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https://doi.org/10.5109/13026

出版情報：統計数理研究. 12 (3/4), pp.61-73, 1967-03. Research Association of Statistical Sciences
バージョン：
権利関係：

SOME WEIGHTED ESTIMATES IN THE NONPARAMETRIC CASES

By

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(Received October 22, 1964)

§ **1. Introduction.** The procedure of weighted estimates has been first proposed by Huntsberger [5] and recently developed by Asano [1] and Asano-Sugimura [2]. They have however dealt with only the cases where the observations were obtained from the normal populations. There does not exist any literature, so far as the author is aware, about weighted methods in the nonparametric cases where the type of the underlying distribution is not assumed. The author has recently discussed in [6], [7] and [8] some asymptotic properties about the "sometimes pool estimates" in the nonparametric standpoint which are closely connected with our present problem. We are now in the situations to consider the weighted methods for estimation in the same standpoint.

The procedure for weighted estimates are respectively in Section 2 formulated for median, shift and dispersion parameters. In Section 3, some Lemmas are proved in order to establish the main results in later sections Section 4 and 5 are respectively concernned with the weighted estimates for median, shift and dispersion parameters.

§ **2. Formulation of weighted methods.**

(A) **Median.** Consider two continuous and symmetrical c.d.f. $F(x-\xi)$ and $F(y-\eta)$ where ξ and η are medians and $\eta-\xi=\delta\geqq 0$. Let O_{m_2}: $Y_1,\cdots,Y_{m_2}$ and $O_{m_3}: Z_1,\cdots, Z_{m_3}$ be respectively random samples from the populations with $F(x-\xi)$ and $F(y-\eta)$. And let $\hat{\xi}_Y(\hat{\eta}_Z)$ be some suitable estimator of $\xi(\eta)$ obtained from the sample $O_{m_2}(0_{m_3})$ and $\hat{\zeta}_{Y,Z}$ be that of ξ when we combined two samples O_{m_2} and O_{m_3} under the supposition $\delta=0$. As the test statistic U for testing the hypothesis $\xi=\eta$, the Wilcoxon statistic is used,

$$U=(m_2m_3)^{-1}\sum_{j=1}^{m_2}\sum_{k=1}^{m_3}\psi(Y_j, Z_k) \tag{1}$$

, where $\psi(y, z)=1(0)$ for $y<z$ (otherwise).
Then we define the following weighted estimator for ξ

$$W(U)=\phi(U)\hat{\xi}_Y+\{1-\phi(U)\}\hat{\zeta}_{Y,Z} \tag{2}$$

, where ϕ is called as the weighting function.
If we are limited $\hat{\zeta}_{Y,Z}$ to the linear forms of $\hat{\xi}_Y$ and $\hat{\eta}_Z$, the form among

them

(3) $$\hat{\zeta}_{Y,Z}=\lambda_{23}\hat{\xi}_Y+(1-\lambda_{23})\hat{\eta}_Z,\ m_i+m_j=N_{ij},\ \lambda_{ij}=m_i/N_{ij},$$

has the most preferable property when $\delta=0$. On the other hand, we may also use, for example, $\hat{\zeta}_{Y,Z}=\text{med}(Y_1,\cdots, Y_{m_2}, Z_1,\cdots, Z_{m_3})$, but it has been shown in [8] that they are asymptotically equivalent for small δ. From these considerations, we shall use the following as the weighted estimator for ξ

(4) $$W(U)=\phi(U)\hat{\xi}+\{1-\phi(U)\}\{\lambda_{23}\hat{\xi}+(1-\lambda_{23})\hat{\eta}\}$$

, where we write simply $\hat{\xi}(\hat{\eta})$ instead of $\hat{\xi}_Y(\hat{\eta}_Z)$.

(**B**) **Shift and dispersion parametets.** Let $O_{m_1}: X_1,\cdots, X_{m_1}: O_{m_2}: Y_1,\cdots, Y_{m_2}$ and $O_{m_3}: Z_1,\cdots, Z_{m_3}$ be respectively random sample from the continuous c.d.f. $F(x)$, $F(y, \xi)$ and $F(z, \eta)$ where we assume $F(0)=1/2$. The case

$$F(x, \xi)=F(x-\xi) \text{ and } F(y, \eta)=F(y-\eta),\ \eta-\xi=\delta$$

corresponds to the problem for shift parameter. The never pool estimator $\hat{\xi}_{X,Y}(\hat{\eta}_{X,Z})$ of $\xi(\eta)$ by the samples O_{m_1} and $O_{m_2}(O_{m_3})$ has been obtained by Hodges-Lehmann [4] as follows,

(5) $$\hat{\xi}_{X,Y}=\text{med }(Y_j-Y_i),\ \hat{\eta}_{X,Z}=\text{med }(Z_k-X_i)$$
$$1\leqq i\leqq m_1,\ 1\leqq j\leqq m_2,\ 1\leqq k\leqq m_3.$$

For the dispersion problem, we assume that

$$F(x, \xi)=F(x/\xi),\ F(y, \eta)=F(y/\eta),\ \xi,\ \eta>0,\ \eta/\xi=\delta.$$

The author [8] has proposed the estimator $\xi(\eta)$ after the similar considerations as [4]

