ON DECOMPOSABILITY OF PROBABIKITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

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ON DECOMPOSABILITY OF PROBABILITY MEASURES ON A SEPARABLE METRIC SPACE ACTED UPON BY A COMPACT METRIC GROUP

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§1. Summary.

In this paper we consider probability measures on a complete separable metric space T (or on a topological subspace T of a complete separable metric space T^*) with which a compact metric group G of homeomorphisms acting on T is associated.

Let μ be an arbitrary probability measure on T invariant under every $g \in G$. Then it is shown that for an arbitrary small $\varepsilon > 0$, there always exists a compact set $A_0 \subset T$ and a probability measure μ_0 on T invariant under every $g \in G$ such that

(i) $G \cdot A_0 = A_0$.

(ii) The support of μ_0 is a closed subset of A_0.

(iii) For every Borel set A, $|\mu(A) - \mu_0(A)| < \varepsilon$.

(iv) Let V be a Borel set in A_0 such that $G \cdot V = A_0$ and $Gv_1 \cap Gv_2 = \phi$ if $v_1 \neq v_2$, $(v_1, v_2 \in V)$, whose existence is shown in lemma 3. Then the probability measure μ_0 is decomposed into a direct product measure of the normalized Haar measure of G and a probability measure on V.

§2. Decomposition of G-invariant probability measures.

Let T be a complete separable metric space or a topological subspace of a complete separable metric space T^*. Let G be a compact metric group of homeomorphisms acting on T such that $G \cdot T = T$ and the mapping $(g, t) \rightarrow gt$ from $G \times T$ into T is continuous.

For any metric space X we shall denote by $\mathfrak{B}_X$ the σ-field of Borel subsets of X.

Let μ be a probability measure defined on $\mathfrak{B}_T$ and invariant under every $g \in G$, that is,

(1) $$\mu(T) = 1.$$

(2) $$\text{For each } g \in G,\ \mu(gA) = \mu(A),\ A \in \mathfrak{B}_T.$$

Lemma 1. *Let g and t be any elements of G and T respectively but be fixed. Let W be an arbitrary neighbourhood of $gt \in T$. Then we can always find a neighbourhood U of t such that $W = gU$.*

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Proof. $g: t \to gt$ is a homeomorphism from T onto itself and $g^{-1}(W) = g^{-1} \cdot W$ is an inverse image of the open set W under the mapping g. Hence, $g^{-1} \cdot W$ is open.

Since $t = g^{-1} \cdot gt \in g^{-1}W$, $g^{-1}W$ is a neighbourhood of t. Thus, by writing $U = g^{-1} \cdot W$, we see that $W = gU$.

Lemma 2. *Let Λ be the support of the probability measure μ.*

Then $G \cdot \Lambda = \Lambda$.

Proof. Λ is a set of all elements $t \in T$ such that for any neighbourhood U of t, $\mu(U) > 0$. (See [2], page 28).

Let W be an arbitrary neighbourhood of $gt \in T$, where g and t are any fixed elements of G and Λ respectively.

Then from lemma 1, we can choose a neighbourhood U of t such that $W = gU$. Thus, we see that $\mu(W) = \mu(gU) = \mu(U) > 0$. This implies that $gt \in \Lambda$ for all $g \in G$ and hence $G \cdot \Lambda \subset \Lambda$.

Since $\Lambda \subset G \cdot \Lambda$, we have proved that $G \cdot \Lambda = \Lambda$.

Lemma 3. *Let $\Gamma \subset T$ be a compact set invariant under every $g \in G$.*

Then there is a Borel set $V \subset T$ such that

$$\Gamma = G \cdot V, \tag{3}$$

$$Gv_1 \cap Gv_2 = \phi \quad \text{if} \quad v_1 \neq v_2, \qquad (v_1, v_2 \in V). \tag{4}$$

Proof. For any two points $t_1, t_2 \in \Gamma$, we shall say that $t_1 \sim t_2$ if there exists $g \in G$ such that $gt_1 = t_2$. "$\sim$" is an equivalence relation. Let M be the space of all such equivalence classes. Let $[t]$ denote the equivalence class containing t. Then, since the mapping $t \to [t]$ from Γ into M is continuous under the quotient topology and is onto, we see that M is a continuous image of the compact set Γ under this mapping and hence M is a compact metric space.

