GENERALISED MEASURE OF DISPERSION AND ANALYSIS OF DISTRIBUTION PATTERN

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https://doi.org/10.5109/13031

出版情報：統計数理研究. 13 (1/2), pp.53-71, 1968-03. Research Association of Statistical Sciences
バージョン：
権利関係：

GENERALISED MEASURE OF DISPERSION AND ANALYSIS OF DISTRIBUTIONAL PATTERN[1]

By

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(Received December 25, 1967)

§ 1. Introduction.

For measuring dispersion of distribution of numbers of individuals in spatial area and analysing their distributional pattern, a certain index of dispersion has been advocated and used by many research workers since Simpson [6] including ecologists and entomologists, and its theoretical aspects have been deeply discussed by an animal ecologist, Morishita [2], [3], [4] and [5]. The author of the present paper has been for these several years in close contact with Morishita in this research area of the problems, and it is the purpose of the present paper to explore certain mathematical principles, which can secure us to establish a unified standpoint in dealing with a generalised measure of dispersion, and which, at the same time, may provide us a certain set of tools as substantial techniques in getting penetrated observations on distributional pattern.

In section 2 we shall introduce three basic notions upon which our formulation of ϕ-dispersions will be established: (1) multistage decomposition system $D^{(m)}(A)$ of the set A; (2) x-variates associated with $D^{(m)}(A)$; (3) d-depth ϕ-dispersion of a function f on the set $A_{i_1 i_2 \cdots i_h}$ with respect to $D^{(d)}(A_{i_1 i_2 \cdots i_h})$ where $A_{i_1 i_2 \cdots i_h}$ is a set belonging to it. In section 3 another essential set-up is introduced which is called as distributional pattern with one parameter generalised exponential type distributions associated with the decomposition $D^{(m)}(A)$. There are also two important notions, homogenity and inheritance, in this connection. A few examples given in section 3 will serve to show that important special cases discussed by Morishita [1] and Smith [5] can be explained in term of these notions in our present formulations, including binomial, non-negative and Poisson distributions as our special examples. The main purpose of this paper is then to evalute the conditional expectation of various ϕ-dispersions.

For this purpose there is a crucial property of the concurrence function ϕ with respect to the assigned additive family of generalised exponential type distributions, which we shall define in section 4 as being reproductive. After these preparations we are now in the position to give the conditional expectation of ϕ-dispersion for such a reproductive ϕ-function. The main result in this section is enunciated in Proposition 6 of which Propositions 4 and 5 are special cases. They show that these conditional expectations of ϕ-dispersion can be obtained in concise forms. For instance

1) This work was done by the grant Tokutei-Kenkyu of the Ministry of Education for the PT (Productivity of Terrestorial Ecosystem) Section of the International Biological Project.

Proposition 3 gives the simple formula (5.1) for the 1-depth ϕ-dispersion in the set A.

In section 6 we discuss partitional pattern of additive parameters in the decomposition system $D^{(m)}(A)$ with particular reference to two extreme cases, equipartition and monopolistic concentration. In section 7 we turn to discuss ϕ-dispersion as function of size variable associated with the decomposition system $D^{(m)}(A)$. An interpolation procedure for getting size variable to penetrate into finer and global structures is explained in virtue of the notions of homogenity and inheritance previously defined. Section 8 is devoted to explain the relations between our ϕ-dispersion and current index of dispersion introduced and discussed by Morishita [2]~[5]. We shall show that the latter is a particular example of the former apart from a certain multiplier which is suited for Poisson distribution. Two stage and three stage distributional patterns are illustrated in order to show the uses of ϕ-dispersions as functions of size variables with particular reference to binomial, negative binomial and Poisson distributions.

These results which are summarized briefly here may be said to give us some tools for theoretical analysis of various empirical results obtained by ecologists for natural and artificial populations. On the other hand any sampling theory regarding the uses of our ϕ-dispersions is not discussed in this paper, and we expect to have another occasion to develop it.

It is also noted that the formulation of the present paper is partly due to that of additive family of exponential type which was discussed by the author [1], whose detailed results are not required for the understanding of the present paper.

§ 2. ϕ-dispersion in multistage distributional pattern.

For introducing a notion of ϕ-dispersions regarding x-variate in a set A, there are three fundamental notions upon which it can be established.

(1) *Multistage decomposition system $D^{(m)}(A)$ of the set A.* A set A is decomposed into the sum of $q_1 q_2 \cdots q_k$ mutually disjoint subsets $A_{i_1 i_2 \cdots i_m}$ ($i_j = 1, 2, \cdots, q_j$; $j = 1, 2, \cdots, m$) in such a way that

(1°) $A = \sum_{i_1=1}^{q_1} A_{i_1}$

(2°) $A_{i_1 i_2 \cdots i_h} = \sum_{i_{h+1}=1}^{q_{h+1}} A_{i_1 i_2 \cdots i_h i_{h+1}}$, $(i_j = 1, 2, \cdots, q_j;\ j = 1, 2, \cdots, h)$

for $h = 1, 2, 3, \cdots, m-1$.

The set $\{A_{i_1 i_2 \cdots i_m}\}$ is said to constitute the (m-stage) decomposition system $D^{(m)}(A)$ of the set A.

The family of q_1 subsets enunciated in (1°) and those of $q_1 q_2 \cdots q_h$ subsets enunciaied in (2°) are said to constitute the first stage decomposition $D_1^{(m)}(A)$ and the $(h+1)$-th stage decomposition $D_{h+1}^{(m)}(A)$ ($h = 1, 2, \cdots, m-1$) respectively. Under m-stage decomposition system $D^{(m)}(A)$ there are imbedded the lower stage decomposition systems $D^{(n)}(A)$ for $n = 1, 2, \cdots, m-1$, and it is immediate to observe that, for each subset $A_{i_1 i_2 \cdots i_h}$, we can define the l-stage decomposition system $D^{(l)}(A_{i_1 i_2 \cdots i_h})$ and its ν-th stage decomposition $D_\nu^{(l)}(A_{i_1 i_2 \cdots i_h})$ for $\nu = 1, 2, \cdots, l$; $l = 1, 2, \cdots, m-h$. In view of these facts, for the sake of convenience, we may and we shall consider the set A as the unique subset of the set A at the zero-th stage decomposition, say, $D_0^{(m)}(A)$, and consider it as an element of the family of the sets $\{A_{i_1 i_2 \cdots i_h}\}$ corresponding to the case $h=0$.

(2) *x-variates associated with an m-stage decomposition system* $D^{(m)}(A)$. A set of $q_1 q_2 \cdots q_m$ non-negative integers $\{x_{i_1 i_2 \cdots i_m}\}$ $(i_j=1, 2, \cdots, q_j;\ j=1, 2, \cdots, m)$ is assumed to be assigned, and the following set of x-variates is defined in correspondence with the decomposition system $D^{(m)}(A)$:

(1°) $x=\sum_{i_1=1}^{q_1} x_{i_1}$

(2°) $x_{i_1 i_2 \cdots i_h}=\sum_{i_{h+1}=1}^{q_{h+1}} x_{i_1 i_2 \cdots i_h i_{h+1}}, \quad (i_j=1, 2, \cdots, q_j;\ j=1, 2, \cdots, h)$

for $h=1, 2, \cdots, m-1$.

(3) *ϕ-dispersions.* A function $f(x)$ is assumed to be defined in the set $\mathfrak{D}_f$ which is a subset of the set of all non-negative integers. Among such functions let us choose a function ϕ with whose reference we introduce

Definition 1. *A d-depth ϕ-measure of dispersion, abbreviated by ϕ-dispersion, of a function f on the set $A_{i_1 i_2 \cdots i_h}$ with respect to the decomposition system $D^{(d)}(A_{i_1 i_2 \cdots i_h})$ is defined as*

$$\phi_d(f)(A_{i_1 i_2 \cdots i_h}) \equiv \frac{1}{\phi(x_{i_1 i_2 \cdots i_h})} \sum_{i_{h+1}=1}^{q_{h+1}} \sum_{i_{h+2}=1}^{q_{h+2}} \cdots \sum_{i_{h+d}=1}^{q_{h+d}} \phi(x_{i_1 \cdots i_h i_{h+1} \cdots i_{h+d}}) f(x_{i_1 \cdots i_{h+d}}), \tag{2.1}$$

for every $i_j=1, 2, \cdots, q_j$; $j=1, 2, \cdots, h$; $h=0, 1, 2, \cdots, m-1$.

Particularly when $f(x)$ becomes to be identical with a value 1 (identity) over the domain $\mathfrak{D}_f$, we shall write

$$\phi_d[A_{i_1 i_2 \cdots i_h}]=\phi_d(1)(A_{i_1 i_2 \cdots i_h}). \tag{2.2}$$

As a direct consequence of Definition 1, we have

Proposition 1. *We have*

$$\phi_{d_1}(\phi_{d_2}(f))(A_{i_1 i_2 \cdots i_h})=\phi_{d_1+d_2}(f)(A_{i_1 i_2 \cdots i_h}). \tag{2.3}$$

Among various ϕ-dispersion of a function f there are specific important cases which are of our particular interests. These occur when at least one of the following three conditions is satisfied: (i) $A_{i_1 i_2 \cdots i_h}=A$, that is, the case when $h=0$ formally; (ii) the function $f(x)$ is identical with 1 for every x in $\mathfrak{D}_f$; (iii) the depth d is equal to 1. For illustration a few examples are given.