(6) $$\begin{aligned}\hat{\xi}_{X,Y}&=\text{upper quartile of } (Y_j/X_i)\\ \hat{\eta}_{X,Z}&=\text{upper quartile of } (Z_k/X_i)\end{aligned}\qquad 1\leqq i\leqq m_1,\ 1\leqq j\leqq m_2,\ 1\leqq k\leqq m_3.$$

If we denote by U the test statistic for testing the hypothesiss $\xi=\eta$ in either case, we may also express the weighted estimetor for ξ by the following form

(7) $$W(U)=\phi(U)\hat{\xi}_{X,Y}+\{1-\phi(U)\}\{\lambda_{23}\hat{\xi}_{X,Y}+(1-\lambda_{23})\hat{\eta}_{X,Z}\}.$$

As the test statistic U, Wilcoxon statistic (1) and Sukhatme's one (8) are usually applied for the shift and dispersion problem respectively,

(8) $$V=(m_2m_3)^{-1}\sum_{j=1}^{m_2}\sum_{k=1}^{m_3}\varphi(Y_j, Z_k)$$

$$\varphi(y, z)=1(0) \text{ for } 0<y<z \text{ or } 0>y>z \text{ (otherwise)}$$

Throughout (A) and (B), if we define the weighting function ϕ by

$$\phi(u)=\begin{cases}1 & \text{for } u\in R \text{ (rejection region)}\\ 0 & \text{for } u\in R^c \text{(acception region)},\end{cases}$$

then the estimator $W(U)$ may be reduced to the "sometimes pool estimate" with a nonparametric preliminary test which has been already discussed in [6], [7] and [8].

§ **3. Some Lemmas.** We shall in this section prove some Lemmas applicable to the later discussions. Lemma 1 is concerned with the determination of the form of the weighting function ϕ of which considerations are due to Huntsberger [3]. When a constant A be a function of fixed δ, we consider an estimator

$$W_A=A\hat{\xi}+(1-A)\{\lambda\hat{\xi}+(1-\lambda)\hat{\eta}\} \tag{9}$$

, where we denote λ_{23} by λ in this section.

Lemma 1. *Assume that*

(i) $\lambda\mathrm{Var}(\hat{\xi})=(1-\lambda)\mathrm{Var}(\hat{\eta})$

(ii) $\mathrm{cov}(\hat{\xi},\ \hat{\eta})=0$ or $\lambda=1/2$,

then the mean square error D_A^2 of W_A about ξ is minimized when

$$A=\left(\frac{\delta}{c}\right)^2\Big/\left\{1+\left(\frac{\delta}{c}\right)^2\right\}, \text{ where } c \text{ is a constant.} \tag{10}$$

If $\mathrm{Var}(\hat{\xi})$, $\mathrm{Var}(\hat{\eta})$ *and* $\mathrm{cov}(\hat{\xi},\ \hat{\eta})$ *are all of order N^{-1} and δ is any constant independent on N, we obtain $A=1$ which is a trivial case corresponding to the never pool estimate.*

Proof. Let be $\mathrm{Var}(\hat{\xi})=\sigma_1^2$, $\mathrm{Var}(\hat{\eta})=\sigma_2^2$ and $\mathrm{cov}(\hat{\xi},\ \hat{\eta})=\sigma_{12}$. Then

$$D_A^2=E(W_A-\xi)^2=A^2\sigma_1^2+(1-A)^2\{\lambda^2\sigma_1^2+(1-\lambda)^2\sigma_2^2+(1-\lambda(^2\delta_1+2\lambda(1-\lambda)\sigma_{12}\}$$
$$+2A(1-A)\{\lambda\sigma_1^2+(1-\lambda)\sigma_{12}\}.$$

Therefore D_A^2 attains the minimum value when

$$A(\sigma_1^2+\sigma_2^2-2\sigma_{12}+\delta^2)=\sigma_2^2-\frac{\lambda}{1-\lambda}\sigma_1^2-\frac{1-2\lambda}{1-\lambda}\sigma_{12}+\delta^2.$$

Under the assumptions (i) and (ii), we obtain (10).

We here note that the assumptions will be both satisfied for our weighted estimators later. Now it is easy to get for small δ

$$E\left(U-\frac{1}{2}\right)\sim k\,\delta,\quad k=\int_{-\infty}^{\infty}f^2(x)\,dx$$

$$\mathrm{Var}_I\sqrt{N}\left(U-\frac{1}{2}\right)\sim[12\lambda(1-\lambda)]^{-1}\qquad \text{for Wilcoxon } U$$

$$E\left(V-\frac{1}{4}\right)\sim h\,\delta \quad h=\int_0^{\infty} xf^2(x)\,dx-\int_{-\infty}^{0} xf^2(x)\,dx$$

for Sukhatme V

$$\mathrm{Var}_1\sqrt{N}\left(V-\frac{1}{4}\right)\sim[48\lambda(1-\lambda)]^{-1}$$

