A MULTIVARIATE NORMAL TEST WITH TWO-SIDED ALTERNATIVE

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http://hdl.handle.net/2324/13033
A MULTIVARIATE NORMAL TEST WITH TWO-SIDED ALTERNATIVE

By

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(Received January 30th, 1968)

§ 1. Introduction.

The works related to the present one are Kudô [1], and Kudô and Fujisawa [2], [3], and the present paper follows the principle stated in § 5 of [3]. In the present paper, we are concerned with a multivariate normal population with a mean vector

\[ \theta : \theta' = (\theta_1, \theta_2, \ldots, \theta_k) \quad k \geq 2 \]

and a known variance unit matrix \( I \). The problem is to test hypothesis \( H_0 : \theta_i = 0 \) \( (i = 1, 2, \ldots, k) \) against the alternative hypothesis \( H_1 : (\theta_i \geq 0, i = 1, 2, \ldots, k) \) or \( (\theta_i \leq 0, i = 1, 2, \ldots, k) \), where the inequality is strict for at least one value of \( i \) in either case. For the test we shall define a statistic, \( \chi^2 \), based on the likelihood ratio criterion and derive the probability distribution function of the \( \chi^2 \) statistic and also present a table of the percentage points appropriate to the test.

§ 2. The likelihood ratio criterion.

We are concerned with the testing problem mentioned above. Let \( X(1)' = (X_{11}, X_{21}, \ldots, X_{1j}, \ldots, X_{kj}) \) is distributed according to the \( k \)-variate normal distribution with the mean vector

\[ \theta' = (\theta_1, \theta_2, \ldots, \theta_k) \]

and a known unit variance matrix, for \( j = 1, \ldots, n \). Let \( (X_1', X_2', \ldots, X_n') \) be a random sample of size \( n \) and \( \bar{X} \) be the random sample mean. In this section we shall derive and discuss the computation of the likelihood ratio criterion.

The joint distribution of the sample under the alternative hypothesis is given by

\[
\begin{align*}
\lambda(+) &= \max_{\theta_i \geq 0} f \quad \frac{\lambda(-)}{\max_{\theta_i \geq 0} f} \\
\lambda(+) &= \frac{1}{(\sqrt{2\pi})^{kn}} \exp \left[ -\frac{1}{2} \sum_{j=1}^{n} (X^{(j)}-\theta)'(X^{(j)}-\theta) \right] \\
\lambda(-) &= \frac{1}{(\sqrt{2\pi})^{kn}} \exp \left[ -\frac{1}{2} \sum_{j=1}^{n} (X^{(j)}-\bar{X})'(X^{(j)}-\bar{X}) + n(\bar{X}-\theta)'(\bar{X}-\theta) \right]
\end{align*}
\]

Consider

\[ \lambda(+) = \max_{\theta_i \geq 0} f \quad \frac{\lambda(-)}{\max_{\theta_i \geq 0} f} \quad \lambda(+) \quad i = 1, \ldots, k \]

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\[ \lambda(\cdot) = \max_{\theta \leq 0} \frac{\max f}{\max f_{\theta \equiv 0}} \]

\[ = \exp \left[ -\frac{1}{2} \sum_{i=1}^{k} n(X_i - \theta) (X_i - \theta)^{'} \right] \quad \text{(2.2)} \]

The likelihood ratio test is based on whether \( \lambda(\cdot, \lambda(-)) \)

is too small or not. Evaluation of (2.4) is equivalent to examining the following statistic

\[ \tilde{\chi}^2 = \max \{ \chi^2(+), \chi^2(-) \} \quad \text{(2.5)} \]

where

\[ \chi^2(+) = n \{ \bar{X}^{'} A^{-1} \bar{X} - \min_{\theta \geq 0} (\bar{X} - \theta)^{'} (\bar{X} - \theta) \} \quad \text{(2.6)} \]

and

\[ \chi^2(-) = n \{ \bar{X}^{'} A^{-1} \bar{X} - \min_{\theta \equiv 0} (\bar{X} - \theta)^{'} (\bar{X} - \theta) \} . \quad \text{(2.7)} \]

We shall call (2.5), (2.6) and (2.7), respectively, \( \chi^2 \)-statistic, \( \chi^2(+) \)-statistic and \( \chi^2(-) \)-statistic.

For the calculation of the probability that \( \chi^2 \) exceeds \( c^2 \), \( \Pr(\chi^2 \geq c^2) \), when the null hypothesis is true, we need the following lemma by Kudô [1], and we shall state them without proof.