Thus, by a theorem of Federer and Morse (see [2], page 23, Theorem 4.2), it follows that there exists a Borel set $V \subset \Gamma$ satisfying (3) and (4).

Now, we have the following:

Lemma 4. *Let $\Gamma \subset T$ be the set considered in lemma 3 and suppose for any $g \in G$, $g \neq e$, there is no fixed point in Γ.*

Then the mapping

$$\xi : (g, v) \to gv \tag{5}$$

is a Borel isomorphism between $G \times V$ and Γ.

Proof. From lemma 3, we see that the mapping $\xi : G \times V \to \Gamma$ is onto. Since there is no fixed point in Γ for any $g \in G$, $g \neq e$, ξ is one-one. From our assumption, it is clear that the mapping ξ is continuous.

In particular, ξ is measurable. Hence, by a theorem of Kuratowski (See [2], page 21, theorem 3.9), ξ^{-1} is measurable.

Thus, we have proved lemma 4.

Now we have the following theorem.

Theorem 1. *Let the support Λ of μ be compact. Let $V \subset T$ be a Borel set such that $G \cdot V = \Lambda$ and $Gv_1 \cap Gv_2 = \phi$ if $v_1 \neq v_2$, $(v_1, v_2 \in V)$ and the mapping $\xi : (g, v) \to gv$ be a Borel isomorphism between $G \times V$ and Λ.*

Let C be a Borel set in T and A and B Borel subsets in G and V respectively such that

$$\xi(A\times B)=C\cap\Lambda\in\mathfrak{B}_\Lambda\,.$$

Then,

$$\mu(C)=\nu(A)\cdot\rho(B)\,,$$

where ν is the normalized Haar measure on G and ρ a probability measure on $\mathfrak{B}_V$.

Proof. Let us notice that Λ is a compact set in T and hence $\Lambda\in\mathfrak{B}_T$. Therefore $\mathfrak{B}_\Lambda$ is the σ-field of subsets of Λ of the form $A\cap\Lambda$, $A\in\mathfrak{B}_T$, that is, $\mathfrak{B}_\Lambda=\{\tilde{A}=A\cap\Lambda\mid A\in\mathfrak{B}_T\}$.

Now let us define a set function $\tilde{\mu}$ on $\mathfrak{B}_\Lambda$ in such a way that:

For each $\tilde{A}=A\cap\Lambda\in\mathfrak{B}_\Lambda$, $A\in\mathfrak{B}_T$,

$$\tilde{\mu}(\tilde{A})=\mu(A)\,.$$

It is easy to see that $\tilde{\mu}$ is a probability measure on $\mathfrak{B}_\Lambda$. Indeed, for any $\tilde{A}\in\mathfrak{B}_\Lambda$, $0\leqq\tilde{\mu}(\tilde{A})=\mu(A)\leqq 1$ and $\tilde{\mu}(\Lambda)=\mu(\Lambda)=1$.

Let $\{\tilde{A}_\nu=A_\nu\cap\Lambda,\ A_\nu\in\mathfrak{B}_T,\ \nu=1,2,\cdots\}$ be a sequence of disjoint sets in $\mathfrak{B}_\Lambda$. However $\{A_\nu;\ \nu=1,2,\cdots,A_\nu\in\mathfrak{B}_T\}$ are not necessarily disjoint.

Let us notice that $\bigcup_{\nu=1}^\infty A_\nu=\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda)\cup\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda^c)$, and $\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda)$ and $\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda^c)$ are disjoint. Since $\bigcup_{\nu=1}^\infty\tilde{A}_\nu=\bigcup_{\nu=1}^\infty A_\nu\cap\Lambda$, we have

$$\begin{aligned}\tilde{\mu}\Big(\bigcup_{\nu=1}^\infty\tilde{A}_\nu\Big)&=\mu\Big(\bigcup_{\nu=1}^\infty A_\nu\Big)\\&=\mu\Big(\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda)\Big)+\mu\Big(\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda^c)\Big)\,.\end{aligned}$$

