

GENERALIZATION OF NEYMAN-PEARSON FUNDAMENTAL LEMMA TO VECTOR-VALUED DECISION FUNCTIONS AND ITS APPLICATION TO UNBIASED SLIPPAGE TESTS

Yeh, Neng-Che
Tamkang College of Arts and Sciences | Kyushu University

<https://doi.org/10.5109/13032>

出版情報 : 統計数理研究. 13 (1/2), pp.73-83, 1968-03. Research Association of Statistical
Sciences
バージョン :
権利関係 :



GENERALIZATION OF NEYMAN-PEARSON FUNDAMENTAL LEMMA TO VECTOR-VALUED DECISION FUNCTIONS AND ITS APPLICATION TO UNBIASED SLIPPAGE TESTS

By

Neng-che YEH

(Received January 30th, 1968)

§ 1. Introduction.

Slippage problems have been discussed by various authors such as Mostellar [7], Paulson [9], Traux [10], Doornbos and Prins [1], Kudô [5], Karlin and Traux [4], and others.

In particular a class of slippage problems has been mainly treated in a manner analogous to the treatment of problems in hypotheses testing (Lehmann [9]). In general, the problem is as follows. Assuming that we have a populations with densities $p(x; \theta_i)$ ($i=1, \dots, a$) and we wish to test the hypothesis $H_0: \theta_1 = \dots = \theta_a$ against a alternatives $H_i: \theta_1 = \dots = \theta_i - \Delta = \dots = \theta_a$ with a zero-one type loss function, where $\Delta > 0$.

Hall and Kudô [2] gave a generalization of Neyman-Pearson lemma so as to be useful in solving slippage problems where the null hypothesis is simple and the alternatives are simple. They also applied their generalized lemma to slippage problems where the alternatives are composite for the population distribution for which satisfying certain conditions closely connected with monotone likelihood ratio. The existence of an uniformly most powerful test under some conditions was also given. Furthermore, Hall, Kudô and Yeh [3] treated a similar test for the slippage problem.

In this paper we shall start with a new generalization of Neyman-Pearson lemma with respect to vector-valued decision function which will be given in Theorem 1 in section 2. This Theorem 1 is crucially important for establishing unbiased slippage test as we shall show in section 3. In view of these results the author proceed to give another generalization of Neyman-Pearson lemma which will be useful in treating a certain type of testing hypothesis whose formulation is given in section 4. This certain type of testing hypothesis is: under the hypothesis H_0 , a random variable X has a density equal to one of the densities $h_i(x)$ ($i=1, \dots, k$), while each of its alternatives H_i has the density $h_{k+i}(x)$ ($i=1, \dots, a$). For the particular case, i. e. $a=k$, is given in section 5.

§ 2. A generalization of the fundamental lemma to vector-valued decision functions (I).

Let \mathfrak{X} be a Euclidean space and \mathfrak{A} be a σ -field of subsets of \mathfrak{X} with a measure μ on \mathfrak{A} . Let $f_1, f_2, \dots, f_a, h_0, h_1, \dots, h_a$ be real-valued functions defined on the space \mathfrak{X} and integrable μ . Suppose that for a given constants c_0 and c_i ($i=1, 2, \dots, a$) there exists a vector-valued decision function $\varphi(x) = (\varphi_0(x), \varphi_1(x), \dots, \varphi_a(x))$ with $\sum_{i=0}^a \varphi_i(x) = 1$, $0 \leq \varphi_i \leq 1$ for $i=0, 1, \dots, a$, satisfying

$$(2.1) \quad \int \varphi_0(x) h_0(x) d\mu(x) = c_0 \quad \text{and} \quad \int \varphi_i(x) f_i(x) d\mu(x) = c_i$$

for $i=1, 2, \dots, a$.

Denote by C the class of vector-valued decision functions φ for which (2.1) holds. Unless otherwise explicitly expressed, all integrals \int considered in the present and following sections are $\int_{\mathfrak{X}}$. Then we have the following theorem.

THEOREM 1.

- (i) *Existence.* $\text{Max}_{\varphi \in C} \left[\sum_{i=1}^a \int \varphi_i(x) h_i(x) d\mu(x) \right]$ exists.
- (ii) *Sufficiency.* If a decision function φ in C satisfies

$$(2.2) \quad \begin{aligned} \varphi_0(x) &= 1, \xi(x), 0 \quad \text{when} \quad k_0 h_0 >, =, < \max_i [(k_i f_i + h_i)], \\ \varphi_j(x) &= \eta_j(x), 0 \quad \text{when} \quad k_j f_j + h_j =, < \max_i [(k_i f_i + h_i)], \end{aligned}$$

where $\xi(x)$ and $\eta_j(x)$ are arbitrary, subject to the decision function φ , for some k_0 and k_i ($i=1, 2, \dots, a$), then this φ maximizes

$$(2.3) \quad \sum_{i=1}^a \int \varphi_i(x) h_i(x) d\mu(x)$$

among C .

(iii) The set D of points in $(a+1)$ -dimensional space whose coordinates are $(\int \varphi_0 h_0 d\mu, \int \varphi_1 f_1 d\mu, \dots, \int \varphi_a f_a d\mu)$ for some decision function φ is convex and closed.