With every \( X \) in the space \( R \) of the sample mean vector, we associate a point \( X^* \), termed the minimum point of

\[ \min_{m \in M, \theta} (X - m)^{'} A^{-1} (X - m) = (X - X^*)^{'} A^{-1} (X - X^*) , \quad \text{(2.8)} \]

where \( A \) is non-singular known matrix and \( m' = (m_1, \ldots, m_k) \). We divide \( R \) into \( 2^k \) disjoint subsets, \( R = \bigcup_{\phi \subseteq M \subseteq K} R_M \), where \( R_M \) is the totality of points \( X \) such that

\[ x_i^* = 0 \quad (i \in M) , \quad x_i^* > 0 \quad (i \in M) , \quad \text{(2.9)} \]

where \( X^* = (x_1^*, x_2^*, \ldots, x_k^*) \) is the minimum point of (2.8) associated with \( X \). Let

\[ \bar{U}^2(+) = \{ X^{'} A^{-1} X - \min_{m \in M, \theta} (X - m)^{'} A^{-1} (X - m) \} . \quad \text{(2.10)} \]

**Lemma (Kudô [1]).** Let \( X_1, \ldots, X_k \) be distributed in a multivariate normal distribution with mean \( m' = (m_1, \ldots, m_k) \) and variance-covariance matrix \( A \). The probability that the value of \( \bar{U}^2(+) \)-statistic, computed taking the null hypothesis as \( H_0: m_i = 0 \) \((i = 1, \ldots, k)\) and the alternative as \( H(+) : m_i \geq 0 \) \((i = 1, \ldots, k)\) where the inequality is strict for at least one value of \( i \), exceeds \( \tilde{\alpha}_0 \) is given by

\[ \Pr(\bar{U}^2(+) \geq \tilde{\alpha}_0) = \sum_{\phi \subseteq M \subseteq K} \Pr(\chi^2(M \subseteq \phi \geq 0) \Pr((A_M)^{-1}) \Pr(A_M, m') \quad \text{(2.11)} \]

where the summation runs over all the subsets \( M \) of \( K = \{1, \ldots, k\} \), \( n(M) \) is the number
of elements in $M, M'$ is the complement of $M, A_M$ is the variance matrix of $x_i, i \in M, A_{M',N}$ is the same under the condition $x_j = 0, j \in M$, and $P\{\Sigma\}$ is the probability that the variables distributed in a multivariate normal distribution with means zero and variance-covariance matrix $\Sigma$ are all positive. $\chi^2_{n(M)}$ has the chi-square distribution with n(M) degrees of freedom, where $\chi^2_{n(\phi)}$ is to be understood as a constant zero, $P(A_0, K) = 1$ and $P((A_K)^{-1}) = P((A_0)^{-1}) = 1$.

By using of the Lemma we shall have the following.

**Theorem.** Let $(X_1, \ldots, X_k)$ be distributed in a multivariate normal distribution with mean $\theta' = (\theta_1, \ldots, \theta_k)$, and variance-covariance matrix $I$. The probability that the $\chi^2$-statistic defined in (2.5), exceeds $c^2$ is given by

$$
Pr(\chi^2 \geq c^2) = \frac{1}{2^{k-1}} \sum_{m=1}^{k} C_{m} Pr(\chi^2_{m} \geq c^2) - \frac{1}{2^{k}} \sum_{m=1}^{k-1} C_{m} Pr(\chi^2_{m} \geq c^2) Pr(\chi^2_{k-m} \geq c^2),
$$

where $\chi^2_{m}$ has the chi-square distribution with $m$ degrees of freedom.

**Proof.** Because of the symmetry, the distributions of $\chi^2(+) \text{ and } \chi^2(-)$ are the same (Kudo [3]). The distribution of $\chi^2$ is thus

$$
Pr(\chi^2 \geq c^2) = Pr[\max(\chi^2(+), \chi^2(-)) \geq c^2] = 2Pr(\chi^2(+) \geq c^2) - Pr(\chi^2(+) \geq c^2, \chi^2(-) \geq c^2) = 2 \times \frac{1}{2^k} \sum_{m=1}^{k} C_{m} Pr(\chi^2_{m} \geq c^2) - \frac{1}{2^k} \sum_{m=1}^{k-1} C_{m} Pr(\chi^2_{m} \geq c^2, \chi^2_{k-m} \geq c^2) = \frac{1}{2^{k-1}} \sum_{m=1}^{k-1} C_{m} Pr(\chi^2_{m} \geq c^2) - \frac{1}{2^k} \sum_{m=1}^{k-1} C_{m} Pr(\chi^2_{m} \geq c^2, \chi^2_{k-m} \geq c^2).
$$

§ 3. Table.

The table provides the values of the percentage points $k = 2(1)15$. The values are expected to be correct up to 4 decimal places.

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§ 4. Acknowledgements.

The author is deeply indebted to: Professor T. Kitagawa for his encouragement and advice; and Professor A. Kudô for his valuable suggestions and guidances. The author's thanks are also to Miss Yasuko Fujita and Miss Yasuko Tanaka for their help in computation.

References