A BAYESION APPROACH FOR TESTING HYPOTHESES WITH CONTINUATION OF OBSERVATIONS FROM A APOPULATION IN A FAMILY OF POPULATIONS WITH ONE PARAMETER EXPONENTIAL DISTRIBUTIONS

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https://doi.org/10.5109/13024

出版情報：統計数理研究. 12 (3/4), pp.35-46, 1967-03. Research Association of Statistical Sciences
バージョン：
権利関係：

A BAYESIAN APPROACH FOR TESTING HYPOTHESES WITH CONTINUATION OF OBSERVATIONS FROM A POPULATION IN A FAMILY OF POPULATIONS WITH ONE PARAMETER EXPONENTIAL DISTRIBUTIONS

By

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§1. Summary.

We shall consider the problem to decide between two hypotheses concerning the unknown population parameter when our population is distributed in one parameter exponential distribution, under the following set up of sampling.

A sample of prescribed size m is decided to be drawm from our population, and moreover we (A) may or (B) may not draw a second independent sample of another prescribed size n from the same population. This last choice decision between two alternatives (A) and (B) is assumed to be made in view of the risk defined in terms of the expected loss due to the error with reference to a given Bayesian distribution of the population parameter and the costs due to samplings and observations.

It is noted that somewhat similar but not identical problem was discussed by Anderson [1] for normal distributions. The purpose of our paper is to show that the notion of additive family of sufficient statistics is useful in getting concrete results through the comparisons of the risks associated two alternatives. We shall be concerned with the following two cases when the continuous distribution is distributed (I) over all real line, and (II) over all positive part of it.

§2. Introduction.

Let u_m and u_n be two independent statistics of the same parameter τ having their probability density function with respect to a common measure μ over the real line R such that

$$f_m(u_m,\tau)\,d\mu(u_m)=\exp\{\tau u_m+b_m(\tau)+a_m(u_m)\}\,d\mu(u_m) \tag{2. 1}$$

$$f_n(u_n,\tau)\,d\mu(u_n)=\exp\{\tau u_n+b_n(\tau)+a_n(u_n)\}\,d\mu(u_n), \tag{2. 2}$$

respectively. Let us assume that $a_l(x)$ $(l=m,\ n)$ is a function with a continuous first derivative with regard to x in R.

Testing a null hypothesis $H_1:\ \tau=\tau_1$ against an alternative $H_2:\ \tau=\tau_2$,

($\tau_1<\tau_2$) by use of a certain statistics u_l is done in the following way.

(1) We accept H_1, if

$$f_l(u_l,\ \tau_2)/f_l(u_l,\ \tau_1)<a, \tag{2.3}$$

and

(2) We accept H_2, if

$$f_l(u_l,\ \tau_2)/f_l(u_l,\ \tau_1)\geqq a, \tag{2.4}$$

where a is a certain positive constant. The determination of the critical value a may be given in several different ways.

We shall be concerned with the case when a priori distribution regarding population parameter τ is given. Let ξ be an a priori probability that the hypothesis H_1 is true and then $1-\xi$ be the a priori probability that the hypothesis H_2 is true. Let W_i be a loss which is caused by the incorrect decision to accept $H_j (j\neq i)$ when H_i is true $(i,\ j=1,\ 2)$, respectively. Let C_m and C_n be the cost due to the sampling and observations for the first sample of size m and for the second sample of size n respectively. Then the expected risk using u_m alone for an assigned non-negative a is given by

$$\begin{aligned} R_A(a\,;\ \xi)=&W_1\xi P[f_m(u_m,\ \tau_2)/f_m(u_m,\ \tau_1)\geqq a\mid H_1] \\ &+W_2(1-\xi)P[f_m(u_m,\ \tau_2)/f_m(u_m,\ \tau_1)<a\mid H_2]+C_m. \end{aligned} \tag{2.5}$$

Similarly, the expected risk using u_m+u_n is given by

$$\begin{aligned} R_B(a\,;\ \xi)=&W_1\xi P[f_{m+n}(u_m+u_n,\ \tau_2)/f_{m+n}(u_m+u_n,\ \tau_1)\geqq a\mid H_1] \\ &+W_2(1-\xi)P[f_{m+n}(u_m+u_n,\tau_2)/f_{m+n}(u_m+u_n,\ \tau_1)<a\mid H_2]+C_m+C_n. \end{aligned} \tag{2.6}$$

With regard to these risks, we shall employ the following Bayesian procedure. This runs as follows. Generally, if the likelihood ratio Z has a probability density function under each of two hypotheses, the Bayes procedure is to adopt the criterion defined by the critical value a which minimizes the following risk

$$R(a\,;\ \xi)=W_1\xi P[Z\geqq a\mid H_1]+W_2(1-\xi)P[Z<a\mid H_2]+C. \tag{2.7}$$

In order to obtain this critical value a, we shall make use of two lemmas which will be enunciated in the section 3.