Though the value of δ is unknown, we may adopt the form of ϕ by considering the results above and (10)

$$\phi(U_1)=U_1^2/(1+U_1^2) \tag{11}$$

where

$$U_1=\begin{cases}\sqrt{12\lambda_{23}(1-\lambda_{23})N_{23}}\left(U-\frac{1}{2}\right) & \text{for median, shift}\\ \sqrt{48\lambda_{23}(1-\lambda_{23})N_{23}}\left(V-\frac{1}{4}\right) & \text{for dispersion}\end{cases} \tag{12}$$

Lemma 2. *Assume that the asymptotic joint distribution of the statistics* U_1, $U_2=(\hat{\xi}-\xi)/\sigma_1$ *and* $U_3=(\hat{\eta}-\eta)/\sigma_2$ *be nonsingular normal distribution* $N(\boldsymbol{\mu}, \Sigma)$ *where*

$$\boldsymbol{\mu}=(\omega,\ 0,\ 0),\ \Sigma=(\rho_{ij}),\ \rho_{ii}=1$$

, *then the following identities hold*

$$\begin{aligned}
&(i) && E\phi(U_1)=1-H(\omega),\ E\phi^2(U_1)=1-2H(\omega)+G(\omega)\\
&(ii) && E\phi(U_1)U_j=-\rho_{1j}H'(\omega) \qquad j=2,3\\
&(iii) && E\phi(U_1)U_j^2=1-H(\omega)-\rho_{1j}^2H''(\omega)\\
(13)\ &(iv) && E\phi^2(U_1)U_j=\rho_{1i}\{G'(\omega)-2H'(\omega)\}\\
&(v) && E\phi^2(U_1)U_j^2=1-2H(\omega)+G(\omega)+\rho_{1j}^2\{G''(\omega)-2H''(\omega)\}\\
&(vi) && E\phi(U_1)U_2U_3=-|\Sigma_{23}|\{1-H(\omega)\}-\frac{\rho_{12}}{|\Sigma_{33}|}(|\Sigma_{13}|+\rho_{12}|\Sigma_{23}|)\\
& && \times\{1-H(\omega)-H''(\omega)\}
\end{aligned}$$

where

$$\begin{aligned}
(vii)\quad E\phi^2(U_1)U_2U_3=&-|\Sigma_{23}|\{1-2H(\omega)+G(\omega)\}-\frac{\rho_{12}}{|\Sigma_{33}|}(|\Sigma_{13}|+\rho_{12}|\Sigma_{23}|)\\
&\times\{1-2H(\omega)+G(\omega)+G''(\omega)-2H''(\omega)\}
\end{aligned}$$

where

$$\begin{aligned}
&H(\omega)=\exp(-\omega^2/2)\sum_{n=0}^{\infty}\left(\frac{\omega^2}{2}\right)^n A_n/n!\\
(14)\quad&\\
&G(\omega)=\frac{1}{2}\exp(-\omega^2/2)\left[1+\sum_{n=1}^{\infty}\left(\frac{\omega^2}{2}\right)^n(A_{n-1}-A_n)/n!\right]
\end{aligned}$$

$$A_n = \sqrt{e}\int_1^\infty x^{-2n}\exp(-x^2/2)\,dx,$$
(15)
$$A_n = (1-A_{n-1})/(2n-1)$$

Σ_{ij} *is the cofactor of* ρ_{ij} *in* Σ.

Proof. Since the identities (i)∼(v) and (vi)∼(vii) are respectively by the similar computations, we deal with only (v) and (vi). U_1 and U_i are jointly asymptotically normally distributed and hence we get

$$I = E\phi^2(U_1)U_j^2 = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho_{1j}^2}}\int_{-\infty}^\infty\int_{-\infty}^\infty \exp\left[-\frac{1}{2(1-\rho_{1j}^2)}\{(u_1-\omega)^2+u_j^2 - 2\rho_{1j}(u_1-\omega)u_j\}\right]u_j^2\left(\frac{u_1^2}{1+u_1^2}\right)^2 du_1du_j$$

Let

$$u_1 = v_1,\ u_j = \rho_{1j}(u_1-w)v_j$$

and integrate regarding to v_j, then

$$I = (1-\rho_{1j}^2)\int_{-\infty}^\infty\left(1-\frac{1}{1+v_1^2}\right)^2\varphi(v_1-\omega)\,dv_1 + \rho_{1j}^2\int_{-\infty}^\infty\left(1-\frac{1}{1+v_1^2}\right)^2(v_1-\omega)^2\varphi(v_1-\omega)\,dv_1$$

, where $\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$.

Define the functions

(16) $$H(\omega) = \int_{-\infty}^\infty (1+u^2)^{-1}\varphi(u-\omega)\,du,\quad G(\omega) = \int_{-\infty}^\infty (1+u^2)^{-2}\varphi(u-\omega)\,du.$$

Then we get

$$H''(\omega) = \int_{-\infty}^\infty (1+u^2)^{-1}(u-\omega)^2\varphi(u-\omega)\,du - H(\omega),$$

$$G''(\omega) = \int_{-\infty}^\infty (1+u^2)^{-2}(u-\omega)^2\varphi(u-\omega)\,du - G(\omega).$$