From the fact that $\mu\Big(\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda^c)\Big)\leqq\mu(\Lambda^c)=0$, it follows that

$$\begin{aligned}\tilde{\mu}\Big(\bigcup_{\nu=1}^\infty\tilde{A}_\nu\Big)&=\mu\Big(\bigcup_{\nu=1}^\infty(A_\nu\cap\Lambda)\Big)\\&=\sum_{\nu=1}^\infty\mu(A_\nu\cap\Lambda)\\&=\sum_{\nu=1}^\infty\tilde{\mu}(\tilde{A}_\nu)\,.\end{aligned}$$

Thus, we have proved that $\tilde{\mu}$ is a probability measure on $\mathfrak{B}_\Lambda$.

By μ^* let us denote the probability measure on $\mathfrak{B}_{G\times V}$ induced by the mapping ξ^{-1} from Λ onto $G\times V$, that is, μ^* is the probability measure such that for any $A\times B\in\mathfrak{B}_{G\times V}$, $(A\in\mathfrak{B}_G,\ B\in\mathfrak{B}_V)$,

$$\mu^*(A\times B)=\tilde{\mu}(\xi(A\times B))\,.$$

Since both $G\times V$ and G are complete separable metric spaces, these are automatically separable standard Borel spaces (see [2], page 133).

Let us consider the mapping $\pi:(g,v)\to g$ from $G\times V$ onto G. Then it is clear that π is measurable, since for any $E\in\mathfrak{B}_G$, $\pi^{-1}(E)=E\times V\in\mathfrak{B}_{G\times V}$.

Thus, it follows that there exists a regular conditional probability distribution of μ^* given π, which we shall denote by $\tilde{m}_g(A\times B)$, $g\in G$, $A\times B\in\mathfrak{B}_{G\times V}$. This satisfies the following conditions:

(i) For each $g\in G$, $\tilde{m}_g(\cdot)$ is a probability measure on $\mathfrak{B}_{G\times V}$.

(ii) For each $A\times B\in\mathfrak{B}_{G\times V}$, the mapping $g\to\tilde{m}_g(A\times B)$ is $\mathfrak{B}_G$-measurable and

(iii) $\mu^*(A\times B)=\int_G \tilde{m}_g(A\times B)d\nu(g)$,

where $\nu(E)=\mu^*(\pi^{-1}(E))=\mu^*(E\times V)$, $E\in\mathfrak{B}_G$. (See [2], page 146).

It is obvious that ν is a probability measure on $\mathfrak{B}_G$.

In particular, we have for any $E\in\mathfrak{B}_G$,

$$\mu^*(A\times B\cap\pi^{-1}(E))=\mu^*((A\cap E)\times B) \tag{6}$$

$$=\int_E \tilde{m}_g(A\times B)d\nu(g), \qquad A\times B\in\mathfrak{B}_{G\times V}.$$

Let us write

$$m_g(B)=\tilde{m}_g(G\times B), \qquad B\in\mathfrak{B}_V.$$

Then we have by putting $A=G$ in (6),

$$\mu^*(E\times B)=\int_E m_g(B)d\nu(g), \tag{7}$$

for any $E\in\mathfrak{B}_G$ and any $B\in\mathfrak{B}_V$, where $m_g(\cdot)$ is a probability measure on $\mathfrak{B}_V$.

Let $A\in\mathfrak{B}_G$ and $B\in\mathfrak{B}_V$ and $C=\{\eta\in T\mid\eta=g't,\ g'\in A,\ t\in B\}$. Then for any $g\in G$,

$$\mu^*(gA\times B)=\tilde{\mu}(\xi(gA\times B))=\tilde{\mu}(gC)=\tilde{\mu}(C)$$

$$=\tilde{\mu}(\xi(A\times B))=\mu^*(A\times B).$$

Hence it follows by putting particularly $B=V$ in (7) that

$$\nu(gA)=\mu^*(gA\times V)=\mu^*(A\times V)=\nu(A) \qquad \text{for any} \quad A\in\mathfrak{B}_G.$$

This implies that ν is the normalized Haar measure on G.