(iv) *Necessity.* A decision function in C satisfying (2.2) is attainable, provided (c_0, c_1, \dots, c_a) is an inner point of D , and if a member of C maximizes (2.3) then for some k_0 and k_i , it satisfies (2.2) a.e. μ .

PROOF. (i) Consider a sequence $\{\varphi^{(n)}(x)\}$ of the set C such that $\sum_{i=1}^a \int \varphi_i^{(n)}(x) h_i(x) d\mu(x)$ tends to $\text{Sup}_{\varphi \in C} \left[\sum_{i=1}^a \int \varphi_i(x) h_i(x) d\mu(x) \right]$. By the generalization of the weak compactness theorem to vector-valued functions (Kudô [6]), there is a subsequence $\{\varphi^{(n_j)}(x)\}$ of $\{\varphi^{(n)}(x)\}$ and $\varphi^0(x)$ such that

$$(2.4) \quad \int \varphi_i^{(n_j)} f_i(x) d\mu(x) \xrightarrow{n_j} \int \varphi_i^0 f_i(x) d\mu(x) \quad (i=0, 1, \dots, a)$$

for all integrable function $f(x)$ with respect to μ . Since

$$(2.5) \quad \int \varphi_0^{(n_j)}(x) h_0(x) d\mu(x) \xrightarrow{n_j} \int \varphi_0^{(0)}(x) h_0(x) d\mu(x)$$

$\varphi^{(0)}(x)$ belongs to C . On the other hand,

$$(2.6) \quad \sum_{i=1}^a \int \varphi_i^{(n_j)}(x) h_i(x) d\mu(x) \xrightarrow{n_j} \sum_{i=1}^a \int \varphi_i^{(0)}(x) h_i(x) d\mu(x).$$

By the uniqueness,

$$(2.7) \quad \sum_{i=1}^a \int \varphi_i^{(0)}(x) h_i(x) d\mu(x) = \sup_{\varphi \in C} \left[\sum_{i=1}^a \int \varphi_i(x) h_i(x) d\mu(x) \right].$$

Hence $\max_{\varphi \in C} \left[\sum_{i=1}^a \int \varphi_i(x) h_i(x) d\mu(x) \right]$ exists.

(ii) Let φ in C satisfy (2.2) and $\hat{\varphi}$ be any other one in C . We consider

$$\sum_{i=1}^a \int \varphi_i(x) h_i(x) d\mu(x) - \sum_{i=1}^a \int \hat{\varphi}_i(x) h_i(x) d\mu(x) \equiv \int g(x) d\mu(x),$$

where

$$(2.8) \quad g(x) = \left[\varphi_0 k_0 h_0 + \sum_{i=1}^a \varphi_i (k_i f_i + h_i) \right] - \left[\hat{\varphi}_0 k_0 h_0 + \sum_{i=1}^a \hat{\varphi}_i (k_i f_i + h_i) \right].$$

We divide the sample space \mathfrak{X} into three disjoint regions:

$$\{x: \varphi_0(x) = 1\}, \quad \{x: \varphi_0(x) = \xi(x)\} \quad \text{and} \quad \{x: \varphi_0(x) = 0\}.$$

We shall verify that the $g(x) \geq 0$ in each of these regions.

For $\{x: \varphi_0(x) = 1\}$, we have

$$g(x) \geq (1 - \hat{\varphi}_0) k_0 h_0 - \max_i (k_i f_i + h_i) \sum_{i=1}^a \hat{\varphi}_i > (1 - \hat{\varphi}_0) k_0 f_0 - k_0 h_0 \sum_{i=1}^a \hat{\varphi}_i = 0,$$

for $\{x: \varphi_0(x) = \xi(x)\}$,

$$g(x) \geq (\xi - \hat{\varphi}_0) k_0 h_0 + \sum_{i=1}^a \varphi_i (k_i f_i + h_i) - \max_i (k_i f_i + h_i) \sum_{i=1}^a \hat{\varphi}_i = k_0 h_0 - k_0 h_0 \sum_{i=1}^a \hat{\varphi}_i = 0.$$

And by the same argument we have $g(x) \geq 0$ for x in $\{x: \varphi_0(x) = 0\}$. Therefore

$$(2.9) \quad \sum_{i=0}^a \int \varphi_i(x) h_i(x) d\mu(x) \geq \sum_{i=1}^a \int \hat{\varphi}_i(x) h_i(x) d\mu(x).$$

The assertion of (ii) is thus proved.

