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A MULTIVARIATE NORMAL TEST WITH TWO-SIDED ALTERNATIVE

By

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§ 1. Introduction.

The works related to the present one are Kudô [1], and Kudô and Fujisawa [2], [3], and the present paper follows the principle stated in § 5 of [3]. In the present paper, we are concerned with a multivariate normal population with a mean vector

$$\theta : \theta' = (\theta_1, \theta_2, \cdots, \theta_k) \qquad k \geqq 2$$

and a known variance unit matrix I. The problem is to test hypothesis H_0; $\theta_i = 0$ $(i = 1, 2, \cdots, k)$ against the alternative hypothesis H_1: $(\theta_i \geqq 0,\ i = 1, 2, \cdots, k)$ or $(\theta_i \leqq 0,\ i = 1, 2, \cdots, k)$, where the inequality is strict for at least one value of i in either case. For the test we shall define a statistic, $\bar{\chi}^2$, based on the likelihood ratio criterion and derive the probability distribution function of the $\bar{\chi}^2$ statistic and also present a table of the percentage points appropriate to the test.

§ 2. The likelihood ratio criterion.

We are concerned with the testing problem mentioned above. Let $X^{(j)\prime} = (X_{1j}, X_{2j}, \cdots, X_{ij}, \cdots, X_{kj})$ is distributed according to the k-variate normal distribution with the mean vector

$$\theta' = (\theta_1, \theta_2, \cdots, \theta_k)$$

and a known unit variance matrix, for $j = 1, \cdots, n$. Let $(X^{(1)}, X^{(2)}, \cdots, X^{(n)})$ be a random sample of size n and $\bar{X}$ be the random sample mean. In this section we shall derive and discuss the computation of the likelihood ratio criterion.

The joint distribution of the sample under the alternative hypothesis is given by

$$\begin{aligned} f &= \frac{1}{(\sqrt{2\pi})^{kn}} \exp\Big[-\frac{1}{2}\sum_{j=1}^{n}(X^{(j)}-\theta)'(X^{(j)}-\theta)\Big] \\ &= \frac{1}{(\sqrt{2\pi})^{kn}} \exp\Big[-\frac{1}{2}\sum_{j=1}^{n}(X^{(j)}-\bar{X})'(X^{(j)}-\bar{X}) + n(\bar{X}-\theta)'(\bar{X}-\theta)\Big] \end{aligned} \tag{2.1}$$

Consider

$$\lambda(+) = \max_{\substack{\theta_i = 0 \\ i=1,\cdots,k}} f \Big/ \max_{\substack{\theta_i \geqq 0 \\ i=1,\cdots,k}} f$$

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$$= \exp\left[-\frac{1}{2}n\bar{X}'\bar{X}\right] \Big/ \max_{\substack{\theta_i \geqq 0 \\ i=1,\cdots,k}} \exp\left[-\frac{1}{2}\{n(\bar{X}-\theta)'(\bar{X}-\theta)\}\right] \tag{2.2}$$

$$\lambda(-) = \max_{\substack{\theta_i=0 \\ i=1,\cdots,k}} f \Big/ \max_{\substack{\theta_i \leqq 0 \\ i=1,\cdots,k}} f$$

$$= \exp\left[-\frac{1}{2}n\bar{X}'\bar{X}\right] \Big/ \max_{\substack{\theta_i \leqq 0 \\ i=1,\cdots,k}} \exp\left[-\frac{1}{2}\{n(\bar{X}-\theta)'(\bar{X}-\theta)\}\right]. \tag{2.3}$$

The likelihood ratio test is based on whether

$$\lambda = \max(\lambda(+), \lambda(-)) \tag{2.4}$$

is too small or not. Evaluation of (2.4) is equivalent to examining the following statistic

$$\bar{\chi}^2 = \max(\bar{\chi}^2(+), \bar{\chi}^2(-)), \tag{2.5}$$

where

$$\bar{\chi}^2(+) = n\{\bar{X}'\bar{X} - \min_{\substack{\theta_i \geqq 0 \\ i=1,\cdots,k}} (\bar{X}-\theta)'(\bar{X}-\theta)\} \tag{2.6}$$

and

$$\bar{\chi}^2(-) = n\{\bar{X}'\bar{X} - \min_{\substack{\theta_i \leqq 0 \\ i=1,\cdots,k}} (\bar{X}-\theta)'(\bar{X}-\theta)\}. \tag{2.7}$$

We shall call (2.5), (2.6) and (2.7), respectively, $\bar{\chi}^2$-statistic, $\bar{\chi}^2(+)$-statistic and $\bar{\chi}^2(-)$-statistic.