Thus, we may write

$$\mu^*(A\times B)=\int_A m_g(B)dg.$$

Now, for any $A\in\mathfrak{B}_G$, $B\in\mathfrak{B}_V$ and $h\in G$,

$$\begin{aligned}\mu^*(A\times B)&=\int_A m_g(B)dg\\ &=\mu^*(hA\times B)\\ &=\int_{hA} m_g(B)dg\\ &=\int_A m_{h^{-1}g}(B)dg.\end{aligned}$$

This implies that for any $B\in\mathfrak{B}_V$ and any $g, g'\in G$,

$$m_g(B)=m_g(B),$$

in other words, $m_g(B)$ is a probability measure on $\mathfrak{B}_V$ independent of $g\in G$.

Let us write

$$\rho(B)=m_g(B), \qquad B\in\mathfrak{B}_V.$$

Then we have for any $C \in \mathfrak{B}_T$ such that $\xi(A \times B) = C \cap A$, $A \times B \in \mathfrak{B}_{G \times V}$,

$$\begin{aligned} \mu(C) &= \tilde{\mu}(C \cap A) \\ &= \tilde{\mu}(\xi(A \times B)) \\ &= \mu^*(A \times B) \\ &= \int_A \rho(B) dg \\ &= \nu(A) \cdot \rho(B) . \end{aligned}$$

Thus, we have proved theorem 1.

Now, let us consider the case where the support of the probability measure μ is not necessarily compact in T.

Lemma 5. *For any $\varepsilon > 0$, there exists a compact set A_0 in T such that*

(8) $$G \cdot A_0 = A_0 .$$

(9) $$\mu(A_0) > 1 - \varepsilon/3 .$$

Proof. Every probability measure on a complete separable metric space or on a topological subspace which is a Borel subset of such a metric space is tight (See [2], page 29). Hence, for any small $\varepsilon > 0$, there is a compact set $C \subset T$ such that

$$\mu(C) > 1 - \varepsilon/3 .$$

From a theorem of Tychonoff, it follows that $G \times C$ is compact, since it is a topological product space of a compact metric group G and a compact metric space C. From our assumption, the mapping $\gamma : (g, t) \rightarrow gt$ from $G \times T$ into T is continuous and $G \cdot C = \gamma(G \times C)$ is a continuous image of a compact set $G \times C$. Hence $G \cdot C$ is compact.

Let us write $A_0 = G \cdot C$. Then A_0 is invariant under every $g \in G$ and $\mu(A_0) \geqq \mu(C) > 1 - \varepsilon/3$.

Let us consider a probability measure μ_0 on $\mathfrak{B}_T$ defined by

(10) $$\mu_0(A) = \frac{\mu(A \cap A_0)}{\mu(A_0)} , \qquad A \in \mathfrak{B}_T .$$

Then, since $\mu_0(A_0) = 1$, the support of μ_0 is compact.

Since $G \cdot A_0 = A_0$, we see that for any $g \in G$ and $A \in \mathfrak{B}_T$, $gA \cap A_0 = g(A \cap A_0)$. Thus, it is clear that for any $g \in G$, $\mu_0(gA) = \mu_0(A)$, $A \in \mathfrak{B}_T$.

Now, we have the following theorem:

Theorem 2. *For an arbitrary small $\varepsilon > 0$, there exists a compact set $A_0 \subset T$ and a probability measure μ_0 on $\mathfrak{B}_T$ such that*

(11) *The support of μ_0 is a closed subset of A_0.*

(12) *For any $g \in G$ and $A \in \mathfrak{B}_T$,*

$$\mu_0(gA) = \mu_0(A) ,$$

and

(13) *For any $A \in \mathfrak{B}_T$,*

$$| \mu(A) - \mu_0(A) | < \varepsilon .$$

Proof. We have already shown that the existence of the compact set A_0 and the probability measure μ_0 satisfying (11) and (12).

Now, we shall prove (13).

Since for any $A \in \mathfrak{B}_T$,

$$\mu(A) = \mu(A \cap A_0) + \mu(A \cap A_0^c),$$

we have the following inequality:

$$\begin{aligned}
&|\mu(A) - \mu_0(A)| \\
&= \left| \mu(A \cap A_0^c) + \mu(A \cap A_0) - \frac{\mu(A \cap A_0)}{\mu(A_0)} \right| \\
&\leqq \mu(A \cap A_0^c) + \mu(A \cap A_0) \cdot (1 - \mu(A_0))/\mu(A_0) \\
&= \mu(A_0^c) \cdot (1 + 1/\mu(A_0)) .
\end{aligned}$$

As long as ε is not greater than 1, $1 + 1/\mu(A_0)$ is less than 5/2.