Example 2.1. The d-depth dispersion of identity

$$\phi_d(1)(A)=\phi_d[A]=\frac{1}{\phi(x)} \sum_{i_1=1}^{q_1} \sum_{i_2=1}^{q_2} \cdots \sum_{i_d=1}^{q_d} \phi(x_{i_1 i_2 \cdots i_d}) \tag{2.4}$$

and

$$\phi_d(1)(A_{i_1 i_2 \cdots i_h})=\phi_d[A_{i_1 i_2 \cdots i_h}]=\frac{1}{\phi(x_{i_1 i_2 \cdots i_h})} \sum_{i_{h+1}=1}^{q_{h+1}} \cdots \sum_{i_{h+d}=1}^{q_{h+d}} \phi(x_{i_1 \cdots i_h i_{h+1} \cdots i_{h+d}}). \tag{2.5}$$

Example 2.2. The 1-depth dispersion of function f

$$\phi_1(f)(A)=\frac{1}{\phi(x)} \sum_{i_1=1}^{q_1} \phi(x_{i_1}) f(x_{i_1}), \tag{2.6}$$

$$\phi_1(f)(A_{i_1 i_2 \cdots i_h})=\frac{1}{\phi(x_{i_1 \cdots i_h})} \sum_{i_{h+1}=1}^{q_{h+1}} \phi(x_{i_1 \cdots i_h i_{h+1}}) f(x_{i_1 \cdots i_h i_{h+1}}). \tag{2.7}$$

Example 2.3. There are a lot of possibilities in choosing the basic function ϕ in defining our ϕ-dispersion (2.1). The following particular examples are important:

$$\phi^{[j]}(x)=x(x-1)\cdots(x-j+1)=x^{[j]}, \qquad \mathfrak{D}\phi^{[j]}=[x\,;\,x\geqq j] \tag{2.8}$$

for $j\geqq 2$. Indeed the current index of dispersion discussed by Morishita [2]~[4] is in our notation

$$I_\delta=q_1\phi_1^{[2]}[A]=\frac{q_1}{\phi_1^{[2]}(x)}\sum_{i_1=1}^{q_1}\phi(x_{i_1})=I_1^{[2]}[A]\,. \tag{2.9}$$

In a more general aspect a d-depth $[j]$-index of dispersion of the function f on the set $A_{i_1i_2\cdots i_h}$ is defined by

$$I_d^{[j]}(f)(A_{i_1i_2\cdots i_h})=(q_{h+1}q_{h+2}\cdots q_{h+d})^{j-1}\phi_d^{[j]}(A_{i_1i_2\cdots i_h}) \tag{2.10}$$

in virtue of (2.1). In particular when the function f becomes an identity, (2.10) becomes

$$\begin{aligned} I_d^{[j]}(1)(A_{i_1i_2\cdots i_h}) &= I_d^{[j]}[A_{i_1i_2\cdots i_h}] \\ &= \frac{(q_{h+1}q_{h+2}\cdots q_{h+d})^{j-1}}{x_{i_1i_2\cdots i_h}^{[j]}}\sum_{i_{h+1}=1}^{q_{h+1}}\cdots\sum_{i_{h+d}=1}^{q_{h+d}}x_{i_1i_2\cdots i_hi_{h+1}\cdots i_{h+d}}^{[j]}\,. \end{aligned} \tag{2.11}$$

Multistage distributional patterns have been discussed by some ecologists. In fact, Morishita [2] for instance, discussed intraclump distribution in analysis of distributional patterns by I_δ. In this connection it seems to us to be indispensable to introduce a multistage decomposition system $\{A_{i_1i_2\cdots i_m}\}$, a set of x-variates $\{x_{i_1i_2\cdots i_m}\}$, as we have just done, in order to prepare a mathematical tool by which to explore a multistage distributional pattern.

In some instances which follow, when any confusion may not occur but convenience can be expected to make clear multistage structure, we may use the notation $x_{i_1i_2\cdots i_h\cdots}$ and $x\ldots$ in the place of $x_{i_1i_2\cdots i_h}$ and x respectively. At the same time we may and we shall simplify our notation so as to concentrate our discussion in lower stage decompositions associated with our original decomposition $D^{(m)}(A)$. Thus we may and we shall denote sometimes x and q instead of $x\cdots$ and q_1 originally defined in the system $D^{(m)}(A)$, when we are particularly concerned with the first stage decomposition.

§ 3. Distributional pattern with one parameter generalised exponential type distributions associated with the m-stage decomposition system $D^{(m)}(A)$ of the set A.

We start with

Definition 2. *A set of $q_1q_2\cdots q_k$ stochastic variables $\{X_{i_1i_2\cdots i_m}\}$ ($i_h=1, 2, \cdots q_h$, ; $h=1, 2, \cdots, m$) is assumed to satisfy the following conditions:*

(1) *Each stochastic variable $X_{i_1i_2\cdots i_m}$ is associated with the set $A_{i_1i_2\cdots i_m}$.*

(2) *The $q_1q_2\cdots q_m$ stochastic variables $\{X_{i_1i_2\cdots i_m}\}$ are mutually independent.*

(3) *The probability density function of $X_{i_1i_2\cdots i_m}$ belongs to the same family of exponential type distributions with the probability element*

$$\begin{aligned} &g(x\,;\,k_{i_1i_2\cdots i_m}|A_{i_1i_2\cdots i_m})d\mu_{i_1i_2\cdots i_m}(x) \\ &\qquad\equiv \exp\{\lambda x+k_{i_1i_2\cdots i_m}b(\lambda)\}a_{i_1i_2\cdots i_m}(x)d\mu_{k_{i_1i_2\cdots i_m}}(x) \end{aligned} \tag{3.1}$$

with the additive parameter $k_{i_1i_2\cdots i_m}$, *which will be denoted briefly by* $g(x\,;\,k_{i_1i_2\cdots i_m}|A_{i_1i_2\cdots i_m})$, *where* $\mu_{k_{i_1i_2\cdots i_m}}(x)$ *is a step function having jump solely at integral values of* x.

(4) *The set of stochastic variables* $\{X_{i_1i_2\cdots i_h}\}$ $(h=1, 2, \cdots, m-1)$ *and* X *is defined in the following way:*

(1°) $X=\sum_{i_1=1}^{q_1} X_{i_1}$

(2°) $X_{i_1i_2\cdots i_h}=\sum_{i_{h+1}=1}^{q_{h+1}} X_{i_1i_2\cdots i_hi_{h+1}}$, $(i_j=1, 2, \cdots, q_j\,;\, j=1, 2, \cdots, h)$

for $h=1, 2, 3, \cdots, m-1$.

(5) *The set of additive constants* $\{k_{i_1i_2\cdots i_h}\}$ $(i_j=1, 2, \cdots, q_j\,;\, j=1, 2, \cdots, h\,;\, h=1, 2, \cdots, m-1)$ *and* k *is defined in the following way:*

(1°) $k=\sum_{i_1=1}^{q_1} k_{i_1}$

(2°) $k_{i_1i_2\cdots i_h}=\sum_{i_{h+1}=1}^{q_{h+1}} k_{i_1i_2\cdots i_hi_{h+1}}$, $(i_j=1, 2, \cdots, q_j\,;\, j=1, 2, \cdots, h)$

for $h=1, 2, 3, \cdots, m-1$.

Then the set of $q_1q_2\cdots q_m$ *stochastic variables* $\{X_{i_1i_2\cdots i_h}\}$ *is said to constitute an additive family of one parameter exponential type distributions* $\{g(x\,;\,k_{i_1i_2\cdots i_m}|A_{i_1i_2\cdots i_m})\}$ *associated with the m-stage decomposition system* $D^{(m)}(A)$ *of the set* A, *and it is denoted by*

$$\{X_{i_1i_2\cdots i_m}\}\in E_\lambda(k_{i_1i_2\cdots i_m}\,;\,D^{(m)}(A))\,. \tag{3.2}$$

Now it is immediate to observe

Proposition 2. *Under the assumption* (3.2), *the set of* $q_1q_2\cdots q_h$ *stochastic variables* $\{X_{i_1i_2\cdots i_h}\}$ *constitutes an additive family of one parameter exponential type distributions* $\{g(x\,;\,k_{i_1i_2\cdots i_h}|A_{i_1i_2\cdots i_h})\}$

$$\{X_{i_1i_2\cdots i_h}\}\in E_\lambda(k_{i_1i_2\cdots i_h}\,;\,D^{(h)}(A))\,, \tag{3.3}$$

for $h=1, 2, \cdots, m-1$. *Moreover the stochastic variable* X *is distributed in* $g(x\,;\,k|\lambda)$.

We shall give here a few examples which are important in our practical applications.

Example 3.1. *Binomial distribution.* This is the case when each stochastic variable $X_{i_1i_2\cdots i_m}$ is distributed according to the distribution

$$Pr.\{X_{i_1i_2\cdots i_h}=x\}=\binom{k_{i_1i_2\cdots i_m}}{x}p^x(1-p)^{k_{i_1\cdots i_m}-x} \tag{3.4}$$
$$=g_B(x\,;\,k_{i_1i_2\cdots i_m}|A_{i_1i_2\cdots i_m})\,,\qquad \text{say,}$$

for $x=0, 1, \cdots, k_{i_1i_2\cdots i_m}$. The transformation of the parameter p into λ by means of $\log\{p/(1-p)\}=\lambda$ reduces (3.4) to the exponential type (3.1) with

$$b(\lambda)=\log(e^\lambda+1)^{-1}\,, \tag{3.5}$$

$$a_{k_{i_1i_2\cdots i_m}}(x)=\binom{k_{i_1i_2\cdots i_m}}{x} \tag{3.6}$$

and $d\mu_{k_{i_1i_2\cdots i_m}}(x)=1$, for $x=0, 1, 2, \cdots, k_{i_1i_2\cdots i_m}$ and is equal to zero otherwise.

Example 3.2. *Negative binomial distribution.* This is the case when each $X_{i_1i_2\cdots i_m}$ is distributed according to the distribution

$$(3.7)\qquad Pr.\{X_{i_1i_2\cdots i_m}=x\}=\binom{k_{i_1i_2\cdots i_m}+x-1}{x}p^x(1-p)^{k_{i_1i_2\cdots i_m}}$$
$$=g_{NB}(x\,;\ k_{i_1i_2\cdots i_m}|A_{i_1i_2\cdots i_m})$$

for $x=0, 1, 2, \cdots$. The transformation of the parameter p into λ by means of $\lambda=\log p$ reduces (3.7) to the exponential type (3.1) with

$$(3.8)\qquad b(\lambda)=\log(1-e^{\lambda}),$$

$$(3.9)\qquad a_{k_{i_1i_2\cdots i_m}}(x)=\binom{k_{i_1i_2\cdots i_m}+x-1}{x}$$

and $d\mu_{k_{i_1i_2\cdots i_m}}(x)=1$, for $x=0, 1, 2, \cdots$, and is equal to zero otherwise.