Substituting these equalities into I, we get (v).
To prove (vi), let the inverse of the covariance matrix Σ be $\Sigma^{-1} = (\rho^{ij})$ and the cofactor of ρ^{ij} be Σ_{ij}^{-1}. From the asymptotic nonsingular normality,

$$J = E\phi(U_1)U_2U_3 = \left(\frac{1}{\sqrt{2\pi}}\right)^3\frac{1}{|\Sigma|^{1/2}}\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty \phi(u_1)u_2u_3\exp\Bigg[-\frac{1}{2}\Bigg\{\rho^{11}(u_1-\omega)^2 + \rho^{22}u_2^2 + 2\rho^{12}(u_1-\omega)u_2 + \rho^{33}\left(u_3+\frac{\rho^{13}}{\rho^{33}}(u_1-\omega)+\frac{\rho^{23}}{\rho^{33}}u_2\right)^2 - \rho^{33}\left(\frac{\rho^{13}}{\rho^{33}}(u_1-\omega)+\frac{\rho^{23}}{\rho^{33}}u_2\right)^2\Bigg\}\Bigg]du_1du_2du_3$$

$$=\frac{1}{2\pi}\frac{1}{(|\Sigma|\,|\Sigma_{11}^{-1}|)^{1/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(u_1)\exp\left\{\frac{-1}{2}(v^2+(u_1-\omega)^2)\right\}\left[\frac{\rho^{23}}{\rho^{33}}\left\{\frac{v}{\sqrt{\frac{|\Sigma_{11}^{-1}|}{\rho^{33}}}}\right.\right.$$

$$\left.\left.+\frac{|\Sigma_{12}^{-1}|}{|\Sigma_{11}^{-1}|}(u_1-\omega)\right\}^2+\frac{\rho^{13}}{\rho^{33}}(u_1-\omega)\left\{\frac{v}{\sqrt{\frac{|\Sigma_{11}^{-1}|}{\rho^{33}}}}+\frac{|\Sigma_{12}^{-1}|}{|\Sigma_{11}^{-2}|}(u_1-\omega)\right\}\right]du_1dv$$

$$=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(u_1)\exp\left\{-\frac{1}{2}(u_1-\omega)^2\right\}\left[\rho^{23}|\Sigma|+\frac{\rho^{12}}{\rho^{33}}(\rho_{12}\rho^{23}+\rho^{13})(u_1-\omega)^2\right]du_1$$

$$=\frac{\rho_{12}}{\rho^{33}}(\rho_{12}\rho^{23})+\rho^{13}H''(\omega)-\left\{|\Sigma|\rho^{23}+\frac{\rho_{12}}{\rho^{33}}(\rho_{12}\rho^{23}+\rho^{13})\right\}\{1-H(\omega)\}.$$

It follows (vi) from the relation $\rho^{ij}=|\Sigma_{ij}|/|\Sigma|$.

Lemma 3. *If the rank of the covariance matrix Σ is 2 in* Lemma 2, *there exists a linear relation between U_i's with a probability one*

$$(17)\qquad |\Sigma_{13}|(U_1-\omega)+|\Sigma_{23}|U_2+|\Sigma_{33}|U_3=0.$$

Furthermore it holds that

$$E\phi(U_1)U_2U_3=-\frac{|\Sigma_{13}|}{|\Sigma_{33}|}\rho_{12}\{1-H(\omega)-H''(\omega)\}$$

$$-\frac{|\Sigma_{23}|}{|\Sigma_{33}|}\{1-H(\omega)-\rho_{12}^2H''(\omega)\}.$$

(18)

$$E\phi^2(U_1)U_2U_3=-\frac{1}{|\Sigma_{33}|}(|\Sigma_{23}|+\rho_{12}|\Sigma_{13}|)\{1-2H(\omega)+G(\omega)\}$$

$$-\frac{\rho_{12}}{|\Sigma_{33}|}(|\Sigma_{13}|+\rho_{12}|\Sigma_{23}|)\{G''(\omega)-2H''(\omega)\}.$$

Proof. Following Cramer's theory [3] for singular distributions, we have a linear relation with a probability one

$$U_3=\alpha_{13}(U_1-\omega)+\alpha_{23}U_2$$

Multiply both hands by $U_1-\omega$ or U_2 and compute the expected values, then we get the equations

$$\alpha_{13}+\alpha_{23}\rho_{12}=\rho_{13},\quad \alpha_{13}\rho_{12}+\alpha_{23}=\rho_{23}.$$

Hence

$$\alpha_{13}=-|\Sigma_{13}|/|\Sigma_{33}|,\quad \alpha_{23}=-|\Sigma_{23}|/|\Sigma_{33}|.$$

To prove the identities (18), the relations (17) and (13) are applied.

$$E\phi(U_1)U_2U_3=-\frac{|\Sigma_{13}|}{|\Sigma_{33}|}E\phi(U_1)(U_1-\omega)U_2-\frac{|\Sigma_{23}|}{|\Sigma_{33}|}E\phi(U_1)U_2^2$$

And by the similar computations as Lemma 2

$$E\phi(U_1)(U_1-\omega)U_2=\rho_{12}\int_{-\infty}^{\infty}(u-\omega)^2\frac{u^2}{1+u^2}\varphi(u-\omega)du$$
$$=\rho_{12}\{1-H(\omega)-H''(\omega)\}.$$

From (iii) in (13),

$$E\phi(U_1)U_2^2=1-H(\omega)-\rho_{12}^2H''(\omega)$$

These values lead to the first part of (18). The computations are similar for the second part.