(iii) The closedness of the set D follows from the generalization of the weak compactness theorem to vector-valued decision functions and the convexity of the set D follows from

$$\sum_{i=0}^a [\lambda \varphi_i + (1 - \lambda) \hat{\varphi}_i] = 1, \quad 0 \leq \lambda \leq 1.$$

We note that the set E of points in $(a+2)$ -dimensional space whose coordinates are $\left(\int \varphi_0 h_0 d\mu, \int \varphi_1 f_1 d\mu, \dots, \int \varphi_a f_a d\mu, \sum_{i=1}^a \varphi_i h_i d\mu \right)$ for some decision function is also convex and closed. The points of E , the first $(a+1)$ coordinates of which are c_0, c_1, \dots, c_a , form a closed interval $[c^*, c^{**}]$. Since $(c_0, c_1, \dots, c_a, c^{**})$ is a boundary point of E , there is a hyperplane through it such that every point of E lies below or on this hyperplane. Let us write the equation of the hyperplane

$$k_0 \int \varphi_0 h_0 d\mu + \sum_{i=1}^a k_i \int \varphi_i f_i d\mu + k_{a+1} \int \sum_{i=1}^a \varphi_i h_i d\mu = \sum_{i=0}^a k_i c_i + k_{a+1} c^{**}.$$

Since (c_0, c_1, \dots, c_a) is an inner point of D , the $k_{a+1} \neq 0$ and

$$(2.10) \quad \int \left[\sum_{i=1}^a (h_i + k_i f_i) \varphi_i + k_0 h_0 \varphi_0 \right] d\mu \leq \int \left[\sum_{i=1}^a (h_i + k_i f_i) \varphi_i^{**} + k_0 h_0 \varphi_0^{**} \right] d\mu$$

for all points in E , where φ^{**} is the decision function given rise to the point $(c_0, c_1, \dots, c_a, c^{**})$. Thus φ^{**} maximizes the left-hand side of (2.10). Since the integral is maximized by putting φ as the form of (2.2), φ^{**} satisfies (2.2) a.e. μ the assertions (iii) and (iv) are thus proved.

In the present section we restrict $h_i(x)$ ($i=0, 1, \dots, a$) to be probability density of X with respect to μ , and denote $\int \varphi_i(x) h_i(x) d\mu(x)$ by $E_i \varphi_i(X)$ ($i=0, 1, \dots, a$).

DEFINITION 1. A decision function $\varphi(x) = (\varphi_0(x), \varphi_1(x), \dots, \varphi_a(x))$ is of size α if

$$(2.11) \quad E_0 \varphi_0(X) \geq 1 - \alpha.$$

It is of exact size α if the equality holds.

DEFINITION 2. $\varphi(x)$ is symmetric in power if

$$(2.12) \quad E_1 \varphi_1(X) = \dots = E_a \varphi_a(X).$$

The common value of (2.12) is called the power of φ . It is called the most powerful symmetric of size α (MPSS α) if it maximizes each term of (2.12) subject to (2.11) and (2.12).

Throughout this paper, we consider a measurable transformation group G on \mathfrak{X} , which is isomorphic to the permutation group, $\Pi = \{\pi_g\}$ on $(1, 2, \dots, a)$ or its transitive subgroup.

DEFINITION 3. $\varphi(x)$ is invariant under the group $G = \{g\}$ if

$$(2.13) \quad \varphi_0(x) = \varphi_0(gx) \quad \text{and} \quad \varphi_i(x) = \varphi_{\pi_g i}(gx) \quad \text{for all } g \text{ and } i,$$

where π_g is a permutation on $(1, 2, \dots, a)$ corresponding to g .

Now we assume that there is a measurable transformation group G on \mathfrak{X} , which is isomorphic to the permutation group, π , or to its transitive subgroup, such that $f_i(x) = f_{\pi_g i}(gx)$, $h_i(x) = h_{\pi_g i}(gx)$ and $h_0(x) = h_0(gx)$ for all g and i , and $\mu(A) = \mu(gA)$ for all $A \in \mathfrak{A}$ and g . Then the following Corollary follows.

COROLLARY 1. Consider the class of decision functions satisfying

$$(2.14) \quad E_0 \varphi_0(X) = 1 - \alpha \quad \alpha \in (0, 1) \quad \text{and} \quad \int \varphi_i(x) f_i(x) d\mu(x) = c \quad \text{for all } i.$$

Let C_α be the class of φ for which (2.14) holds.

(i) Given any α , the most powerful symmetric of size α (MPSS α) decision function exists among C_α , furthermore, it is invariant.

(ii) If $\varphi \in C_\alpha$ satisfies

$$(2.15) \quad \begin{aligned} \varphi_0(x) &= 1, \xi, 0 && \text{when } k_0 h_0 >, =, < \max_i [(k f_i + h_i)], \\ \varphi_j(x) &= \frac{1 - \varphi_0(x)}{J(x)}, 0, && \text{when } k f_j + h_j =, < \max_i [(k f_i + h_i)], \end{aligned}$$

where $J(x)$ is the number of times that $\max_i [(k f_i + h_i)]$ is attained, for some k_0 and k , then it is a MPSS α decision function among C_α .

(iii) k_0 and k in (ii) is attainable for $\varphi \in C_\alpha$ if the point $(1-\alpha, c, \dots, c)$ is an inner point of the set D , and if φ is MPSS α among C_α then it satisfies (2.15) a.e. μ .

