \in L_3 \mid 0 \leqq x^2+y^2+z^2 \leqq \alpha^2\},$$

where $\alpha$ is a finite real number.

Let us consider a set

$$V=\{(\sqrt{c^2+r^2}, r, 0, 0) \mid 0 \leqq r \leqq \alpha\} \subset \Gamma.$$

Then $V$ satisfies the conditions (3) and (4) in lemma 3.

Suppose that for any $C \in \mathfrak{B}_\Gamma$, $\mu(C)$ is proportional to the area of $C$. Then it is written in the view of theorem 1 as follows:

$$\int_C d\mu(x, y, z) = \int_{A_1 \times A_2 \times B} d\mu(r, \theta, \varphi)$$
$$= \int_{A_1 \times A_2} \frac{1}{4\pi} \sin\theta d\theta d\varphi \cdot \int_B \frac{\beta r^2}{\sqrt{c^2+r^2}} dr ,$$

where $C$ is a set of all points $(t, x, y, z)$ in $\Gamma$ such that

$$x = r \cdot \sin\theta \cdot \cos\varphi ,$$
$$y = r \cdot \sin\theta \cdot \sin\varphi ,$$
$$z = r \cdot \cos\theta ,$$
$$t = \sqrt{c^2+r^2} ,$$

$(\theta, \varphi) \in A_1 \times A_2 \in \mathfrak{B}_G$ and $r \in B \in \mathfrak{B}_V$, and $\beta = \left[\int_0^\alpha r^2/(c^2+r^2)^{1/2} dr\right]^{-1}$.

## § 3. Acknowledgment.

The author would like to express his heartiest thanks to Professor K. R. Parthasarathy and Professor J. Gani for their kind suggestions and encouragements while this paper was being prepared.

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