§ **4. The weighted methods for median.** Though the sample median $\tilde{Y}$ is usually used as the never pool estimate for median ξ, the following is more efficientas shown by Hodges-Lehmann [4]

$$\hat{Y}=\text{med}\left[\frac{1}{2}(Y_i+Y_j)\right]\quad 1\leqq i<j\leqq m_2. \tag{19}$$

We shall discuss about the weighted estimates by both $\tilde{Y}$ and $\hat{Y}$.

4. 1. The methods by $\tilde{Y}$. Let the Sign test statistics be

$$\begin{aligned} U_2'(Y)&=m_2^{-1}\sum_{j=1}^{m_2}\psi(Y_j)\\ U_3'(Z)&=m_3^{-1}\sum_{k=1}^{m_3}\psi(Z_k)\end{aligned} \tag{20}$$

where $\psi(y)=1(0)$ for $y>0$ (otherwize).

The following probability relation has been proved in [4],

$$Pr(\tilde{Y}\leqq a)=Pr[U_2'(Y-a)<1/2] \tag{21}$$

Define the normalized statistics

$$U_2=2f(0)\sqrt{m_2}(\tilde{Y}-\xi),\ U_3=2f(0)\sqrt{m_3}(\tilde{Z}-\eta)$$

, then we may get Theorem 1.

Theorem 1. *The asymptotic joint distribution of U_1, U_2 and U_3 is given by the nonsingular normal distribution $N(\mu, \Sigma)$ with the mean vector $\mu=(\omega, 0, 0)$ and covariance matrix*

$$\Sigma=\begin{pmatrix} 1 & -\sqrt{3(1-\lambda_{23})}/2 & \sqrt{3\lambda_{23}}/2\\ -\sqrt{3(1-\lambda_{23})}/2 & 1 & 0\\ \sqrt{3\lambda_{23}}/2 & 0 & 1\end{pmatrix} \tag{22}$$

, *where* $\delta=\gamma/\sqrt{N_{23}}k$, $\omega=\sqrt{12\lambda_{23}(1-\lambda_{23})}\gamma$.

Proof. The asymptotic normality may be established from (21) and the theory of U-statistics, that is,

$$Pr[U_1 \leqq u_1,\ U_2 \leqq u_2,\ U_3 \leqq u_3]$$

$$= Pr\left[U^1 \leqq u_1,\ U\left(Y-\xi-\frac{u_2}{2f(0)\sqrt{m_2}}\right) \leqq \frac{1}{2},\ U_3'\left(Z-\eta-\frac{u_3}{2f(0)\sqrt{m_3}}\right) \leqq \frac{1}{2}\right]$$

, where U_1, U_j' are well-known U-statistics.
It is obvious that

$$\mathrm{Var}\ U_j \sim 1, \quad \mathrm{cov}(\tilde{Y}, \tilde{Z}) = 0.$$

Secondly we compute $\rho_{12} = \mathrm{cov}(U_1, U_2) \sim -\sqrt{3(1-\lambda_{23})}/2$. In fact,

$$\mathrm{cov}(U, U_2') = \frac{1}{m_2^2 m_3} E\left[\Sigma\Sigma\psi(Y_j, Z_k)\Sigma\psi\left(Y_j-\xi-\frac{u_2}{2f(0)\sqrt{m_2}}\right)\right]$$
$$-E\psi(Y, Z)\ E\psi\left(Y-\xi-\frac{u_2}{2f(0)\sqrt{m_2}}\right)$$

And

$$E\Sigma\Sigma\psi(Y_j, Z_k)\Sigma\psi\left(Y_j-\xi-\frac{u_2}{2f(0)\sqrt{m_2}}\right)$$
$$= m_2 m_3 E\psi(Y, Z)\psi\left(Y-\xi-\frac{u_2}{2f(0)\sqrt{m_2}}\right)$$
$$+ m_2 m_3 (m_2-1) E\psi(Y, Z) E\psi\left(Y-\xi-\frac{u_2}{2f(0)\sqrt{m_2}}\right)$$
$$\sim \frac{1}{8} m_2 m_3 + \frac{1}{4} m_2 m_3 (m_2-1)$$

Thus

$$\rho_{12} \sim \sqrt{12\lambda_{23}(1-\lambda_{23})N_{23}m_2}\left(-\frac{1}{8m_2}\right)/\sqrt{\mathrm{Var}U_2'}$$
$$= -\sqrt{3(1-\lambda_{23})}/2.$$

The computations are also similar for ρ_{13}. Now when we define the weighted estimator

$$W(U_1) = \phi(U_1)\tilde{Y} + \{1-\phi(U_1)\}\{\lambda_{23}\tilde{Y} + (1-\lambda_{23})\tilde{Z}\}, \tag{23}$$

we use the mean square error D_w^2 as a criterion of goodness to determine whether or not $W(U_1)$ offers any advantages over the never pool or sometimes pool estimate for ξ.