PROOF. (i) By (i) of Theorem 1, $\text{Max}_{\varphi \in C_\alpha} \left[\sum_{i=1}^a E_i \varphi_i(X) \right]$ exists, say, $\hat{\varphi}(x)$. As the group G is finite, there is a right invariant probability measure ν on (G, β) measurable space, where β is a σ -field of subsets of G . We may define φ as

$$(2.16) \quad \hat{\varphi}_i(x) = \int \hat{\varphi}_{\pi_{gi}}(gx) d\nu(g) \quad i = 0, 1, \dots, a.$$

This $\hat{\varphi}$ is invariant and belongs C_α . We see that

$$\begin{aligned} \sum_{i=1}^a E_i \hat{\varphi}_i(X) &= \sum_{i=1}^a \iint \hat{\varphi}_{\pi_{gi}}(gx) h_i(x) d\mu(x) d\nu(g) \\ &= \int \left[\sum_{i=1}^a \int \hat{\varphi}_{\pi_{gi}}(gx) h_{\pi_{gi}}(gx) d\mu(gx) \right] d\nu(g) \\ &= \int \left[\sum_{i=1}^a E_{\pi_{gi}} \hat{\varphi}_{\pi_{gi}}(X) \right] d\nu(g) \\ &= \int \left[\sum_{i=1}^a E_i \hat{\varphi}_i(X) \right] d\nu(g) = \sum_{i=1}^a E_i \hat{\varphi}_i(X). \end{aligned}$$

Hence $\hat{\varphi}$ is MPSS α among C_α .

(ii) The proof is analogous to that of (ii) of Theorem 1, we shall omit it.

(iii) By (iii) of Theorem 1, there are k_0, k_1, \dots, k_a . We note that the (2.10) of Theorem 1 implies

$$\begin{aligned} (2.17) \quad & \int \left[\sum_{i=1}^a (h_{\pi_{gi}} + k_i f_{\pi_{gi}}) \varphi_{\pi_{gi}} + k_0 h_0 \varphi_0 \right] d\mu \\ & \leq \int \left[\sum_{i=1}^a (h_i + k_i f_i) \varphi_i^{**} + k_0 h_0 \varphi_0^{**} \right] d\mu \quad \text{for all } g \in G. \end{aligned}$$

We can see this by putting $\psi_i(g^{-1}x) = \varphi_{\pi_{gi}}(x)$ and $\psi_0(g^{-1}x) = \varphi_0(x)$. Then by (ii) we have

$$\begin{aligned} (2.18) \quad & \int \left[\sum_{i=1}^a \left(h_i + \frac{k_1 + \dots + k_a}{a} f_i \right) \varphi_i + k_0 h_0 \varphi_0 \right] d\mu \\ & \leq \int \left[\sum_{i=1}^a (h_i + k_i f_i) \varphi_i^{**} + k_0 h_0 \varphi_0^{**} \right] d\mu. \end{aligned}$$

Thus the assertion is proved by putting $k_1 + \dots + k_a = k$.

§ 3. Unbiasedness on slippage test.

Let θ be a parameter in a parameter space Ω and X a random variable with probability density $p_\theta(x)$ with respect to a σ -additive measure μ on a σ -field of subsets of a sample space \mathfrak{X} . Suppose the parameter space is divided into $a+1$ disjoint subsets $\Omega_0, \Omega_1, \dots, \Omega_a$. We shall consider a decision problem involving $a+1$ possible decisions denoted by D_i ($i = 0, 1, \dots, a$) corresponding to $a+1$ hypotheses $H_0: \theta \in \Omega_0$,

$H_1: \theta \in \Omega_1, \dots, H_a: \theta \in \Omega_a$, respectively, where under H_0 , X has density equal $p_\theta(x)$, $\theta \in \Omega_0$ and under H_i the density $p_\theta(x)$, $\theta \in \Omega_i$ for $i=1, \dots, a$. The vector-valued decision functions of the form $\varphi(x) = (\varphi_0(x), \varphi_1(x), \dots, \varphi_a(x))$ with $\varphi_i(x)$ denotes the probability of marking D_i for $i=0, 1, \dots, a$, as the random variable X taking value x , are considered. Of course we have

$$\sum_{i=0}^a \varphi_i(x) = 1, \quad 0 \leq \varphi_i \leq 1.$$

Because of similarity to the theory of a testing hypothesis, we shall call H_0 the null hypothesis, H_i ($i=1, \dots, a$) the alternative, and this problem is called hereafter a test of H_0 against H_i for $i=1, \dots, a$.

DEFINITION. A decision function $\varphi(x)$ is unbiased of size α if

$$(3.1) \quad E_{\theta_0} \varphi_1(X) = \dots = E_{\theta_0} \varphi_a(X), \quad \sum_{i=1}^a E_{\theta_0} \varphi_i(X) \leq \alpha \quad \text{for } \theta_0 \in \Omega_0$$

and

$$(3.2) \quad \sum_{i=1}^a E_{\theta_i} \varphi_i(X) \geq \alpha \quad \text{for } \theta_i \in \Omega_i, \quad i=1, \dots, a.$$

Let X be a random variable with probability density

$$(3.3) \quad p_\theta(x) = C(\theta) \exp [\theta' T(x)]$$

with respect to μ , where θ and $T(x)$ both are the points in a -dimensional Euclidean space, i.e. $\theta' = (\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(a)})$ and $T'(x) = (T_1(x), \dots, T_a(x))$. Assume that the parameter space is divided into a curves and a point which are given by $\theta = \theta_i(\tau)$, $-\infty < \tau < \infty$, $i=1, \dots, a$ and a common starting point $\theta_0 = \theta_i(0)$. We also assume that there are transformation groups $G = \{g\}$ and $\Pi = \{\pi_g\}$ as stated in the previous section such that $p_{\theta_0}(x) = p_{\theta_0}(gx)$, $p_{\theta_i(\tau)}(x) = p_{\theta_{\pi_g i(\tau)}(gx)}$ and $T_i(x) = T_{\pi_g i}(gx)$ for all $i, \tau, g \in G$. Then we have the following theorem.