For the calculation of the probability that $\bar{\chi}^2$ exceeds c^2, $Pr(\bar{\chi}^2 \geqq c^2)$, when the null hypothesis is true, we need the following lemma by Kudô [1], and we shall state them without proof.

With every X in the space R of the sample mean vector, we associate a point X°, termed the minimum point of

$$\min_{\substack{m_i \geqq 0 \\ i=1,\cdots,k}} (X-m)'\Lambda^{-1}(X-m) = (X-X^\circ)'\Lambda^{-1}(X-X^\circ), \tag{2.8}$$

where Λ is non-singular known matrix and $m' = (m_1, \cdots, m_k)$. We devide R into 2^k disjoint subsets, $R = \bigcup_{\phi \leqq M \leqq K} R_M$, where R_M is the totality of points X such that

$$\begin{aligned} x_i^\circ &= 0 \qquad (i \notin M), \\ x_i^\circ &> 0 \qquad (i \in M), \end{aligned} \tag{2.9}$$

where $X^{\circ\prime} = (x_1^\circ, x_2^\circ, \cdots, x_k^\circ)$ is the minimum point of (2.8) associated with X. Let

$$\bar{U}^2(+) = \{X'\Lambda^{-1}X - \min_{\substack{m_i \geqq 0 \\ i=1,\cdots,k}} (X-m)'\Lambda^{-1}(X-m)\}. \tag{2.10}$$

LEMMA (Kudô [1]). *Let* $(X_1, \cdots, X_k)$ *be distributed in a multivariate normal distribution with mean* $m' = (m_1, \cdots, m_k)$ *and variance-covariance matrix* Λ. *The probability that the value of* $\bar{U}^2(+)$*-statistic, computed taking the null hypothesis as* $H_0: m_i = 0$ $(i = 1, \cdots, k)$ *and the alternative as* $H(+): m_i \geqq 0$ $(i = 1, \cdots, k)$ *where the inequality is strict for at least one value of i, exceeds* $\bar{u}_0^2$ *is given by*

$$Pr(\bar{U}^2(+) \geqq \bar{u}_0^2) = \sum_{\phi \leqq M \leqq K} Pr(\chi^2_{n(M)} \geqq \bar{u}_0^2) Pr\{(\Lambda_{M'})^{-1}\} Pr\{\Lambda_{M;M'}\} \tag{2.11}$$

where the summation runs over all the subsets M *of* $K = \{1, \cdots, k\}$, $n(M)$ *is the number* *of elements in* M, M' *is the complement of* M, Λ_M *is the variance matrix of* x_i, $i \in M$, $\Lambda_{M:M'}$ *is the same under the condition* $x_j=0$, $j \notin M$, *and* $P\{\Sigma\}$ *is the probability that the variables distributed in a multivariate normal distribution with means zero and variance-covariance matrix* Σ *are all positive.* $\chi^2_{n(M)}$ *has the chi-square distribution with* $n(M)$ *degrees of freedom, where* $\chi^2_{n(\phi)}$ *is to be understood as a constant zero,* $P\{\Lambda_{\phi:K}\}=1$ *and* $P\{(\Lambda_{K'})^{-1}\}=P\{(\Lambda_{\phi})^{-1}\}=1$.

By using of the Lemma we shall have the following.

THEOREM. *Let* $(X_1, \cdots, X_k)$ *be distributed in a multivariate normal distribution with mean* $\theta'=(\theta_1, \cdots, \theta_k)$, *and variance-covariance matrix* I. *The probability that the* $\bar{\chi}^2$-*statistic defined in* (2.5), *exceeds* c^2 *is given by*

$$Pr(\bar{\chi}^2 \geqq c^2)=\frac{1}{2^{k-1}}\sum_{m=1}^{k} {}_kC_m Pr(\chi^2_m \geqq c^2)-\frac{1}{2^k}\sum_{m=1}^{k-1} {}_kC_m Pr(\chi^2_m \geqq c^2)Pr(\chi^2_{k-m} \geqq c^2), \quad (2.12)$$