Theorem 2. *The asymptrtic mean value and mean square error D_w^2 of $W(U_1)$ about ξ are given by the followings*

$$EW(U_1) = \xi + \frac{\sqrt{3(1-\lambda_{23})}}{4f(0)\sqrt{m_2}} H'(\omega) + \frac{1-\lambda_{23}}{\sqrt{N_{23}}}\frac{r}{k}H(\omega) + o(N^{-1/2}) \tag{24}$$

$$(25)\quad D_w^2=\frac{1}{4f^2(0)N_{23}}\left[1+\frac{1-\lambda_{23}}{\lambda_{23}}\left\{1-2H(\omega)+G(\omega)+\frac{3}{4}[G''(\omega)-2H''(\omega)]\right\}\right]$$
$$+\frac{\sqrt{3\lambda_{23}(1-\lambda_{23})}}{2f(0)N_{23}}\frac{1-\lambda_{23}}{\lambda_{23}}\frac{r}{k}\{G'(\omega)-H'(\omega)\}+\frac{(1-\lambda_{23})^2}{4N_{23}}\frac{r^2}{k^2}G(\omega).$$

Proof. From (23),

$$\sqrt{N_{23}}(W-\xi)=\phi(U_1)\frac{U_2}{2f(0)\sqrt{\lambda_{23}}}+\{1-\phi(U_1)\}\left\{\frac{\sqrt{\lambda_{23}}}{2f(0)}U_2\right.$$
$$\left.+\frac{\sqrt{1-\lambda_{23}}}{2f(0)}U_3+(1-\lambda_{23})\frac{r}{k}\right\}$$

Applying Theorem 1 and Lemma 2, we may obtain the results.
Consider the case of $r=0$, then we get from (25)

$$(26)\quad D_{w_0}^2=\frac{1}{4f^2(0)N_{23}}\left[1+\frac{1-\lambda_{23}}{\lambda_{23}}\left\{1-2H(0)+G(0)+\frac{3}{4}[G''(0)-2H''(0)]\right\}\right]$$

The identities (14) and (15) show that

$$H(0)=A_0=.656 \qquad G(0)=.5 \qquad G'(0)=H'(0)=0$$
$$H''(0)=1-2A_0=-.312 \qquad G''(0)=A_0-1=-.344$$

and we get the inequalities after substituting these values into (26)

$$(27)\qquad \frac{1}{4f^2(0)N_{23}}<D_{w_0}^2<\frac{1}{4f^2(0)m_2}.$$

The mean square error of the "sometimes pool estimate" of ξ has been given for $r=0$, $\lambda_{23}=1/2$ in [6] by

$$(28)\qquad \left[2-\Phi(z_\alpha)+\frac{3}{4}z_\alpha\varphi(z_\alpha)\right]/4f^2(0)N_{23}$$
$$1-\Phi(z_\alpha)=\alpha.$$

The asymptotic efficiency $e_{w,s}$ of the weighted estimator with regard to the "sometimes pool estimate" is given by

$$e_{w,s}=\left\{2-\Phi(z_\alpha)+\frac{3}{4}z_\alpha\varphi(z_\alpha)\right\}/\left[2-2H(0)+G(0)+\frac{3}{4}\{G''(0)-2H''(0)\}\right].$$

z_α	∞	1.30	1	0.50	0	$-\infty$
e	0.72	0.90	0.96	1.03	1.07	1.43

4.2. Use of Wilcoxon one sample test statistics. As the estimates of ξ and η, we use the followings instead of the sample medians

$$(29)\qquad \begin{aligned} \hat{Y} &= \mathrm{med}\left[\frac{1}{2}(Y_i+Y_j)\right] \qquad 1\leqq i<j\leqq m_2 \\ \hat{Z} &= \mathrm{med}\left[\frac{1}{2}(Z_i+Z_j)\right] \qquad 1\leqq i<j\leqq m_3 \end{aligned}$$

which are derived from Wileoxon one sample test statistics. There also exists a similar relation as (21) which is useful to obtain the asymptotic distribution of $\hat{Y}$. The weighted estimation for ξ is defined by

$$(30)\qquad W(U_1)=\phi(U_1)\hat{Y}+\{1-\phi(U_1)\}\{\lambda_{23}\hat{Y}+(1-\lambda_{23})\hat{Z}\}.$$

When we define

$$(31)\qquad \hat{U}_2=\sqrt{12m_2}k(\hat{Y}-\xi),\quad \hat{U}_3=\sqrt{12m_3}k(\hat{Z}-\eta),$$

the following values have been computed in [7]

$$\mathrm{Var}\,\hat{U}_j\sim 1,\ \mathrm{cov}(U_1,\ \hat{U}_2)\sim-\sqrt{1-\lambda_{23}},\ \mathrm{cov}(U_1,\ \hat{U}_3)=\sqrt{\lambda_{23}}.$$

Thus the asymptotic covariance matrix is given by

$$(32)\qquad \Sigma=\begin{pmatrix} 1 & -\sqrt{1-\lambda_{23}} & \sqrt{\lambda_{23}} \\ -\sqrt{1-\lambda_{23}} & 1 & 0 \\ \sqrt{\lambda_{23}} & 0 & 1 \end{pmatrix}$$

and the rank of Σ is 2. From Lemma 3, we get the linear relation with a probability one

$$(33)\qquad \hat{U}_3=\frac{1}{\sqrt{\lambda_{23}}}(U_1-\omega)+\sqrt{\frac{1-\lambda_{23}}{\lambda_{23}}}\hat{U}_2$$

and

$$E\phi(U_1)\hat{U}_2\hat{U}_3=\sqrt{\lambda_{23}(1-\lambda_{23})}H''(\omega)$$

$$E\phi^2(U_1)\hat{U}_2\hat{U}_3=-\sqrt{\lambda_{23}(1-\lambda_{23})}\{G''(\omega)-2H''(\omega)\}.$$