THEOREM 2. Let $\theta_i(\tau) = (\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(a)} + \tau, \theta^{(1)}, \dots, \theta^{(a)})$, where $\theta^{(i)} + \tau$ is the i -th coordinate of the point $\theta_i(\tau)$, and $-\infty < \tau < \infty$. Suppose $C(\theta_i(\tau))$ depends upon τ only. Then the following φ constitutes a MPSS α unbiased test uniformly in τ for testing

$$H_0: \theta = \theta_0 = \theta_i(0) \text{ (known)}$$

against

$$H_i: \theta = \theta_i(\tau), \quad -\infty < \tau < \infty, \quad \tau \neq 0 \quad i=1, \dots, a.$$

The φ is given by

$$(3.4) \quad \begin{aligned} \varphi_0(x) &= 1 \quad \text{when both } \min_i T_i > K_0 \text{ and } \max_i T_i < K, \quad K_0 < K \\ &= 0 \quad \text{when } \min_i T_i < K_0 \text{ or } \max_i T_i > K \\ &= \hat{\xi} \quad \text{when both } \min_i T_i = K_0 \text{ and } \max_i T_i \leq K \text{ or both} \\ &\quad \min_i T_i \geq K_0 \text{ and } \max_i T_i = K \\ \varphi_j(x) &= \frac{1 - \varphi_0(x)}{J(x)} \quad \text{when } T_j = \max_i T_i \geq K \text{ or } T_j = \min_i T_i \leq K_0 \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

where $J(x)$ is the number of times that $\max T_i$, in addition $\min T_i$ are attained. K 's and ξ are determined by

$$(3.5) \quad E_{\theta_0} \varphi_0(X) = 1 - \alpha \quad \text{and} \quad E_{\theta_0} [T_i(X) \varphi_i(X)] = \frac{\alpha}{a} E_{\theta_0} [T_i(X)] \quad \text{for all } i.$$

PROOF. By Theorem 9 of Chap. 2 in Lehman [8], $E_{\theta_i} \varphi_i(X)$ is analytic function of $\theta^{(j)}$ ($j=1, \dots, a$), where $\theta^{(j)}$ is the j -th coordinate of the vector θ_i therefore unbiasedness implies

$$E_{\theta_0} \varphi_0(X) = 1 - \alpha \quad \text{and} \quad E_{\theta_0} [T_i(X) \varphi_i(X)] = E_{\theta_0} [\varphi_i(X)] E_{\theta_0} [T_i(X)] \quad \text{for all } i.$$

We note that

$$E_{\theta_0} [T_i(X)] = E_{\theta_{\pi_{g_0}}} [T_{\pi_{g_0} i}(gx)] = E_{\theta_0} [T_j(x)] \quad \text{and} \quad E_{\theta_0} [\varphi_i(X)] = E_{\theta_0} [\varphi_i(X)].$$

Hence, by the corollary 1 in § 2, there is a MPSS α test for each τ , given by

$$\varphi_0(x) = 1, \xi, 0 \quad \text{when } k_0(\tau) p_{\theta_0}(x) >, =, < \max_i [k(\tau) T_i(x) p_{\theta_0}(x) + p_{\theta_i}(x)],$$

$$\varphi_j(x) = \frac{1 - \varphi_0(x)}{J(x)}, 0$$

$$\text{when } k(\tau) T_j(x) p_{\theta_0}(x) + p_{\theta_j}(x) =, < \max_i [k(\tau) T_i(x) p_{\theta_0}(x) + p_{\theta_i}(x)].$$

We note that $p_{\theta_i(\tau)}(x) = \frac{c(\theta_i(\tau))}{c(\theta_0)} p_{\theta_0}(x) e^{\tau T_i(x)}$, then (3.4) immediately follows.

Example. Let $X = (X_1, X_2)$ be distributed according to the bivariate normal distribution with mean $\theta = (\theta^{(1)}, \theta^{(2)})$ and covariance matrix I . Suppose the parameter space is $\Omega = \{(0, 0), (\tau, 0), (0, \tau); \tau \neq 0\}$. Then a most powerful symmetric of size α unbiased test uniformly in τ is given by

$$\begin{aligned} \varphi_0(x) &= 1 \quad \text{when } -c < x_1, x_2 < c \\ &= 0 \quad \text{when } \min(x_1, x_2) < -c \text{ or } \max(x_1, x_2) > c \\ &= \xi \quad \text{when } (x_1, x_2) \text{ is on the side of the square with center} \\ &\quad \text{at the origin and side length } 2c. \end{aligned}$$

$$\begin{aligned} \varphi_j(x) &= \frac{1 - \varphi_0(x)}{J(x)} \quad \text{when } x_j = \max(x_1, x_2) \geq c \text{ or } x_j = \min(x_1, x_2) \leq -c \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

c is chosen such that $p_r[-c < x_1, x_2 < c | (0, 0)] = 1 - \alpha$.