Example 3.3. *Poission distribution.* This is the case when each $X_{i_1i_2\cdots i_m}$ is distributed according to the distribution

$$(3.10)\qquad Pr.\{X_{i_1i_2\cdots i_m}=x\}=\exp\{-k_{i_1i_2\cdots i_m}\theta\}(k_{i_1i_2\cdots i_m}\theta)^x/x!$$
$$=g_p(x\,;\ k_{i_1i_2\cdots i_m}|A_{i_1i_2\cdots i_m})$$

for $x=0, 1, 2, \cdots$. The transformation of the parameter θ into λ by means of $\log\theta=\lambda$ reduces (3.10) to the exponential type distribution with

$$(3.11)\qquad b(\lambda)=-e^{\lambda},$$

$$(3.12)\qquad a_{k_{i_1i_2\cdots i_m}}(x)=\frac{k^x_{i_1i_2\cdots i_m}}{x!}.$$

Remarks on possible modifications of notations on stochastic variables $\{X_{i_1i_2\cdots i_h}\}$ and $\{k_{i_1i_2\cdots i_h}\}$ should be added here, similarly as we have made on the last part of section 2. Indeed we may use, in some cases when we consider it adequate and useful, the notations $X_{i_1i_2\cdots i_h\cdots}$, $k_{i_1i_2\cdots i_h\cdots}$, $x_{\cdots\cdots}$ and $k_{\cdots\cdots}$ instead of the original notations $X_{i_1i_2\cdots i_h}$, $k_{i_1i_2\cdots i_h}$, x and k just introduced respectively.

We proceed to introduce

Definition 3. *A set of $q_1q_2\cdots q_k$ stochastic variables $\{X_{i_1i_2\cdots i_m}\}$ $(i_h=1, 2, \cdots, q_h;$ $h=1, 2, \cdots, m)$ is said to have an m-stage distributional pattern of one parameter generalised exponential type distributions with the set of additive parameters $\{k^{(\nu)}_{i_1i_2\cdots i_\nu}\}$ $(\nu=1, 2, \cdots, m)$ when the following conditions are satisfied:*

(1) *The set of q_1 stochastic variables $\{X_{i_1}\}$ $(i_1=1, 2, \cdots, q_1)$ has a joint probability element*

$$(3.13)\qquad Pr.\Big\{\prod_{i_1=1}^{q_1}X_{i_1}=x_{i_1}\Big\}=\prod_{i_1=1}^{q_1}g^{(1)}(x_{i_1}\,;\ k^{(1)}_{i_1}|A_{i_1})d\mu^{(1)}(x_{i_1}).$$

(2) *The conditional joint probability element of q_h stochastic variables $\{X_{i_1i_2\cdots i_{h-1}i_h}\}$ $(i_h=1, 2, \cdots, q_h)$ under the condition that the stochastic variable $X_{i_1i_2\cdots i_{h-1}}$ is assigned as to be equal to $x_{i_1i_2\cdots i_{h-1}}$ is given by*

$$(3.14)\qquad Pr.\Big\{\prod_{i_h=1}^{q_h}(X_{i_1i_2\cdots i_h}=x_{i_1i_2\cdots i_h})\,|\,X_{i_1i_2\cdots i_{h-1}}=x_{i_1i_2\cdots i_{h-1}}\Big\}$$
$$=\frac{\prod_{i_h=1}^{q_h}g^{(h)}_{i_1i_2\cdots i_{h-1}}(x_{i_1i_2\cdots i_h}\,;\ k^{(h)}_{i_1i_2\cdots i_h}|A_{i_1i_2\cdots i_h})d\mu^{(h)}_{i_1\cdots i_{h-1}}(x_{i_1i_2\cdots i_h})}{g^{(h)}_{i_1i_2\cdots i_{h-1}}(x_{i_1i_2\cdots i_{h-1}}\,;\ k^{(h)}_{i_1i_2\cdots i_{h-1}}|A_{i_1i_2\cdots i_{h-1}})d\mu^{(h)}_{i_1\cdots i_{h-1}}(x_{i_1i_2\cdots i_{h-1}})}$$
$$=dg^{(h)}_{i_1i_2\cdots i_{h-1}}((x_{i_1i_2\cdots i_h})|x_{i_1i_2\cdots i_{h-1}}\,;\ k^{(h)}_{i_1i_2\cdots i_h}|A_{i_1i_2\cdots i_h}),\qquad say.$$

(3) *The conditional joint probability element of* $q_{h_1}q_{h_1+1}\cdots q_{h_2}$ *stochastic variables* $\{X_{i_1i_2\cdots i_h}\}$ $(i_j=1, 2, \cdots, q_j;\ j=h_1, h_1+1, \cdots, h_2)$ *under the condition that* $X_{i_1i_2\cdots i_{h_1-1}} = x_{i_1i_2\cdots i_{h_1-1}}$, *for* $i_j=1, 2, \cdots, q_j$, $j=1, 2, \cdots, h_1-1$, *is given by*

$$Pr.\{E^{(2)}(h_1, h_2)\,|\,E^{(1)}(h_1-1)\} = \prod_{i_{h_1}=1}^{q_{h_1}} \prod_{i_{h_1+1}=1}^{q_{h_1+1}} \cdots \prod_{i_{h_2}=1}^{q_{h_2}} dG(i_{h_1}i_{h_1+1}\cdots i_{h_2}\,|\,i_1i_2\cdots i_{h_1-1}), \tag{3.15}$$

where the left-hand side of (3.15) *denotes the probability element of the occurance of the event* $E^{(2)}(h_1, h_2)$ *under the condition that the event* $E^{(1)}(h_1-1)$ *occurs where*

$$E^{(2)}(h_1, h_2):\ \prod_{i_{h_1}=1}^{q_{h_1}} \cdots \prod_{i_{h_2}=1}^{q_{h_2}} (X_{i_1\cdots i_{h_1-1}i_{h_1}\cdots i_{h_2}} = x_{i_1\cdots i_{h_1-1}i_{h_1}\cdots i_{h_2}}), \tag{3.16}$$

$$E^{(1)}(h_1-1):\ \prod_{i_1=1}^{q_1} \cdots \prod_{i_{h_1-1}=1}^{q_{h_1-1}} (X_{i_1i_2\cdots i_{h_1-1}} = x_{i_1i_2\cdots i_{h_1-1}}) \tag{3.17}$$

and in the right-hand side of (3.15) *we have put*

$$dG(i_{h_1}i_{h_1+1}\cdots i_{h_2}\,|\,i_1i_2\cdots i_{h_1-1}) = \prod_{j=h_1}^{h_2} dg^{(j)}_{i_1i_2\cdots i_{j-1}}((x_{i_1\cdots i_{h_1-1}i_{h_1}\cdots i_j})\,|\,x_{i_1\cdots i_{j-1}};\ k^{(j)}_{i_1i_2\cdots i_j}\,|\,A_{i_1i_2\cdots i_j}). \tag{3.18}$$

It is readily seen that the stochastic scheme defined by Definition 2 is an extremely specialised case of the general scheme introduced in Definition 3. Between these two cases there are a lot of types of multistage distributional patterns which are worth while to be noticed. For a systematic description of these situations it is adequate to introduce two fundamental notions, homegeneity and inheritance, in the following way:

Definition 4. *In a set of* $q_1q_2\cdots q_k$ *stochastic variables* $\{X_{i_1i_2\cdots i_m}\}$ $(i_h=1, 2, \cdots, q_h;\ h=1, 2, \cdots, m)$ *having a multistage distributional pattern of generalised exponential type distribution with the set of additive parameter* $\{k^{(\nu)}_{i_1i_2\cdots i_\nu}\}$ $(\nu=1, 2, \cdots, m)$, *a set of* $q_{h+1}\cdots q_{h+d}$ *stochastic variables* $\{X_{i_1i_2\cdots i_h\cdots i_{h+d}}\}$ $(i_j=1, 2, \cdots, q_j;\ j=h+1, \cdots, h+d)$ *is said to be homogeneous in the d-depth in the set* $A_{i_1i_2\cdots i_h}$ *when there exists a sequence of functions* $\{g^{(j)}_{i_1i_2\cdots i_h}\}$ *and* $\{d\mu^{(j)}_{i_1i_2\cdots i_h}\}$ *such that*

$$dg^{(j)}_{i_1i_2\cdots i_{j-1}}((x_{i_1i_2\cdots i_j})\,|\,x_{i_1i_2\cdots i_{j-1}};\ k^{(j)}_{i_1i_2\cdots i_j}\,|\,A_{i_1i_2\cdots i_j}) = dg^{(j)}_{i_1i_2\cdots i_h}((x_{i_1i_2\cdots i_j})\,|\,x_{i_1i_2\cdots i_{j-1}};\ k^{(j)}_{i_1i_2\cdots i_j}\,|\,A_{i_1i_2\cdots i_j}), \tag{3.19}$$

for $i_j=1, 2, \cdots, q_j;\ j=h+1, \cdots, h+d$.

Definition 5. *A set of* $q_{h+1}\cdots q_{h+d}$ *stochastic variables* $\{X_{i_1i_2\cdots i_{h-1}i_h\cdots i_{h+d}}\}$ $(i_j=1, 2, \cdots, q_j;\ j=h+1, \cdots, h+d)$ *is said to be inherited with the d-depth in the set* $A_{i_1i_2\cdots i_h}$ *when the following two conditions are satisfied:*

(1°) $k^{(j)}_{i_1i_2\cdots i_{j-1}} = k^{(j-1)}_{i_1i_2\cdots i_{j-1}}$, $(j=h+1, \cdots, h+d)$

(2°) $g^{(j)}_{i_1i_2\cdots i_{j-1}}(x_{i_1i_2\cdots i_{j-1}};\ k^{(j)}_{i_1i_2\cdots i_{j-1}}\,|\,A_{i_1i_2\cdots i_{j-1}})d\mu^{(j)}_{i_1i_2\cdots i_{j-1}}(x_{i_1i_2\cdots i_{j-1}})$
$= g^{(j-1)}_{i_1i_2\cdots i_{j-2}}(x_{i_1i_2\cdots i_{j-1}};\ k^{(j-1)}_{i_1i_2\cdots i_{j-1}}\,|\,A_{i_1i_2\cdots i_{j-1}})d\mu^{(j-1)}_{i_1i_2\cdots i_{j-2}}(x_{i_1i_2\cdots i_{j-1}})$

for every $x_{i_1i_2\cdots i_{j-1}}$ *and* $k^{(j)}_{i_1i_2\cdots i_{j-1}}$ *in their domains of definition, for* $j=h+1, \cdots, h+d$.