In fact, it is easily verified in Lemma 3 from the values

$$|\Sigma_{33}|=\lambda_{23},\ |\Sigma_{13}|=-\sqrt{\lambda_{23}},\ |\Sigma_{23}|=-\sqrt{\lambda_{23}(1-\lambda_{23})}\ \text{and}\ \rho_{12}=-\sqrt{1-\lambda_{23}}.$$

Theorem 3. *The asymptotic mean value and mean squre error of* W (30) *may be given as follows,*

$$(34)\qquad E(W)=\xi+\frac{1-\lambda_{23}}{\sqrt{N_{23}}}\frac{r}{k}H(\omega)+\frac{\sqrt{1-\lambda_{23}}}{\sqrt{12m_2}\,k}H'(\omega)$$

$$(35)\quad \mathrm{M.S.E.}(W)=(12N_{23}k^2)^{-1}\left[1+\frac{1-\lambda_{23}}{\lambda_{23}}\{1-2H(\omega)+G(\omega)+G''(\omega)-2H''(\omega)\}\right]+\frac{(1-\lambda_{23})^2}{N_{23}}\frac{r^2}{k^2}G(\omega)+\frac{(1-\lambda_{23})^2}{\sqrt{3\lambda_{23}(1-\lambda_{23})}\,N_{23}}\frac{r}{k^2}\{G'(\omega)-H'(\omega)\}.$$

The proof is easy by using Lemma 3 and the similar computations as in Theorem 2.

§ 5. The weighted methods for shift and dispersion qarameters.

5.1. Shift parameter. As stated in Section 2, we assume the distributions be $F(x)$, $F(y-\xi)$ and $F(z-\eta)$ where $\eta-\xi=\delta$ and $F(0)=1/2$ without loss of generality. The never pool estimates of ξ and η are given by

$$\text{(36)}\qquad \begin{aligned}\hat{\xi}_{x,y}&=\text{med }(Y_j-X_i)\\ \hat{\eta}_{x,z}&=\text{med }(Z_k-X_i)\end{aligned}\qquad 1\leqq i\leqq m_1,\ 1\leqq j\leqq m_2,\ 1\leqq k\leqq m_3.$$

We define the statistics

$$\text{(37)}\qquad \begin{aligned}U_{12}&=\sqrt{N_{12}}\,k_{12}(\hat{\xi}_{x,y}-\xi)\\ U_{13}&=\sqrt{N_{13}}\,k_{13}(\hat{\eta}_{x,z}-\eta)\end{aligned}$$

, where $\quad k_{ij}=\sqrt{12\lambda_{ij}(1-\lambda_{ij})}\int_{-\infty}^{\infty}f^2(x)dx$

and we here express U_1 in (12) by U_{23}. Then the asymptotic covariance of U_{ij}'s are given for amall δ from the results in [7] as follows

$$\text{(38)}\qquad \begin{aligned}&\text{cov }(U_{23},\ U_{12})\sim-\sqrt{\lambda_{12}(1-\lambda_{23})}\\ &\text{cov}(U_{23},\ U_{13})\sim\ \sqrt{\lambda_{13}\lambda_{23}}\\ &\text{cov}(U_{12},\ U_{13})\sim\ \sqrt{(1-\lambda_{12})(1-\lambda_{13})}.\end{aligned}$$

Thus the covariance matrix is given by

$$\text{(89)}\qquad \Sigma=\begin{pmatrix}1 & -\sqrt{\lambda_{12}(1-\lambda_{23})} & \sqrt{\lambda_{13}\lambda_{23}}\\ -\sqrt{\lambda_{12}(1-\lambda_{23})} & 1 & \sqrt{(1-\lambda_{12})(1-\lambda_{13})}\\ \sqrt{\lambda_{13}\lambda_{23}} & \sqrt{(1-\lambda_{12})(1-\lambda_{13})} & 1\end{pmatrix}$$

and we may get that Σ is of rank 2 after some computations. From Lemma 3, we obtain the relation

$$\text{(40)}\qquad U_{13}=\alpha_{23}(U_{23}-\omega_{23})+\alpha_{12}U_{12}$$

where

$$\begin{aligned}&\alpha_{23}=\{\sqrt{\lambda_{12}\lambda_{23}}+\sqrt{\lambda_{12}(1-\lambda_{12})(1-\lambda_{13})(1-\lambda_{23})}\}/\{1-\lambda_{12}(1-\lambda_{23})\}\\ &\alpha_{12}=\{\sqrt{(1-\lambda_{12})(1-\lambda_{13})}+\lambda_{12}\sqrt{\lambda_{23}(1-\lambda_{23})}\}/\{1-\lambda_{12}(1-\lambda_{23})\},\\ &\delta=\gamma/\sqrt{N_{12}}\,k_{12},\qquad \omega_{23}=\sqrt{\frac{N_{23}\lambda_{23}(1-\lambda_{23})}{N_{12}\lambda_{12}(1-\lambda_{12})}}\,\gamma.\end{aligned}$$