§ 4. A generalization of the fundamental lemma to vector-valued decision functions (II).

Let \mathfrak{X} be a sample space and let \mathfrak{H} be a σ -field of subsets of \mathfrak{X} with a countable number of generators. Suppose we have $k+a$ densities $h_i(x)$, $i=1, \dots, k, k+1, \dots, k+a$, with respect to a σ -additive measure μ on \mathfrak{H} . We shall consider a decision problem involving $a+1$ decisions D_0, D_1, \dots, D_a corresponding to $a+1$ hypotheses H_0, H_1, \dots, H_a , respectively, where under H_0 , X has a density equal to one of the $h_i(x)$ ($i=1, 2, \dots, k$) and under H_i the density $h_{k+i}(x)$ ($i=1, 2, \dots, a$). We shall denote by $E_i(\cdot)$ the expectation of (\cdot) when the X have probability density $h_i(x)$.

THEOREM 3. Let $C_0 = \{\varphi(x); E_i \varphi_0(X) \geq c_0, i = 1, \dots, k\}$.

(i) $\sup_{\varphi \in C_0} \sum_{i=1}^a E_{k+i} \varphi_i(X)$ is attained by some one in C_0 .

(ii) If a decision function $\varphi(x)$ satisfies $E_i \varphi_0(X) = c_0$ for $i = 1, \dots, k$ with the form

$$(4.1) \quad \begin{aligned} \varphi_0(x) &= 1, \xi(x), 0 \quad \text{when} \quad \max_{i=1, \dots, a} h_{k+i}(x) <, =, > M_0 \cdot \frac{1}{k} \cdot \sum_{i'=1}^k h_{i'}(x) \\ \varphi_j(x) &= \eta_j(x), 0 \quad \text{when} \quad h_{k+j}(x) =, < \max_{i=1, \dots, a} h_{k+i}(x), \end{aligned}$$

where $\xi(x)$ and $\eta_j(x)$ are arbitrary, subject to the decision function φ and $M_0 (\geq 0)$ is a constant.

Then

$$(4.2) \quad \sum_{i=1}^a E_{k+i} \varphi_i(X) \geq \sum_{i=1}^a E_{k+i} \hat{\varphi}_i(X)$$

for any other $\hat{\varphi}(x)$ in C_0 .

PROOF. (i) Consider a sequence $\{\varphi^{(n)}(x)\}$ of the set C_0 such that $\sum_{i=1}^a E_{k+i} \varphi_i^{(n)}(X)$ tends to $\sup_{\varphi \in C_0} \sum_{i=1}^a E_{k+i} \varphi_i(X)$. By the generalization of the weak compactness theorem to vector-valued functions (Kudô [6]), there is a subsequence $\{\varphi^{(n_j)}(x)\}$ of $\{\varphi^{(n)}(x)\}$ and $\varphi^{(0)}(x)$ such that

$$(4.3) \quad \int \varphi_i^{(n_j)} f(x) d\mu(x) \rightarrow \int \varphi_i^{(0)} f(x) d\mu(x)$$

for all integrable $f(x)$ w.r.t. μ . Since

$$(4.4) \quad \int \varphi_0^{(n_j)}(x) h_i(x) d\mu(x) \rightarrow \int \varphi_0^{(0)}(x) h_i(x) d\mu(x) \quad \text{for } i = 1, \dots, k,$$

$\varphi^{(0)}(x)$ belongs to C_0 . On the other hand,

$$(4.5) \quad \sum_{i=1}^a E_{k+i} \varphi_i^{(n_j)}(X) \rightarrow \sup_{\varphi \in C_0} \sum_{i=1}^a E_{k+i} \varphi_i(X).$$

Hence $\varphi^{(0)}(x)$ maximizes $\sum_{i=1}^a E_{k+i} \varphi_i(X)$ among C_0 .

(ii) Consider the integral $\int g(x) d\mu(x)$, where

$$(4.6) \quad g(x) = (\varphi_0(x) - \hat{\varphi}_0(x)) M_0 \cdot \frac{1}{k} \cdot \sum_{i=1}^k h_i(x) + \sum_{i=1}^a (\varphi_i(x) - \hat{\varphi}_i(x)) h_{k+i}(x),$$

$\varphi(x)$ satisfies $E_i \varphi_0(x) = c_0$ ($i = 1, \dots, k$) with the form (4.1) and $\hat{\varphi}(x)$ is any other decision function in C_0 .