where χ^2_m *has the chi-square distribution with* m *degrees of freedom.*

PROOF. Because of the symmetry, the distributions of $\bar{\chi}^2(+)$ and $\bar{\chi}^2(-)$ are the same (Kudô [3]). The distribution of $\bar{\chi}^2$ is thus

$$\begin{aligned} Pr(\bar{\chi}^2 \geqq c^2) &= Pr[\max(\bar{\chi}^2(+), \bar{\chi}^2(-)) \geqq c^2] \\ &= 2Pr(\bar{\chi}^2(+) \geqq c^2)-Pr(\bar{\chi}^2(+) \geqq c^2, \bar{\chi}^2(-) \geqq c^2) \\ &= 2\times\frac{1}{2^k}\sum_{m=1}^{k} {}_kC_m Pr(\chi^2_m \geqq c^2)-\frac{1}{2^k}\sum_{m=1}^{k-1} {}_kC_m Pr(\chi^2_m \geqq c^2, \chi^2_{k-m} \geqq c^2) \\ &= \frac{1}{2^{k-1}}\sum_{m=1}^{k} {}_kC_m Pr(\chi^2_m \geqq c^2)-\frac{1}{2^k}\sum_{m=1}^{k-1} {}_kC_m Pr(\chi^2_m \geqq c^2)Pr(\chi^2_{k-m} \geqq c^2). \end{aligned}$$

§ 3. Table.

The table provides the values of the percentage points $k=2(1)15$. The values are expected to be correct up to 4 decimal places.

The table of percentage points of the $\bar{\chi}^2$-distribution $(=c^2)$, $P=Pr(\bar{\chi}^2 \geqq c^2)$.

k \ P	0.995	0.990	0.975	0.950	0.900	0.750	0.500
2		0.017600	0.044480	0.090347	0.185750	0.509298	1.237167
3	0.058079	0.092871	0.174962	0.285756	0.476192	0.995993	1.970671
4	0.158015	0.227416	0.371017	0.546282	0.821061	1.500473	2.666056
5	0.301074	0.406138	0.611455	0.845426	1.195046	2.010388	3.335849
6	0.478319	0.618785	0.881359	1.170240	1.587008	2.522094	3.987574
7	0.681947	0.856379	1.173852	1.512313	1.990527	3.034307	4.625837
8	0.907528	1.114665	1.482866	1.868657	2.403370	3.546615	5.253643
9	1.151017	1.389552	1.805624	2.235390	2.823487	4.058533	5.872720
10	1.408669	1.676457	2.139649	2.610755	3.248326	4.570431	6.484957
11	1.678800	1.975309	2.482728	2.993148	3.677997	5.081872	7.091235
12	1.960324	2.283959	2.833697	3.382208	4.111211	5.593107	7.692466
13	2.251002	2.601252	3.191178	3.775851	4.547896	6.103959	8.289210
14	2.549784	2.925883	3.554927	4.174239	4.987405	6.614581	8.882090
15	2.856056	3.257410	3.924384	4.576888	5.429311	7.124913	9.471535

k \ P	0.250	0.100	0.050	0.025	0.010	0.005	0.001
2	2.504990	4.216146	5.530868	6.857860	8.627439	9.974322	13.120620
3	3.487964	5.414450	6.852472	8.283967	10.171157	11.595250	14.894316
4	4.376893	6.474591	8.013589	9.531038	11.515578	13.005230	16.436602
5	5.209464	7.454297	9.080385	10.672670	12.743499	14.290052	17.839055
6	6.004717	8.379923	10.083954	11.743855	13.892551	15.492421	19.147621
7	6.772482	9.266090	11.041499	12.763613	14.984097	16.631992	20.386448
8	7.519122	10.122129	11.963691	13.743574	16.030479	17.724974	21.570913
9	8.249308	10.954700	12.858296	14.691682	17.041853	18.778470	22.712043
10	8.965457	11.767043	13.729092	15.613656	18.023029	19.800487	23.815345
11	9.670070	12.562561	14.579915	16.513041	18.979214	20.794046	24.891200
12	10.364378	13.344200	15.414848	17.394307	19.914012	21.767103	25.938789
13	11.050619	13.986919	16.234849	18.258557	20.830615	22.719015	26.963537
14	11.729114	14.871327	17.041859	19.108457	21.729998	23.652587	27.967998
15	12.400684	15.620167	17.837099	19.945195	22.615817	24.570807	28.952593

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