Thus, we have $|\mu(A) - \mu_0(A)| < (5/6)\varepsilon < \varepsilon$, for any $A \in \mathfrak{B}_T$.

This completes the proof.

Thus, in generally, every probability measure on T invariant under every $g \in G$ can be approximated by a direct product probability measure of the normalized Haar measure on G and a probability measure on a Borel set V satisfying the conditions stated in lemma 3 as closely as possible.

Example 1. Let $T = R_2 = \{(\theta, r) |\ 0 \leqq r < \infty,\ 0 \leqq \theta < 2\pi\}$ and $G = SO(2)$. Let us consider a compact set $\Gamma = \{(\theta, r) | 0 \leqq r \leqq \alpha,\ 0 \leqq \theta \leqq 2\pi\}$ and a set $V = \{(0, r) |\ 0 \leqq r \leqq \alpha\}$, where α is a finite real number.

Then it is obvious that V satisfies the conditions (3) and (4) in lemma 3.

Let μ be a probability measure on $\mathfrak{B}_T$ and $\mu(\Gamma) = 1$. If for any $A \in \mathfrak{B}_\Gamma$, $\mu(A)$ is proportional to the area of the set A, then the probability measure μ is invariant under every $g \in SO(2)$ and it is written in the view of theorem 1 as follows:

$$\int_{A \times B} d\mu(\theta, r) = \int_A \frac{1}{2\pi} d\theta \cdot \int_B \frac{2}{\alpha^2} r dr ,$$

where $A \in \mathfrak{B}_G$ and $B \in \mathfrak{B}_V$.

Example 2. Let $L_3 = \{(t, x, y, z) | t^2 - (x^2 + y^2 + z^2) = c^2\}$, where c is a real number and (t, x, y, z)'s are points in 4-dim. Euclidean space R_4. Let G be a subgroup of the proper Lorentz group of the order (3, 1) such that its elements are of the following form:

$$g = \begin{pmatrix} 1, & 0 \\ 0, & h \end{pmatrix}, \qquad h \in SO(3) ,$$

where 0 in the above expression denotes 3×1-zero vector and also 1×3-zero vector.

Let μ be a probability measure on L_3 invariant under every $g \in G$. Let the support of the measure μ be such that

$$\Gamma = \{(t, x, y, z) \in L_3 | 0 \leqq x^2 + y^2 + z^2 \leqq \alpha^2\} ,$$

where α is a finite real number.

Let us consider a set

$$V = \{(\sqrt{c^2 + r^2}, r, 0, 0) | 0 \leqq r \leqq \alpha\} \subset \Gamma .$$

Then V satisfies the conditions (3) and (4) in lemma 3.

Suppose that for any $C \in \mathfrak{B}_\Gamma$, $\mu(C)$ is proportional to the area of C. Then it is written in the view of theorem 1 as follows:

$$\int_C d\mu(x, y, z) = \int_{A_1 \times A_2 \times B} d\mu(r, \theta, \varphi)$$
$$= \int_{A_1 \times A_2} \frac{1}{4\pi} \sin\theta d\theta d\varphi \cdot \int_B \frac{\beta r^2}{\sqrt{c^2 + r^2}} dr ,$$

where C is a set of all points (t, x, y, z) in Γ such that

$$x = r \cdot \sin\theta \cdot \cos\varphi ,$$
$$y = r \cdot \sin\theta \cdot \sin\varphi ,$$
$$z = r \cdot \cos\theta ,$$
$$t = \sqrt{c^2 + r^2} ,$$

$(\theta, \varphi) \in A_1 \times A_2 \in \mathfrak{B}_G$ and $r \in B \in \mathfrak{B}_V$, and $\beta = \left[\int_0^\alpha r^2/(c^2 + r^2)^{1/2} dr \right]^{-1}$.

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