In virtue of Definitions 3, 4 and 5 it is immediate to see

Proposition 3. *The set of* $q_1q_2\cdots q_m$ *stochastic variables* $\{X_{i_1i_2\cdots i_m}\} \in E_\lambda(k_{i_1i_2\cdots i_m};\ D^{(m)}(A))$ *is both homogeneous and inherited in the m-depth in the set* A.

Proof. This can be seen by an interpretation to the effect that the assumption (3.2) implies

$$k^{(h)}_{i_1 i_2 \cdots i_h} = k_{i_1 i_2 \cdots i_h}, \tag{3.20}$$

$$\begin{aligned} & g^{(h)}_{i_1 i_2 \cdots i_{h-1}}(x_{i_1 i_2 \cdots i_h};\ k^{(h)}_{i_1 i_2 \cdots i_h} \mid A_{i_1 i_2 \cdots i_h}) d\mu^{(h)}_{i_1 \cdots i_{h-1}}(x_{i_1 i_2 \cdots i_{h-1}}) \\ & = g(x_{i_1 i_2 \cdots i_h};\ k_{i_1 i_2 \cdots i_h} \mid A_{i_1 i_2 \cdots i_h}) d\mu(x_{i_1 i_2 \cdots i_{h-1}}) \end{aligned} \tag{3.21}$$

for $i_j = 1, 2, \cdots, q_j$; $j = 1, 2, \cdots, h$; $h = 1, 2, \cdots, m$.

Particular examples of two and three stage distributional patterns are explained by adopting various cominations of binomial, negative binomial and Poisson distributions in defining functions in Definition 3.

Example 3.4. *Two stage distributional patterns.* Nine sets of two stage distributional patterns can be obtained from all the possible permutations by adopting one of the three fundamental probability density functions as shown in the following Table. We may and we shall denote these two stage distributional patterns by $B\times B$, $B\times NB$, $B\times P$, $NB\times B$, $NB\times NB$, $NB\times P$, $P\times B$, $P\times NB$ and $P\times P$.

Table 1. Probability Element Function (PEF) in Two stage Distributional Pattern.

PEF	Illustrative Examples
$dg^{(1)}(x;\ k^{(1)}_{i_1} \mid A_{i_1})$	B, NB, P
$dg^{(2)}_{i_1}((x_{i_1 i_2}) \mid x_{i_1};\ k^{(2)}_{i_1 i_2} \mid A_{i_1 i_2})$	B, NB, P

A direct consequence of Definitions 3 and 4 yields us the following observations:

(i) Homogeneity. Two stage distributional pattern is homogeneous if and only if there is a function such that

$$dg^{(2)}_{i_1}((x_{i_1 i_2}) \mid x_{i_1};\ k^{(2)}_{i_1 i_2} \mid A_{i_1 i_2}) = dg^{(2)}((x_{i_1 i_2}) \mid x_{i_1};\ k^{(2)}_{i_1 i_2} \mid A_{i_1 i_2}) \tag{3.22}$$

for $i_j = 1, 2, \cdots, q_j$; $j = 1, 2$.

(ii) Inheritance. Two stage distributional pattern is inherited if and only if the following two conditions are satisfied:

(1°) $k^{(2)}_{i_1} = k^{(1)}_{i_1}$ $(i_1 = 1, 2, \cdots, q_1)$

(2°) $g^{(2)}_{i_1}(x_{i_1};\ k^{(2)}_{i_1 i_2} \mid A_{i_1 i_2}) = g^{(1)}(x_{i_1};\ k^{(2)}_{i_1 i_2} \mid A_{i_1 i_2})$

for $i_1 = 1, 2, \cdots, q_j$; $j = 1, 2$.

Example 3.5. *Three stage distributional patterns.* Twenty seven sets of three stage distributions such as $B\times B\times B$, $B\times B\times NB$, $NB\times P\times B$ and so on can be obtained from all the possible permutations of the fundamental probability density functions as shown in the following Table 2.

Table 2. Probability Element Function (PEF) in Three stage Distributional Pattern.

PEF	Illustrative Examples
$dg^{(1)}(x;\ k^{(1)}_{i_1} \mid A_{i_1})$	B, NB, P
$dg^{(2)}_{i_1}((x_{i_1 i_2}) \mid x_{i_1};\ k^{(2)}_{i_1 i_2} \mid A_{i_1 i_2})$	B, NB, P
$dg^{(3)}_{i_1 i_2}((x_{i_1 i_2 i_3}) \mid x_{i_1 i_2};\ k^{(3)}_{i_1 i_2 i_3} \mid A_{i_1 i_2 i_3})$	B, NB, P

In conclusion of section 3 let us introduce some notations for the conditional expected values of stochastic variables. Let $z=\{z_{i_1 i_2\cdots i_h}\}$ $(i_j=1, 2, \cdots q_j\,;\ j=1, 2, \cdots, h)$ be a set of stochastic variables and let $P(z)$ be the multidimensional probability distribution function of z. Let $H(z)$ be a function of z with its domain of definition $\mathfrak{D}(H)$. For an assigned set of non-negative integers $(i_1, i_2, \cdots, i_h)$, an assigned value y and an assigned function u, let us denote by $\mathfrak{D}_{u(i_1 i_2\cdots i_h)(y)}(H)$ the subset of $\mathfrak{D}(H)$ where $u(z_{i_1 i_2\cdots i_h})$ has the value y.

Under these preparations, let us introduce a set of the conditional expectation of the stochastic variable $u(z)$ such that

$$E^*\{H\}=\frac{\displaystyle\int_{z\in\mathfrak{D}(H)} H(z)dP(z)}{\displaystyle\int_{z\in\mathfrak{D}(H)} dP(z)} \tag{3.23}$$

and

$$E^*_{u(i_1 i_2\cdots i_h)(y)}\{H\}=\frac{\displaystyle\int_{z\in\mathfrak{D}_{u(i_1\cdots i_h)(y)}(H)} H(z)dP(z)}{\displaystyle\int_{z\in\mathfrak{D}_{u(i_1\cdots i_h)(y)}(H)} dP(z)}, \tag{3.24}$$

provided that the right-hand side of (3.23) and that of (3.24) exist respectively.

We use also the notion $E^*_G\{H\}$ to specify an underlying probability function G such as binomial (B), negative binomial (NB) and Poisson (P) distributions. For instance $E^*_B\{H\}$, $E^*_{NB}\{H\}$ and $E^*_P\{H\}$ will be used to express the conditional expectation of H.

§ 4. Reproductive function in additive family of one parameter exponential type distributions.

Let us introduce

Definition 6. *A function $\phi(x)$ is said to be reproductive in an additive family of one parameter exponential type distributions $\{g(x\,;k)\}$ with additive parameter k, if there exists a set of a function $\phi(x)$, a non-vanishing function $\varphi(k)$, a transformation T of x, a transformation S of k and a non-vanishing function $c(\phi)$, which is independent of k and x, such that, for any assigned set of x and k belonging to their respective domain of definition for specifying $g(x\,;k)$, we have*

$$\phi(x)g(x\,;k)d\mu(x)=\varphi(k)c(\phi)g(Tx\,;Sk)d\mu(Tx) \tag{4.1}$$

with the following properties:

(1°) *For any set of $x_1, x_2, \cdots, x_{q-1}$ and x_q belonging to the domain of definition $g(x\,;k)$ and for any positive integer i in $1\leqq i\leqq q$, we have*

$$T\Big(\sum_{j=1}^{q}x_j\Big)=\sum_{j=1}^{i-1}x_j+Tx_i+\sum_{l=i+1}^{q}x_l\,. \tag{4.2}$$

(2°) *For any set of non-negative integers $k_1, k_2, \cdots, k_{q-1}$ and k_q and any positive integer i in $1\leqq i\leqq q$, we have*

(4.3) $$S\Big(\sum_{j=1}^{q} k_j\Big)=\sum_{j=1}^{i-1} k_j+Sk_i+\sum_{l=i+1}^{q} k_l\,.$$

We observe

Corollary 1. The function $\phi^{[j]}(x)=x(x-1)\cdots(x-j+1)$ is reproductive in additive family of one parameter exponential type distributions $\{g(x\,;k)\}$ with additive parameter k when $g(x\,;k)$ is one of the three fundamental distributions, namely, binomial, negative binomial and Poisson ones. Indeed we have

(a) *Binomial distribution* $g_B(x\,;k)$. We have

(4.4) $$\phi^{[j]}(x)g_B(x\,;k)=\varphi_B(k)c_B(\phi)g_B(T_Bx\,;S_Bk)\,,$$

where we have put

(4.5) $$\varphi_B(k)=k(k-1)\cdots(k-j+1)\,,$$

(4.6) $$c_B(\phi)=p^j\,,$$

(4.7) $$T_Bx=x-j\,,\qquad S_Bk=k-j\,.$$

(b) *Negative binomial distribution* $g_{NB}(x\,;k)$. We have

(4.8) $$\phi^{[j]}(x)g_{NB}(x\,;k)=\varphi_{NB}(k)c_{NB}(\phi)g_{NB}(T_{NB}x\,;S_{NB}k)\,,$$

where we have put

(4.9) $$\varphi_{NB}(k)=k(k+1)\cdots(k+j-1)\,,$$

(4.10) $$c_{NB}(\phi)=p^j(1-p)^{-j}\,,$$

(4.11) $$T_{NB}x=x-j\,,\qquad S_{NB}=k+j\,.$$

(c) *Poisson distribution* $g_p(x\,;k)$. We have

(4.12) $$\phi^{[j]}(x)g_P(x\,;k)=\varphi_P(k)c_P(\phi)g_P(T_Px\,;S_Pk)\,,$$

where we have put

(4.13) $$\varphi_P(k)=k^j\,,$$

(4.14) $$c_P(\phi)=\lambda^j\,,$$

(4.15) $$T_Px=x-j\,,\qquad S_Pk=k\,.$$

It is convenient to have

Definition 7. *A function $\phi(x)$ is said to be reproductive in a multistage distributional pattern of one parameter generalised exponential type distributions with the set of additive parameters $\{k^{(\nu)}_{i_1i_2\cdots i_\nu}\}$ $(\nu=1,2,\cdots,m)$, or simply reproductive in the system $D^{(m)}(A)$, when all the concurrence-probability distribution density functions are reproductive, that is to say, the relations hold true*