We divide the sample space \mathfrak{X} into three disjoint regions: $\{x: \varphi_0(x) = 1\}$, $\{x: \varphi_0(x) = \xi(x)\}$ and $\{x: \varphi_0(x) = 0\}$. It will be verified that $g(x) \geq 0$ in each of these regions. For $\{x: \varphi_0(x) = 1\}$,

$$\begin{aligned} g(x) &= (1 - \hat{\varphi}_0(x)) M_0 \cdot \frac{1}{k} \cdot \sum_{i=1}^k h_i(x) - \sum_{i=1}^a \hat{\varphi}_i(x) h_{k+i}(x) \\ &\geq (1 - \hat{\varphi}_0(x)) \left(M_0 \cdot \frac{1}{k} \cdot \sum_{i=1}^k h_i(x) - M_0 \cdot \frac{1}{k} \cdot \sum_{i'=1}^k h_{i'}(x) \right) = 0, \end{aligned}$$

for $\{x: \varphi_0(x) = \xi(x)\}$,

$$\begin{aligned}
g(x) &\geq (\varphi_0(x) - \hat{\varphi}_0(x))M_0 \cdot \frac{1}{k} \sum_{i'=1}^k h_{i'}(x) + (1 - \varphi_0(x)) \max_{i=1, \dots, a} h_{k+i}(x) - \max_{i=1, \dots, a} h_{k+i}(x) \sum_{i=1}^a \hat{\varphi}_i(x) \\
&= \max_{i=1, \dots, a} h_{k+i}(x) - \left(\hat{\varphi}_0(x) + \sum_{i=1}^a \hat{\varphi}_i(x) \right) \max_{i=1, \dots, a} h_{k+i}(x) = 0,
\end{aligned}$$

and by the same argument $g(x) \geq 0$ for x in $\{x: \varphi_0(x) = 0\}$. Therefore,

$$(4.7) \quad \int_{i=1}^a (\varphi_i(x) - \hat{\varphi}_i(x)) h_{k+i}(x) d\mu(x) \geq \frac{M_0}{k} \int (\hat{\varphi}_0(x) - \varphi_0(x)) \sum_{i'=1}^k h_{i'}(x) d\mu(x) \geq 0.$$

The proof is thus completed.

We shall cite the following definitions for the present section.

DEFINITION 1. A decision function $\varphi(x) = (\varphi_0(x), \varphi_1(x), \dots, \varphi_a(x))$ is of size α for H_0 if

$$(4.8) \quad E_i \varphi_0(X) \geq 1 - \alpha \quad \text{for } i = 1, 2, \dots, k.$$

It is of exact size α for H_0 if the equality holds for all $i = 1, \dots, k$.

DEFINITION 2. $\varphi(x)$ is symmetric in power for H_i if

$$(4.9) \quad E_{k+1} \varphi_1(X) = E_{k+2} \varphi_2(X) = \dots = E_{k+a} \varphi_a(X).$$

The common value of (4.9) is called the power of φ for H_i . $\varphi(x)$ is called the most powerful symmetric of size α (MPSS α) for H_i , if it maximizes each term of (4.9) subject to (4.8) and (4.9).

Now, we assume that there is a measurable transformation group G on \mathfrak{X} such that G is isomorphic to the permutation group $\Pi = \Pi_K \otimes \Pi_A$, where Π_K and Π_A are the permutation groups on $K = (1, 2, \dots, k)$ or its subgroup transitive on $K = (1, \dots, k)$ and on $A = (1, 2, \dots, a)$ or its subgroup transitive on $A = (1, \dots, a)$, respectively. We further assume that $\mu(A) = \mu(gA)$ for all $A \in \mathfrak{A}$ and all $g \in G$, $h_i(x) = h_{\pi_{Kg}i}(gx)$ for all $i = 1, \dots, k$, and all $g \in G$, and $h_{k+i}(x) = h_{k+\pi_{Ag}i}(gx)$ for all $i = 1, \dots, a$ and all $g \in G$, where $\pi_{Ag} \in \Pi_A$ and $\pi_{Kg} \in \Pi_K$, respectively, corresponding to g . Then we have the following corollary.

COROLLARY 2. There is a MPSS α test for testing H_0 against H_i , $i = 1, 2, \dots, a$. It is given by the form (4.1) in Theorem 3 with $\xi(x)$ a constant and $\eta_j(x) = \frac{1 - \varphi_0(x)}{N(x)}$, where $N(x)$ is the number of times that $\max_{i=1, \dots, a} h_{k+i}(x)$ is attained. i.e.,

$$\begin{aligned}
(4.10) \quad \varphi_0(x) &= 1, \xi, 0 \quad \text{when } \max_{i=1, \dots, a} h_{k+i}(x) <, =, > M_0 \cdot \frac{1}{k} \sum_{i'=1}^k h_{i'}(x) \\
\varphi_j(x) &= \frac{1 - \varphi_0(x)}{N(x)}, 0 \quad \text{when } h_{k+j}(x) =, < \max_{i=1, \dots, a} h_{k+i}(x).
\end{aligned}$$

PROOF. Let

$$P_i \left[\left(\max_{i=1, \dots, a} h_{k+i}(x) / \frac{1}{k} \sum_{i'=1}^k h_{i'}(x) \right) \leq M \right] = \alpha_i(M).$$