Now if we take $\lambda_{23}=1/2$, that is, $m_2=m_3$, the assumptions in Lemma 1 are satisfied and then the weighting function ϕ is determined by (11). Under the assumption $m_2=m_3$, we may obtain

$$U_{13}=(U_{23}-\omega)+U_{12},\quad \omega=\gamma/\sqrt{2\lambda_{12}}$$

and define the weighted estimator for

$$W(U_{23})=\phi(U_{23})\hat{\xi}_{x,y}+\frac{1}{2}\{1-\phi(U_{23})\}(\hat{\xi}_{x,y}+\hat{\eta}_{x,z}). \tag{41}$$

Theorem 4. *The asymptotic mean value and mean square error are given by the followings*

$$E(W)=\xi+\frac{1}{2\sqrt{N_{12}}\,k_{12}}[\gamma H(\omega)+\sqrt{2\lambda_{12}}H'(\omega)] \tag{42}$$

$$\begin{aligned}\text{M.S.E.}(W)=(4N_{12}k_{12}^2)^{-1}[4+\gamma^2G(\omega)+2\lambda_{12}\{G(\omega)-2H(\omega)\\ +G''(\omega)-2H''(\omega)\}\ +4\gamma\sqrt{\lambda_{12}/2}\{G'(\omega)-H'(\omega)\}].\end{aligned} \tag{43}$$

Noticing the identity

$$\begin{aligned}E\phi^2(U_{23})U_{12}U_{13}=(1-\lambda_{12})\{1-2H(\omega)+G(\omega)\}\\ -\frac{1}{2}\lambda_{12}\{G''(\omega)-2H''(\omega)\},\end{aligned}$$

the proof is easy from Lemma 3.

5.2. Dispersion parameter. We slightly change the notations in Section 2, that is, let the c.d.f. be respectively $F(x)$, $F(y/\theta)$ and $F(z/\theta\theta')$ where $F(x)$ is symmetrical about $x=0$ and $\theta'=1+\gamma/\sqrt{N_{23}}h$, $h=\int_0^\infty xf^2dx-\int_{-\infty}^0 xf^2dx$. Our purpose is to estimate the value of the dispersion parameter θ. The never pool estimate $V_{X,Y}$ for θ has been given in [8]

$$V_{X,Y}=\text{upper quartile of } m_1m_2 \text{ set of } (Y_j/X_i).$$

Secondly define the statistics

$$\begin{aligned}V_{12}&=\sqrt{N_{12}}\,h_{12}\left(\frac{V_{X,Y}}{\theta}-1\right)\\ V_{13}&=\sqrt{N_{13}}\,h_{13}\left(\frac{V_{X,Z}}{\theta\theta'}-1\right)\end{aligned} \tag{44}$$

$$h_{ij}=\sqrt{48\lambda_{ij}(1-\lambda_{ij})}\,h$$

and denote U_1 in (12) by V_{23}. Then the following properties may be easily obtain from the results in [8]

(i) V_{1j} is asymptotically normally distributed in $N(0, 1)$

(ii) The joint asymptotic distribution of any two V_{ij} is bivariate normal distribution

(iii) The mean and covariance matrix of V'_{ij} s are given by

$$\mu=(\omega,\ 0,\ 0),\quad \omega=\sqrt{48\lambda_{23}(1-\lambda_{23})}\ \gamma$$

and Σ equals to the form (39).

Thus we may get the similar results as in 5. 1.

In the case that the population medians are unknown, we shall apply the statistic $\hat{V}_{X,Y}$ instead of $V_{X,Y}$

$$\hat{V}_{X,Y}=\text{upper quartile of } [(Y_j-\tilde{Y})/(X_i-\tilde{X})],$$

Then we may obtain the asymptotically equivalent results to the above under the assumption $f(x)$ be bounded in absolute.

References

[1] Asano, C. (1961). *Some considerations on the combination of estimates from differt biological assays.* Bull. Math. Stat. 10 17–32.

[2] Asano, C. and Sugimura, M. (1961). *Some consideration on estimation of podulation variance due to the use of pooling data.* Bull. Math. Stat. 10 33–44.

[3] Cramer, H. (1946). *Mathematical methods of statistics.* Princeton Univ. Press.

[4] Hodges, J. L. Jr. and Lehmann, E. L. (1963). *Estimates of location based on rank tests.* Ann. Math. Statist. 34 598–611.

[5] Huntsberger, D. V. (1955). *A generalization of a preliminary testing procedure for pooling data.* Ann. Math. Statist. 26 734–743.

[6] Tamura, R. (1965) *Nonparametric inferences with a preliminary test.* Bull. Math. Statist. 11 39–61.

[7] Tamura, R. (1965) *Some estimate procedures with a nonparametric preliminary test I.* Bull.Math. Statist. 11 63–71.

[8] Tamura, R. *Some estimate procedures with a nonparametric preliminary test II.* (submit to Bull. Math. Statist.)

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