(4.16) $$\begin{aligned}\phi(x)g^{(h)}_{i_1i_2\cdots i_{h-1}}&(x_{i_1i_2\cdots i_h}\,;\,k^{(h)}_{i_1i_2\cdots i_h}\,|\,A_{i_1i_2\cdots i_h})d\mu^{(h)}_{i_1\cdots i_{h-1}}(x_{i_1i_2\cdots i_h})\\ &=\varphi^{(h)}_{i_1i_2\cdots i_{h-1}}(k^{(h)}_{i_1i_2\cdots i_h})c^{(h)}_{i_1i_2\cdots i_{h-1}}(\phi)g^{(h)}_{i_1i_2\cdots i_{h-1}}(T^{(h)}_{i_1i_2\cdots i_{h-1}}x_{i_1i_2\cdots i_h}\,;\\ &\qquad S^{(h)}_{i_1i_2\cdots i_{h-1}}k^{(h)}_{i_1i_2\cdots i_h})\,d\mu^{(h)}_{i_1i_2\cdots i_{h-1}}(T^{(h)}_{i_1i_2\cdots i_{h-1}}x_{i_1i_2\cdots i_h})\,,\end{aligned}$$

where the functions $c^{(h)}_{i_1i_2\cdots i_{h-1}}(\phi)$, transformations $T^{(h)}_{i_1i_2\cdots i_{h-1}}$ and $S^{(h)}_{i_1i_2\cdots i_{h-1}}$ satisfy the conditions similar to those enunciated in Definition 6.

§ 5. The conditional expectation of ϕ-dispersion for reproductive ϕ-function.

It is the purpose of this section to evaluate various conditional expectations of ϕ-dispersions such as $E^*\{\phi_d[A]\}$ and $E_c^*\{\phi_d[A_{i_1 i_2 \cdots i_h}]\}$ for $d \geqq 1$ and for a certain set of conditions c such as specifying underlying probability density function. In order to make clear the essential aspects of our evaluations which are valid for a fairly general situation, let us be concerned with the simplest two examples.

Proposition 4. *For a reproductive function ϕ in the decomposition system $D^{(1)}(A)$ we have*

$$(5.1) \qquad E^*\{\phi_1[A]\} = \frac{\sum_{i=1}^{q} \varphi(k_i)}{\varphi\left(\sum_{i=1}^{q} k_i\right)} = \frac{\sum_{i=1}^{q} \varphi(k_i)}{\varphi(k)}\,.$$

Proof. We have, in virtue of definition,

$$(5.2) \qquad E^*\{\phi_1[A]\} = \int_{\{x_i\} \in \mathfrak{D}\phi_1[A]} \cdots \iint \frac{\sum_{i=1}^{q} \phi(x_i)}{\phi(x)} \prod_{i=1}^{q} g(x_i\,;\,k_i) d\mu(x_i)\,,$$

where the domain of integration $\mathfrak{D}\phi_1[A]$ is the set of all non-negative integers for which $\phi_1[A]$ is defined and the symbol $\{x_i\} \in \mathfrak{D}\phi_1[A]$ denotes the set of $\{x_i\}$ $(i=1, 2, \cdots, q)$ such that $(x_1, x_2, \cdots, x_q)$ belongs to $\mathfrak{D}\phi_1[A]$.

Since the function ϕ is reproductive in $D^{(1)}(A)$, we have, for $i=1, 2, \cdots, q$,

$$(5.3) \qquad \phi(x_i) \prod_{h=1}^{q} g(x_h\,;\,k_h) d\mu(x_h) = \phi(x_i) g(x_i\,;\,k_i) d\mu(x_i) \prod_{h \neq i} g(x_h\,;\,k_h) d\mu(x_h)$$
$$= \varphi(k_i) c(\phi) g(Tx_i\,;\,Sk_i) d\mu(Tx_i) \prod_{h \neq i} g(x_h\,;\,k_h) d\mu(x_h)\,,$$

in virtue of (4.1). In view of (4.2) and (4.3), it ts readily observed that, when $(x_1, x_2, \cdots, x_q)$ runs through the set for which $\sum_{i=1}^{q} x_i$ is equal to an assigned value x^*, the point $(x_1, x_2, \cdots, x_{i-1}, Tx_i, x_{i+1}, \cdots, x_q)$ belongs to the set for which its sum is equal to $T\left(\sum_{i=1}^{q} x_i\right) = Tx^*$ for any non-negative integer i in $1 \leqq i \leqq q$, and also that the converse is true. After summing up the probability elements (5.3) with respect to $d\mu(x_1) \cdots d\mu(x_q)$ over the set of $(x_1, x_2, \cdots, x_q)$ for which $x \equiv \sum_{i=1}^{q} x_i$ is assigned as a particular value, we have

$$(5.4) \qquad \varphi(k_i) c(\phi) g\left(T\left(\sum_{i=1}^{q} x_i\right);\, S\left(\sum_{i=1}^{q} k_i\right)\right) d\mu\left(T\left(\sum_{i=1}^{q} x_i\right)\right),$$

which, apart from $\varphi(k_i)$, is independent of i. Now again the fact that $\phi(x)$ is reproductive implies, since $c(\phi)\varphi(k) \neq 0$,

$$(5.5) \qquad g(Tx\,;\,Sx) d\mu(Tx) = \frac{\phi(x) g(x\,;\,k) d\mu(x)}{c(\phi)\varphi(k)}\,,$$

which, in combination with (5.4), leads us to

$$(5.6) \qquad E^*\{\phi_1[A]\} = \frac{\sum_{i=1}^{q} \varphi(k_i)}{\varphi(k)} \int_{x \in \mathfrak{D}\phi_1[A]} g(x\,;\,k) d\mu(x)$$

$$\times \frac{1}{\int\limits_{x\in\mathfrak{D}\phi_1[A]} g(x;\,k)d\mu(x)} = \frac{\sum_{i=1}^{q}\varphi(k_i)}{\varphi(k)},$$

as we were to prove.

Example 5.1. (a) Binomial distribution

$$E_B^*\{\phi_1^{[j]}[A]\} = \frac{\sum_{i=1}^{q}\varphi_B(k_i)}{\varphi_B(k)} = \frac{\sum_{i=1}^{q}k_i(k_i-1)\cdots(k_i-j+1)}{k(k-1)\cdots(k-j+1)}. \tag{5.7}$$

(b) Negative binomial distribution

$$E_{NB}^*\{\phi_1^{[j]}[A]\} = \frac{\sum_{i=1}^{q}\varphi_{NB}(k)}{\varphi_{NB}(k)} = \frac{\sum_{i=1}^{q}k_i(k_i+1)\cdots(k_i+j-1)}{k(k+1)\cdots(k+j-1)}. \tag{5.8}$$

(c) Poisson distribution

$$E_P^*\{\phi_1^{[j]}[A]\} = \frac{\sum_{i=1}^{q}\varphi_P(k_i)}{\varphi_P(k)} = \frac{\sum_{i=1}^{q}k_i^j}{k^j}. \tag{5.9}$$

Proposition 5. *For a reproductive function ϕ, in the decomposition system $D^{(2)}(A)$, we have*

$$E^*\{\phi_2[A]\} = \sum_{i_1=1}^{q_1}\frac{\varphi^{(1)}(k_{i_1}^{(1)})}{\varphi^{(1)}(k^{(1)})}\sum_{i_2=1}^{q_2}\frac{\varphi_{i_1}^{(1)}(k_{i_1i_2}^{(2)})}{\varphi_{i_1}^{(2)}(k_{i_1}^{(2)})}. \tag{5.10}$$

Proof. The proof is quite similar to that of Proposition 3. First we note that, for each pair (i_1, i_2), we have

$$\begin{aligned}(5.11)\quad &\phi(x_{i_1i_2})\prod_{j=1}^{q_2}g_{i_1}^{(2)}(x_{i_1j};\,k_{i_1j}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1j}^{(2)})\\ &=\phi(x_{i_1i_2})g_{i_1}^{(2)}(x_{i_1i_2};\,k_{i_1i_2}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1i_2}^{(2)})\prod_{j\neq i_2}g_{i_1}^{(2)}(x_{i_1j};\,k_{i_1j}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1j}^{(2)})\\ &=c^{(2)}(\phi)\varphi_{i_1}^{(2)}(k_{i_1i_2}^{(2)})g_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1i_2};\,S_{i_1}^{(2)}k_{i_1i_2})d\mu_i^{(2)}(T_{i_1}^{(2)}x_{i_1i_2}^{(2)})\cdot\prod_{h\neq i_2}g_{i_1}^{(2)}(x_{i_1h};\,k_{i_1h}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1j}^{(2)}),\end{aligned}$$

which yields us

$$\begin{aligned}(5.12)\quad &\sum_{i_2=1}^{q_2}\phi(x_{i_1i_2})\Big\{\prod_{i_2=1}^{q_2}g_{i_1}^{(2)}(x_{i_1i_2};\,k_{i_1i_2}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1i_2})\Big\}\\ &=c^{(2)}(\phi)\sum_{i_2=1}^{q_2}\varphi_{i_1}^{(2)}(k_{i_1i_2}^{(2)})g_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1i_2};\,S_{i_1}^{(2)}k_{i_1i_2}^{(2)})d\mu_i^{(2)}(T_{i_1}^{(2)}x_{i_1i_2}^{(2)})\\ &\qquad\cdot\prod_{h\neq i_2}\{g_{i_1}^{(2)}(x_{i_1h};\,k_{i_1h}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1h})\}.\end{aligned}$$