The probability is computed under P_i ($i = 1, 2, \dots, k$). We shall verify that $\alpha_i(M)$ is independent of i , i.e. $\alpha_i(M) = \alpha(M)$ for $i = 1, 2, \dots, k$. To see this, let

$$\begin{aligned}
\alpha_i(M) &= P_i \left[\left(\max_{i=1, \dots, a} h_{k+i}(x) / \frac{1}{k} \sum_{i'=1}^k h_{i'}(x) \right) \leq M \right] \\
&= \int_{\max_{i=1, \dots, a} h_{k+i}(x) / \frac{1}{k} \sum_{i'=1}^k h_{i'}(x) \leq M} h_i(x) d\mu(x).
\end{aligned}$$

By letting $y = gx$, this is equal to

$$\begin{aligned}
 & \int \max_{i=1, \dots, a} h_{k+\pi_{Ag}i(y)} / \frac{1}{k} \sum_{i'=1}^k h_{\pi_{Kg}i'(y)} \leq M h_i(g^{-1}y) d\mu(g^{-1}y) \\
 &= \int \max_{i=1, \dots, a} h_{k+\pi_{Ag}i(y)} / \frac{1}{k} \sum_{i'=1}^k h_{\pi_{Kg}i'(y)} \leq M h_{\pi_{Kg}i}(y) d\mu(y) \\
 &= \int \max_{i=1, \dots, a} h_{k+i(y)} / \frac{1}{k} \sum_{i'=1}^k h_{i''(y)} \leq M h_{i''}(y) d\mu(y) \\
 &= \alpha_{i''}(M), \quad i'' \in (1, 2, \dots, k).
 \end{aligned}$$

Therefore $\alpha_i(M) = \alpha(M)$ for $i = 1, 2, \dots, k$.

Since $\alpha(M)$ is upper semi-continuous and monotonically increasing

$$P_i \left[\left(\max_{i=1, \dots, a} h_{k+i}(x) / \frac{1}{k} \sum_{i'=1}^k h_{i'}(x) \right) = M \right] = \alpha(M) - \alpha(M-0).$$

For any $\alpha: 0 < \alpha < 1$, there exists an M_0 such that $\alpha(M_0) \leq \alpha \leq \alpha(M_0-0)$. Put $\xi = [(1-\alpha) - \alpha(M_0-0)] / [\alpha(M_0) - \alpha(M_0-0)]$, and consider the decision function $\varphi(x)$ defined by (4.10). The size of $\varphi(x)$ for H_0 is

$$(4.11) \quad E_i \varphi_0(X) = \alpha(M_0-0) + \frac{(1-\alpha) - \alpha(M_0-0)}{\alpha(M_0) - \alpha(M_0-0)} [\alpha(M_0) - \alpha(M_0-0)] = 1 - \alpha.$$

We note that the $\varphi(X)$ is invariant, and hence

$$E_{k+1} \varphi_1(X) = \dots = E_{k+a} \varphi_a(X).$$

By (ii) of Theorem 1, $\varphi(x)$ is obvious a MPSS α test for H_0 against H_i ($i = 1, 2, \dots, a$).

§ 5. Remark.

We can consider a special case when $k = a$, in this case π is assumed to be the permutation group on $(1, 2, \dots, a)$ or its subgroup transitive on $(1, 2, \dots, a)$. Theorem and corollary above also hold in this case when $\pi_g i = \pi_{Kg} i = \pi_{Ag} i$ for all $i = 1, \dots, k$.

§ 6. Acknowledgements.

The author is deeply indebted to Professor T. Kitagawa and Professor A. Kudô for their valuable suggestions and guidances. The author is now at Kyushu University on leave of absence from the Department of Mathematics, Tamkang College of Arts and Sciences, Taipei, Taiwan China.

Kyushu University and Tamkang
College of Arts and Sciences

References

- [1] Doornbos, R. and Prins, H. J. (1958). *On slippage tests*. Indagationes Mathematicae **29**, 38-55.
- [2] Hall, I. J. and Kudô, A. *On slippage tests—(I) A generalization of Neyman Pearson's lemma*. Iowa State University (1968), submitted to Annals of Mathematical Statistics.

- [3] Hall, I. J., Kudô, A. and Yeh, N. C. *On slippage tests—(II) Similar slippage tests*. Iowa State University and Kyushu University (1968), submitted to *Annals of Mathematical Statistics*.
- [4] Karlin, S. and Traux, D. R. (1960). *Slippage problems*. *Annals of Mathematical Statistics* **31**, 296-324.
- [5] Kudô, A. (1956a). *On the testing of outlying observations*. *Sankaya* **17**, 67-76.
- [6] Kudô, A. (1956b). *On the invariant multiple decision procedures*. *Bulletin of Mathematical Statistics* **6**, 57-68.
- [7] Mostellar, F. (1948). *A k -sample slippage test for an extreme population*. *Annals of Mathematical Statistics* **19**, 58-65.
- [8] Lehmann, E. L. (1959). *Testing statistical hypotheses*. New York, John Wiley and Sons, Inc.
- [9] Paulson, E. (1952). *An optimum solution to the k -sample slippage problem for the normal distribution*. *Annals of Mathematical Statistics* **23**, 610-616.
- [10] Traux, D. R. (1953). *An optimum slippage test for the variance of k normal distributions*. *Annals of Mathematical Statistics* **24**, 669-674.