Now the choice decision between two alternatives (A) and (B) will be done by comparing two risks corresponding to the Bayesian procedure applied to each of two alternatives. The outline of comparison can be done in the following way:

(1) We introduce

$$d(\xi\,;\ A,\ B)=\min_{0<a<\infty} R_B(a\,;\ \xi)-\min_{0<a<\infty} R_A(a\,;\ \xi), \tag{2.8}$$

provided that both the first and the second term in the right hand side of (2.8) exist for every ξ in $0<\xi<1$.

(2) Our choice decision between two alternatives (A) and (B) is given in the following way.

(i) We take (A), if $d(\xi; A, B) \geqq 0$.

(ii) We take (B), if $d(\xi; A, B) < 0$.

However it is not easy to observe whether $d(\xi; A, B)$ is nonnegative or negative for each assigned value of ξ in $0<\xi<1$. In order to observe the sign of $d(\xi; A, B)$ we shall give several theorems in the section 4.

§3. Two lemmas and determination of the critical value a.

We shall make use of the following two lemmas regarding an additive family of sufficient statistics.

Lemma 1. (Kitagawa [2]) *Let us assume that there is a function* $a_{m,n}(\cdot)$ *such that*

$$\exp\{a_{m,n}(u)\} = \int_R \exp\{a_m(u-v)\}\exp\{a_n(v)\}d\mu(v). \tag{3. 1}$$

Then $u_m + u_n$ *is a sufficient statistics for* τ *having the probability density function with respect to the measure* μ *such that*

$$\begin{aligned} &f_{m+n}(u_m+u_n, \tau)d\mu(u_m+u_n) \\ &= \exp\{\tau(u_m+u_n)+b_m(\tau)+b_n(\tau)+a_{m,n}(u_m+u_n)\}d\mu(u_m+u_n). \end{aligned} \tag{3. 2}$$

Lemma 2. (Kitagawa [2]) *Let us assume that there holds the additivity such that*

$$\exp\{a_m(u)\} = \int_R \exp\{a_{m-1}(u-v)\}\exp\{a_1(v\}d\mu(v), \tag{3. 3}$$

and

$$b_m(\tau) = mb(\tau), \text{ (say)}, \tag{3. 4}$$

for all positive integers $m \geqq 1$.

Then $a_{m,n}(u) = a_{m+n}(u)$ *for all integers* $m, n \geqq 1$.

However we shall restrict ourselves to the case when $d\mu(u) = dx$ for all real u. The determination of the critical value a is done in each case of the following.

Case (I) The case when the following three conditions are satisfied:

(1°) The distribution functions are distributed over all real line.

(2°) The range of the parameter τ contains a finite interval $(-A, A)$ with some positive constant A.

(3°) The function $b(\sigma)$ is regular and analytic over the strip $-A \leqq \Re(\sigma) \leqq A$ in the complex σ plane.

Case (II) The case when the following three conditions are satisfied:

(1°) The distribution functions are distributed over all positive half-line.

(2°) The range of the parameter τ contains a finite interval $(-B,$

$-A)$ $(0<A<B)$ belonging to the half line $-\infty<\tau<0$.

(3°) The function $b(\sigma)$ is regular and analytic over the strip $-B\leqq\Re(\sigma)\leqq -A$ in the complex σ plane.

Before the determination of the critical value a, let us introduce the following notations:

$$\delta(\xi)=W_1\xi/(W_2(1-\xi)), \tag{3. 5}$$

$$H(\tau_1, \tau_2)=\{b(\tau_1)-b(\tau_2)\}/(\tau_2-\tau_1), \tag{3. 6}$$

$$r_n=nH(\tau_1, \tau_2), \tag{3. 7}$$

$$K(\tau_1, \tau_2)=\{\tau_2 b(\tau_1)-\tau_1 b(\tau_2)\}/(\tau_2-\tau_1), \tag{3. 8}$$

$$s_n=nK(\tau_1, \tau_2), \tag{3. 9}$$

$$x_m(\xi)=(\tau_2-\tau_1)^{-1}log\delta(\xi)+r_m. \tag{3. 10}$$

(I) Making use of two lemmas stated above, for each ξ in $0<\xi<1$ we have

$$\begin{aligned}\frac{\partial R_A(a;\ \xi)}{\partial a}&=exp\left\{a_m\left(\frac{log\ a}{\tau_2-\tau_1}+mH(\tau_1,\ \tau_2)\right)+mK(\tau_1,\ \tau_2)\right\}\\&\quad\times\left[-W_1\xi exp\left\{\tau_1\frac{log\ a}{\tau_2-\tau_1}\right\}+W_2(1-\xi)exp\left\{\tau_2\frac{log\ a}{\tau_2-\tau_1}\right\}\right]\\&=exp\{a_m((\tau_2-\tau_1)^{-1}log\ a+r_m)+s_m\}W_2(1-\xi)a^{\tau1/(\tau_2-\tau_1)}\\&\quad\times(a-W_1\xi/\{W_2(1-\xi)\}).\end{aligned} \tag{3. 11}$$