In view of (4.2) and (4.3), after summing up the probability elements in (5.12), we have

$$g_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1i_2};\,S_{i_1}^{(2)}k_{i_1i_2}^{(2)})d\mu_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1i_2}^{(2)})\cdot\prod_{h\neq i_2}\{g_i^{(2)}(x_{i_1h};\,k_{i_1h}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1i_2})\} \tag{5.13}$$

with respect to $d\mu_{i_1}^{(2)}(x_{i_11})\cdots d\mu_{i_1}^{(2)}(x_{i_1q_2})$, we obtain

$$g_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1};\,S_{i_1}^{(2)}k_{i_1}^{(2)})d\mu_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1}) \tag{5.14}$$

when $(x_{i_11},\cdots,x_{i_1q_2})$ runs through the set for which $\sum_{i_2=1}^{q_2}x_{i_1i_2}$ is equal to an assiged value x_{i_1} (5.14) leads us to the probability element

(5.15) $$c^{(2)}(\phi)\Big\{\sum_{i_2=1}^{q_2}\varphi_{i_1}^{(2)}(k_{i_1i_2}^{(2)})\Big\}g_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1}\,;\,S_{i_1}^{(2)}k_{i_1}^{(2)})d\mu_{i_1}^{(2)}(T_{i_1}^{(2)}x_{i_1}),$$

Again in virtue of the fact that ϕ is reproductive, (5.15) turns out to be

(5.16) $$\frac{\sum_{i_1=1}^{q_2}\varphi_{i_1}^{(2)}(k_{i_1i_2}^{(2)})}{\varphi_{i_1}^{(2)}(k_{i_1}^{(2)})}\phi(x_{i_1})g_{i_1}^{(2)}(x_{i_1}\,;\,k_{i_1}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1}).$$

In the consequence we have

(5.17) $$E^*\{\phi_2[A]\}$$
$$=\iint_{\{x_{i_1i_2}\}\in\mathfrak{D}\phi_2[A]}\cdots\cdots\int\frac{\sum_{i_1=1}^{q_1}\sum_{i_2=1}^{q_2}\phi(x_{i_1i_2})}{\phi(x)}\cdot\frac{\prod_{i_1=1}^{q_1}\prod_{i_2=1}^{q_2}g_{i_1}^{(2)}(x_{i_1i_2}\,;\,k_{i_1i_2}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1i_2})}{\prod_{i_1=1}^{q_1}g_{i_1}^{(2)}(x_{i_1}\,;\,k_{i_1}^{(2)})d\mu_{i_1}^{(2)}(x_{i_1})}$$
$$\cdot\prod_{i_1=1}^{q_1}g^{(1)}(x_{i_1}\,;\,k_{i_1}^{(1)})\,d\mu^{(1)}(x_{i_1})\times\frac{1}{Pr.\{\mathfrak{D}\phi_2[A]\}}$$
$$=\iint_{\{x_{i_1i_2}\}\in\mathfrak{D}\phi_1[A]}\cdots\cdots\int\frac{1}{\phi(x)}\sum_{i_1=1}^{q_1}\frac{\phi(x_{i_1})}{\varphi_{i_1}^{(2)}(k_{i_1}^{(2)})}\prod_{i_1=1}^{q_1}g^{(1)}(x_{i_1}\,;\,k_{i_1}^{(1)})d\mu^{(1)}(x_{i_1})$$
$$\cdot\sum_{i_2=1}^{q_2}\varphi_{i_1}^{(2)}(k_{i_1i_2}^{(2)})/Pr.\{\{x_{i_1i_2}\}\in\mathfrak{D}\phi_2[A]\}.$$

Here we can deal with a transformation

(5.18) $$\phi(x_{i_1})\prod_{i_1=1}^{q_1}g^{(1)}(x_{i_1}\,;\,k_{i_1}^{(1)}):\ c^{(1)}(\phi)\varphi^{(1)}(k_{i_1}^{(1)})g^{(1)}(T^{(1)}x_{i_1}\,;\,S^{(1)}k_{i_1}^{(1)})\prod_{h\neq i_1}g^{(1)}(x_h\,;\,k_h^{(1)})$$

as in the proof of Proposition 5.

Consequently, after summing up the probability elements in (5.18) with respect to $d\mu^{(1)}(x_1)d\mu^{(1)}(x_2)\cdots d\mu^{(1)}(x_{q_1})$ under the condition that $(x_1, x_2, \cdots, x_{q_1})$ runs through the set for which $\sum_{i_1=1}^{q_1}x_{i_1}$ is equal to an assigned value x, we obtain that the sum of

(5.19) $$c^{(1)}(\phi)\sum_{i_1=1}^{q_1}\varphi^{(1)}(k_{i_1}^{(1)})g^{(1)}(T^{(1)}x_{i_1};\,S^{(1)}k_{i_1}^{(1)})d\mu^{(1)}(Tx_{i_1})\cdot\prod_{h\neq i_1}g^{(1)}(x_h\,;\,k_h^{(1)})d\mu^{(1)}(x_h)$$

is equal to

(5.20) $$\frac{\sum_{i_1=1}^{q_1}\varphi^{(1)}(k_{i_1}^{(1)})}{\varphi^{(1)}(k^{(1)})}\phi(x)g^{(1)}(x\,;\,k^{(1)})d\mu^{(1)}(x),$$

which leads us to the result to be proved.

Now we are in the position to enunciate a general result on the conditional expectation of ϕ-dispersion.

Proposition 6. *For a reproductive function* ϕ *in the decomposition system* $D^{(d)}(A_{i_1i_2\cdots i_h})$ *we have*

(5.21) $$E^*\{\phi_d[A_{i_1i_2\cdots i_h}]\}=\sum_{i_{h+1}=1}^{q_{h+1}}\frac{\varphi_{i_1i_2\cdots i_h}^{(h+1)}(k_{i_1i_2\cdots i_{h+1}}^{(h+1)})}{\varphi_{i_1i_2\cdots i_h}^{(h+1)}(k_{i_1\cdots i_h}^{(h+1)})}\sum_{i_{h+2}=1}^{q_{h+2}}\frac{\varphi_{i_1\cdots i_{h+1}}^{(h+1)}(k_{i_1\cdots i_{h+2}}^{(h+2)})}{\varphi_{i_1\cdots i_{h+1}}^{(h+2)}(k_{i_1\cdots i_{h+1}}^{(h+2)})}\cdots$$

$$\sum_{i_{h+d}=1}^{q_{h+d}} \frac{\varphi^{(h+d)}_{i_1\cdots i_{h+d-1}}(k^{(h+d)}_{i_1 i_2\cdots i_{h+d}})}{\varphi^{(h+d)}_{i_1\cdots i_{h+d-1}}(k^{(h+d)}_{i_1 i_2\cdots i_{h+d-1}})},$$

where the case $h=0$ *can be interpretated as that*

$$E^*\{\phi_d[A]\} = \sum_{i_1=1}^{q_1} \frac{\varphi^{(1)}(k^{(1)}_{i_1})}{\varphi^{(1)}(k^{(1)})} \sum_{i_2=1}^{q_2} \frac{\varphi^{(2)}_{i_1}(k^{(2)}_{i_1 i_2})}{\varphi^{(2)}_{i_1}(k^{(2)}_{i_1})} \sum_{i_3=1}^{q_3} \frac{\varphi^{(3)}_{i_1 i_2}(k^{(3)}_{i_1 i_2 i_3})}{\varphi^{(3)}_{i_1 i_2}(k^{(3)}_{i_1 i_2})} \cdots \sum_{i_d=1}^{q_d} \frac{\varphi^{(d)}_{i_1 i_2\cdots i_{d-1}}(k^{(d)}_{i_1 i_2\cdots i_d})}{\varphi^{(d)}_{i_1 i_2\cdots i_{d-1}}(k^{(d)}_{i_1 i_2\cdots i_{d-1}})}. \tag{5.22}$$

Example 5.2. *Two stage homogeneous distributional pattern.* This is characterized by two systems of probability distribution elements $g^{(1)}d\mu^{(1)}$ in (3.13) and $g^{(2)}d\mu^{(2)}$ in (3.14), which, in this case, may and will be denoted by $Gd\mu^{(1)}\times Ld\mu^{(2)}$, where G(global) and L(local) correspond to $g^{(1)}$ and $g^{(2)}$ respectively.

In general we have for such an two stage homogeneous distributional pattern,

$$E^*_{G\times L}\{\phi_2[A]\} = \sum_{i_1=1}^{q_1} \frac{\varphi_G(k^{(1)}_{i_1})}{\varphi_G(k^{(1)})} \sum_{i_2=1}^{q_2} \frac{\varphi_L(k^{(2)}_{i_1 i_2})}{\varphi_L(k^{(2)}_{i_1})}. \tag{5.23}$$

which turns out to the expression

$$E^*_{G\times L}\{\phi_2[A]\} = \frac{1}{\varphi_G(k^{(2)})} \sum_{i_1=1}^{q_1} \sum_{i_2=1}^{q_2} \varphi_G(k^{(2)}_{i_1 i_2}), \tag{5.24}$$

particularly when two-stage distributional pattern is inherited, as can be readily seen from the fact that $\varphi_G(k)=\varphi_L(k)$ for every k in their common domain of definition.

(a) Inherited binomial distributional pattern ($B\times B$) (c. f. (4.5))

$$E^*_{B\times B}\{\phi^{[j]}_2[A]\} = \sum_{i_1=1}^{q_1} \frac{\varphi_B(k^{(1)}_{i_1})}{\varphi_B(k^{(1)})} \sum_{i_2=1}^{q_2} \frac{\varphi_B(k^{(2)}_{i_1 i_2})}{\varphi_B(k^{(2)}_{i_1})}. \tag{5.25}$$

(b) Inherited negative binomial distributional pattern ($NB\times NB$) (c. f. (4.9))

$$E^*_{NB\times NB}\{\phi^{[j]}_2[A]\} = \sum_{i_1=1}^{q_1} \frac{\varphi_{NB}(k^{(1)}_{i_1})}{\varphi_{NB}(k^{(1)})} \sum_{i_2=1}^{q_2} \frac{\varphi_{NB}(k^{(2)}_{i_1 i_2})}{\varphi_{NB}(k^{(2)}_{i_1})}. \tag{5.26}$$

(c) Inherited Poisson distributional pattern ($P\times P$) (c. f. (4.13))

$$E^*_{P\times P}\{\phi^{[j]}_2[A]\} = \sum_{i_1=1}^{q_1} \frac{\varphi_P(k^{(1)}_{i_1})}{\varphi_P(k^{(1)})} \sum_{i_2=1}^{q_2} \frac{\varphi_P(k^{(2)}_{i_1 i_2})}{\varphi_P(k^{(2)}_{i_1})}. \tag{5.27}$$

Example 5.3. *Three stage homogeneous distributional pattern.* This is characterized by three systems of probability distribution elements $g^{(i)}d\mu^{(i)}$ ($i=1, 2, 3$), which may and will be denoted by $Gd\mu^{(1)}\times Sd\mu^{(2)}\times Ld\mu^{(3)}$ in a symbolic way, abbreviated by $G\times S\times L$, where G(global), SG(or S)(subglobal) and L(local) corresponds to $g^{(1)}$, $g^{(2)}$ and $g^{(3)}$ respectively.

$$E^*_{G\times S\times L}\{\phi_3[A]\} = \sum_{i_1=1}^{q_1} \frac{\varphi_G(k^{(1)}_{i_1})}{\varphi_G(k^{(1)})} \sum_{i_2=1}^{q_2} \frac{\varphi_S(k^{(2)}_{i_1 i_2})}{\varphi_S(k^{(2)}_{i_1})} \sum_{i_3=1}^{q_3} \frac{\varphi_L(k^{(3)}_{i_1 i_2 i_3})}{\varphi_L(k^{(3)}_{i_1 i_2})}, \tag{5.28}$$

which turns out to

$$E^*_{G\times S\times L}\{\phi_3[A]\} = \frac{1}{\varphi_L(k^{(3)})} \sum_{i_1=1}^{q_1} \sum_{i_2=1}^{q_2} \sum_{i_3=1}^{q_3} \varphi_L(k^{(3)}_{i_1 i_2 i_3}), \tag{5.29}$$

particularly when three-stage distributional pattern is inherited in the 3-depth, because $\varphi_G(k)=\varphi_S(k)=\varphi_L(k)$ for every k in the common domain of definition and $k^{(j)}_{i_1 i_2\cdots i_{j-1}} = k^{(j-1)}_{i_1 i_2\cdots i_j}$.

§ 6. Partitional pattern of additive parameters in decomposition system $D^{(m)}(A)$.

We prepare

Definition 8. *A set of additive parameters* $\{k^{(\nu)}_{i_1 i_2 \cdots i_\nu}\}$ $(\nu = h+1, h+2, \cdots, h+d)$ *in the decomposition system* $D^{(m)}(A)$ *is said to be equipartitioned at the d-depth in the set* $A_{i_1 i_2 \cdots i_h}$ *when there hold equalities*

$$k^{(h+\nu)}_{i_1 i_2 \cdots i_h i_{h+1} \cdots i_{h+\nu}} = \frac{k^{(h+\nu)}_{i_1 i_2 \cdots i_h}}{q_{h+1} q_{h+2} \cdots q_{h+\nu}} = \overline{k^{(h+\nu)}_{i_1 i_2 \cdots i_h}} \tag{6.1}$$

for $i_j = 1, 2, \cdots, q_j$; $j = h+1, h+2, \cdots, h+\nu$; $\nu = 1, 2, \cdots, d$.

Definition 9. *A set of additive parameters* $\{k^{(\nu)}_{i_1 i_2 \cdots i_\nu}\}$ $(\nu = h+1, h+2, \cdots, h+d)$ *in the decomposition system* $D^{(m)}(A)$ *is said to be monopolistically concentrated in the set* $A_{i_1 i_2 \cdots i_h}$ *when there exists a sequence of d integers* $(i^0_{h+1}, i^0_{h+2}, \cdots, i^0_{h+d})$ *such that*

$$k^{(h+\nu)}_{i_1 i_2 \cdots i_h i^0_{h+1} \cdots i^0_{h+\nu}} = k^{(h+\nu)}_{i_1 i_2 \cdots i_h}, \qquad (\nu = 1, 2, \cdots, d) \tag{6.2}$$

and hence

$$k^{(h+\nu)}_{i_1 i_2 \cdots i_h i_{h+1} \cdots i_{h+\nu}} = 0, \qquad (\nu = 1, 2, \cdots, d) \tag{6.3}$$

for every $(i_{h+1}, i_{h+2}, \cdots, i_{h+\nu}) \neq (i^0_{h_1+1}, i^0_{h_1+2}, \cdots, i^0_{h+\nu})$.

We observe

Proposition 7. *Let a set of additive parameters* $\{k^{(\nu)}_{i_1 i_2 \cdots i_\nu}\}$ $(\nu = h+1, h+2, \cdots, h+d)$ *in the decomposition system* $D^{(m)}(A)$ *be equipartitioned at the d-depth in the set* $A_{i_1 i_2 \cdots i_h}$. *Then we have*

$$\begin{aligned} E^*\{\phi_d[A_{i_1 i_2 \cdots i_h}]\} = & \sum_{i_{h+1}=1}^{q_{h+1}} \frac{q_{h+1} \varphi^{(h+1)}_{i_1 \cdots i_h}(\overline{k^{(h+1)}_{i_1 i_2 \cdots i_h}})}{\varphi^{(h+1)}_{i_1 i_2 \cdots i_h}(k^{(h+1)}_{i_1 i_2 \cdots i_h})} \\ & \cdot \sum_{i_{h+2}=1}^{q_{h+2}} \frac{q_{h+2} \varphi^{(h+2)}_{i_1 \cdots i_h i_{h+1}}(\overline{k^{(h+2)}_{i_1 i_2 \cdots i_{h+1}}})}{\varphi^{(h+2)}_{i_1 \cdots i_{h+1}}(k^{(h+2)}_{i_1 i_2 \cdots i_{h+1}})} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdot \sum_{i_{h+d}=1}^{q_{h+d}} \frac{q_{h+d} \varphi^{(h+d)}_{i_1 \cdots i_{h+d-1}}(\overline{k^{(h+d)}_{i_1 i_2 \cdots i_{h+d-1}}})}{\varphi^{(h+d)}_{i_1 i_2 \cdots i_{h+d-1}}(k^{(h+d)}_{i_1 i_2 \cdots i_{h+d-1}})} . \end{aligned} \tag{6.4}$$

In particular when the set of $q_{h_1} \cdots q_{h_2}$ *stochastic variables* $\{X_{i_1 \cdots i_h \cdots i_{h+1} \cdots i_{h+d}}\}$ $(i_j = 1, 2, \cdots, q_j; j = h+1, \cdots, h+d)$ *is both homogeneous and inherited with the d-depth to the h-th distributional pattern in the set* $A_{i_1 i_2 \cdots i_h}$, *then we have*

$$E^*\{\phi_d[A_{i_1 i_2 \cdots i_h}]\} = \frac{q_{h+1} \cdots q_{h+d} \varphi^{(h+1)}_{i_1 i_2 \cdots i_h}(\overline{k^{(h+d)}_{i_1 i_2 \cdots i_h}})}{\varphi^{(h+1)}_{i_1 i_2 \cdots i_h}(k^{(h+d)}_{i_1 i_2 \cdots i_h})} . \tag{6.5}$$

Proposition 8. *Let a set of additive parameters* $\{k^{(\nu)}_{i_1 i_2 \cdots i_\nu}\}$ $(\nu = h+1, h+2, \cdots, h+d)$ *in the decomposition system* $D^{(m)}(A)$ *be monopolistically concentrated in* $A_{i_1 i_2 \cdots i_h}$ *at the d-depth. Then we have*

$$E^*\{\phi_d[A_{i_1 i_2 \cdots i_h}]\} = 1 . \tag{6.6}$$

§ 7. ϕ-dispersion as function of size variable.

Let us consider a sequence of ϕ-dispersion $\{\phi_d[A]\}$ $(d = 1, 2, \cdots, m)$ for an assigned decomposition system $D^{(m)}(A)$ of an assigned set A. It is now our chief concern how

$\phi_d[A]$ will change as a function of the depth d. In discussing such problem there are two fundamental tools which we propose to use.

(1°) *Size variable associated with the decomposition system* $D^{(m)}(A)$. Let us assume every subset $A_{i_1 i_2 \cdots i_m}$ is measurable, and let its measure be denoted by $m(A_{i_1 i_2 \cdots i_m})$. Let us denote by a_h the average of all $\{m(A_{i_1 i_2 \cdots i_h})\}$ ($i_j = 1, 2, \cdots, q_j$; $j = 1, 2, \cdots, h$, $h = 1, 2, \cdots, m$).

$$a_h = \frac{1}{q_1 q_2 \cdots q_h} \sum_{i_1=1}^{q_1} \cdots \sum_{i_h=1}^{q_h} m(A_{i_1 i_2 \cdots i_h}), \tag{7.1}$$

which is called the h-th average size associated with $D^{(h)}(A)$.

For the sake of convenienience we add to define

$$a_0 = A. \tag{7.2}$$

Under these preparations we introduce

Definition 10. *A sequence of functions* $\{\Phi(a_i | A)\}$ ($i = 0, 1, 2, \cdots, m$) *defined by*

$$\Phi(a_0 | A) = 1, \tag{7.3}$$

$$\Phi(a_1 | A) = \sum_{i_1=1}^{q_1} \frac{\varphi^{(1)}(k_{i_1}^{(1)})}{\varphi^{(1)}(k^{(1)})}, \tag{7.4}$$

$$\Phi(a_h | A) = \sum_{i_1=1}^{q_1} \frac{\varphi^{(1)}(k_{i_1}^{(1)})}{\varphi^{(1)}(k^{(1)})} \sum_{i_2=1}^{q_2} \frac{\varphi_{i_1}^{(2)}(k_{i_1 i_2}^{(2)})}{\varphi_{i_1}^{(2)}(k_{i_1}^{(2)})} \cdots \sum_{i_h=1}^{q_h} \frac{\varphi_{i_1 \cdots i_{h-1}}^{(h)}(k_{i_1 i_2 \cdots i_h}^{(h)})}{\varphi_{i_1 \cdots i_{h-1}}^{(h)}(k_{i_1 i_2 \cdots i_{h-1}}^{(h)})} \tag{7.5}$$

for $h = 2, 3, \cdots, m$ *is said to constitute a* ϕ*-dispersion system as a function of the average size* a_h ($h = 0, 1, 2, \cdots, m$).

Particularly when all $A_{i_1 i_2 \cdots i_h}$ *have the same measure, i.e.,*

$$m(A_{i_1 i_2 \cdots i_h}) = \frac{m(A)}{q_1 q_2 \cdots q_h} = a_h \tag{7.6}$$

for $i_j = 1, 2, \cdots, q_j$; $j = 1, 2, \cdots, h$; $h = 1, 2, \cdots, m$, *each* a_h *is the size of each subset in the h-th stage decomposition.*

In view of these facts we may and we shall call a_h to be size variable which is monotone decreasing: $a_0 > a_1 > \cdots > a_m$.

(2°) *Homogeneous and inherited multistage distributional pattern interpolated between the h-th stage decomposition* $D_h^{(m)}(A)$ *and the* $(h+1)$*-th stage one* $D_{h+1}^{(m)}(A)$. To explain the principal aspects of our techniques, let us consider the simplest ϕ-dispersion

$$\phi_1[A] = \Phi(a_1 | A) = \frac{1}{\varphi^{(1)}(k^{(1)})} \sum_{i_1=1}^{q_1} \varphi^{(1)}(k_{i_1}^{(1)}). \tag{7.7}$$

Now let us consider the case when the non-negative integer q_1 is decomposed into the product of l_1 non-negative integers $\{q_{1\nu}\}$ ($\nu = 1, 2, \cdots, l_1$) such that $q_1 = q_{11} q_{12} \cdots q_{1l_1}$. Then it is possible to establish one-to-one correspondence defined by $i = \psi(i_1, i_2, \cdots, i_l)$ where the variable (integer) i ranges from 1 to q_1 while $i_j = 1, 2, \cdots, q_{1j}$; $j = 1, 2, \cdots, l_1$. In this connection we define $A_{i_1 i_2 \cdots i_l}^{(1)} = A_{\psi(i_1, i_2, \cdots, i_{l_1})}$ and $k_{i_1 i_2 \cdots i_{l_1}}^{(1)} = k_{\psi(i_1, i_2, \cdots, i_{l_1})}^{(1)}$.

In the consequence we may rewrite (7.7) as

$$\phi_1[A] = \frac{1}{\varphi^{(1)}(k^{(1)})} \sum_{i_1=1}^{q_{11}} \sum_{i_2=1}^{q_{12}} \cdots \sum_{i_{l_1}=1}^{q_{1l_1}} \varphi^{(1)}(k_{i_1 i_2 \cdots i_{l_1}}^{(1)}), \tag{7.8}$$

which can be reformed as

$$\phi_1[A]=\sum_{i_1=1}^{q_{11}}\frac{\varphi^{(1)}(k_{i_1}^{(1)})}{\varphi^{(1)}(k^{(1)})}\sum_{i_2=1}^{q_{12}}\frac{\varphi^{(1)}(k_{i_1i_2}^{(1)})}{\varphi^{(1)}(k_{i_1}^{(1)})}\cdots\sum_{i_l=1}^{q_{1l_1}}\frac{\varphi^{(1)}(k_{i_1i_2\cdots i_l}^{(1)})}{\varphi^{(1)}(k_{i_1i_2\cdots i_{l-1}}^{(1)})}\ . \tag{7.9}$$

The expression (7.9) of $\phi_1[A]$ shows that the 1-depth ϕ-dispersion can be interpretated as the l_1-depth ϕ-dispersion which is homogeneous and inherited in the l_1-depth in the set A in the sense of Definitions 4 and 5.

In view of this fact let us now introduce a sequence of area sizes $\{a_\nu^{(1)}\}$ $(\nu=1, 2, \cdots, l_1)$ defined by

$$a_\nu^{(1)}=\frac{1}{q_{11}q_{12}\cdots q_{1\nu}}\sum_{i_1=1}^{q_{11}}\cdots\sum_{i_\nu=1}^{q_1} m(A_{i_1i_2\cdots i_\nu}^{(1)}) \tag{7.10}$$

and, with reference to this sequence, let us define a sequence of ϕ-dispersions

$$\begin{aligned}\Phi(a_\nu^{(1)}\,|\,A)&=\sum_{i_1=1}^{q_{11}}\frac{\varphi^{(1)}(k_{i_1}^{(1)})}{\varphi^{(1)}(k^{(1)})}\sum_{i_2=1}^{q_{12}}\frac{\varphi^{(1)}(k_{i_1i_2}^{(1)})}{\varphi^{(1)}(k_{i_1}^{(1)})}\cdots\sum_{i_\nu=1}^{q_{1\nu}}\frac{\varphi^{(1)}(k_{i_1i_2\cdots i_\nu}^{(1)})}{\varphi^{(1)}(k_{i_1i_2\cdots i_{\nu-1}}^{(1)})}\\&=\frac{1}{\varphi^{(1)}(k^{(1)})}\sum_{i_1=1}^{q_{11}}\cdots\sum_{i_\nu=1}^{q_{\nu 1}}\varphi^{(1)}(k_{i_1i_2\cdots i_\nu}^{(1)})\end{aligned} \tag{7.11}$$

for $\nu=1, 2, \cdots, l_1$.

It is noted that $a_0>a_1^{(1)}>a_2^{(1)}>\cdots>a_{1-1}^{(1)}>a_1^{(1)}=a_1$ and that the sequence $\{\Phi(a_\nu^{(1)}\,|\,A)\}$ $(\nu=1, 2, \cdots, l_1)$ is an interpolation between $\Phi(a_0|A)$ and $\Phi(a_1|A)$.

The interpolation technique just now explained with respect to the first stage decompostion $D_1^{(m)}(A)$ and the second stage one $D_2^{(m)}(A)$ can be similarly defined in the connection with the h-th etage $D_h^{(m)}(A)$ and the $(h+1)$-th stage $D_{h+1}^{(m)}(A)$ for $h=2, 3, \cdots, m-1$. In this way it may be possible to introduce a sequence of the average sizes $\{a_\nu^{(h)}\}$ $(\nu=1, 2, \cdots, l_\nu\,;\ \nu=1, 2, \cdots, m)$ such that

$$a_{\nu-1}>a_1^{(\nu)}>a_2^{(\nu)}>\cdots>a_{l_\nu}^{(\nu)}=a_\nu \tag{7.12}$$

and to define a sequence of ϕ-dispersions

$$\Phi(a_s^{(\nu)}\,|\,A) \tag{7.13}$$

for $s=1, 2, \cdots, l_\nu\,;\ \nu=1, 2, \cdots, m$, as a whole. In this manner we can have a graph of ϕ-dispersion $\Phi(a|A)$ as a function of a which is actually defined for $a=a_s^{(\nu)}$ just mentioned.

(3°) *Equipartition of additive parameters in the homogeneous and inherited multistage distributional pattern in the decomposition system* $D^{(m)}(A)$ *with* $\{A_{i_1i_2\cdots i_m}\}$ *of equal measure* $m(A_{i_1i_2\cdots i_m})=m(A)/(q_1q_2\cdots q_m)$.

In this case we have

$$k_{i_1i_2\cdots i_h}^{(h)}=k_{i_1i_2\cdots i_h}^{(m)}=K/q_1q_2\cdots q_h=Ka_ha_0^{-1} \tag{7.14}$$

and

$$\Phi(a_h\,|\,A)=\frac{a_h^{-1}\varphi(Ka_ha_0^{-1})}{a_0^{-1}\varphi(K)}=\frac{\varphi(b_hK)}{b_h\varphi(K)}\,,\qquad \text{say}\,, \tag{7.15}$$

by putting

$$b_h=a_ha_0^{-1}\,, \tag{7.16}$$

which is called to be a *relative area variable*, or an area-index. A relative area variable is convenient when the set A is assigned, because in this scale the relative area is equal to 1 for the set A.

A deep insight into the structure of multistage distributional pattern can be obtained by use of the ϕ-dispersion function $\Phi(a|A)$ as a function of size a in the context just explained.

It is noted that the behaviours of the ϕ-dispersion function $\Phi(a|A)$ for each of nine two stage and twenty seven three stage distributional patterns given in Examples 3.4 and 3.5 are worth while to particular numerical investigations. Indeed numerical investigations based upon our theoretical formulation can yield us some of the possible reasons to the experimental results on natural and artificial populations which have been given by Morishita [2].

§ 8. Index functions of dispersion as a function of size variable.

In connection with current uses of index of dispersion, it is useful to introduce

Definition 11. (1°) *$I_d^{[j]}[A_{i_1 i_2 \cdots i_h}]$ defined in (2.11) is said to be the d-depth index of dispersion of the j-th order in the set $A_{i_1, i_2, \cdots, i_h}$.*

(2°) *A sequence of functions $I^{[j]}(a_i|A)$ $(i=0, 1, 2, \cdots, m)$ defined by*

$$I^{[j]}(a_0|A)=(q_1 q_2 \cdots q_m)^{j-1}\,, \tag{8.1}$$

$$I^{[j]}(a_\nu|A)=E^*\{I_\nu^{[j]}[A]\}\,, \tag{8.2}$$

for $\nu=1, 2, \cdots, m$, is said to be an index function of dispersion of the j-th order as a function of size variable which takes the values $\{a_\nu\}$ $(\nu=0, 1, 2, \cdots, m)$.

The current index of dispersion used and advocated by Morishita [2]~[5] and others is the 1-depth index of dispersion of the second order in our terminology. Apart from a multiplier $(q_1 q_2 \cdots q_m)^{j-1}$, the essential aspects of $I_d^{[j]}[A]$ can be observed from our general considerations on $\phi_d[A]$ and those of $I^{[j]}(a_i|A)$ from those of functions $\Phi(a_i|A)$. We do not think the multiplier is so essential in our general approach. The sole merits, so far as the results of the present paper are concerned, can be recognised from the fact that we have

$$E_P^*\{I_d^{[j]}[A]\}=1 \tag{8.3}$$

for a homogeneous and inherited multistage distributional patterns based upon Poisson distribution. Multipliers are due to partly to the tradition and partly to the fact that theoretical emphasis may be placed upon Poisson distribution, as we can observe from

$$E^*\{I_d^{[j]}[A]\}=1 \tag{8.4}$$

for a homogeneous and inherited multistage distributional based upon Poisson distribution.

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