Moreover we have

$$\begin{aligned}(\partial^2/\partial a^2)\{R_A(a;\ \xi)\}&=(\partial/\partial a)\{R_A(\mathrm{a};\ \xi)\}(\partial/\partial a)\{a_m((\tau_2-\tau_1)^{-1}log\ a\\&\quad+r_m)\}+exp\{a_m((\tau_2-\tau_1)^{-1}log\ a+r_m)+s_m\}\\&\quad\times W_2(1-\xi)a^{\tau1/(\tau_2-\tau_1)}[\tau_2 a-\tau_1 W_1\xi/\{W_2(1-\xi)\}]\\&\quad\times\{(\tau_2-\tau_1)a\}^{-1}.\end{aligned} \tag{3. 12}$$

In this case we have

$$\lim_{a\to 0} R_A(a;\ \xi)=W_1\xi, \tag{3. 13}$$

$$\lim_{a\to\infty} R_A(a;\ \xi)=W_2(1-\xi). \tag{3. 14}$$

Furthermore we have

(i) $(\partial/\partial a)\{R_A(a;\ \xi)\}<0$ for all a in $0<a<\delta(\xi)$.
(ii) $(\partial/\partial a)\{R_A(a;\ \xi)\}=0$ for $a=\delta(\xi)$.
(iii) $(\partial/\partial a)\{R_A(a;\ \xi)\}>0$ for all a in $\delta(\xi)<a<\infty$.
(iv) $(\partial^2/\partial a^2)\{R_A(a;\ \xi)\}|_{a=\delta(\xi)}>0$.

Hence it is verified that $\delta(\xi)$ is one and only one value for which the risk function $R_A(a;\ \xi)$ is minimized, that is,

$$\underset{0<a<\infty}{\mathrm{Min}}\ R_A(a;\ \xi)=R_A(\delta(\xi);\ \xi). \tag{3. 15}$$

Similarly we have

(3. 16) $$\min_{0<a<\infty} R_B(a\,;\,\xi)=R_B(\delta(\xi)\,;\,\xi).$$

(II) Let us introduce the following notations:

(3. 17) $$\delta_m=exp\{-r_m(\tau_2-\tau_1)\},\ (m\geqq 1)$$

(3. 18) $$\xi_m=W_2\delta_m/(W_1+W_2\delta_m),\ (m\geqq 1).$$

Making use of two lemmas stated above, we have the following

(1) When $(\tau_2-\tau_1)^{-1}\,log\ a+r_m\leqq 0$, we have

(3. 19) $$R_A(a\,;\,\xi)=W_1\xi+C_m.$$

(2) When $(\tau_2-\tau_1)^{-1}log\ a+r_m>0$, we have

(3. 20) $$R_A(a\,;\,\xi)=W_1\xi\int_{(\tau_2-\tau_1)^{-1}\,log\ a+r_m}^{\infty} f_m(y,\ \tau_1)dy$$
$$+W_2(1-\xi)\int_0^{(\tau_2-\tau_1)^{-1}\,log\ a+r_m} f_m(y,\ \tau_2)dy+C_m.$$

In the case (2), we have

(3. 21) $$(\partial/\partial a)\{R_A(a\,;\,\xi)\}=exp\{a_m((\tau_2-\tau_1)^{-1}log\ a+r_m)\}$$
$$\times W_2(1-\xi)a^{\tau 1/(\tau_2-\tau 1)}(a-W_1\xi[W_2(1-\xi)]^{-1}).$$

By means of the similar discussions to that in the case (I), we have

(3. 22) $$\min_{0<a<\infty} R_A(a\,;\,\xi)=R_A(\delta(\xi)\,;\,\xi).$$

Hence we can display the minimum risk $R_A(\delta(\xi)\,;\,\xi)$ as follows:

(Ai) If $0<\xi\leqq\xi_m$, then we have

(3. 23) $$R_A(\delta(\xi)\,;\,\xi)=W_1\xi+C_m.$$

(Aii) If $\xi_m<\xi<1$, then we have

(3. 24) $$R_A(\delta(\xi)\,;\,\xi)=W_1\xi\int_{x_m(\xi)}^{\infty} f_m(y,\ \tau_1)dy+W_2(1-\xi)\int_0^{x_m(\xi)} f_m(y,\ \tau_2)dy.$$

Similarly the minimum expected risk using u_m+u_n is given as follows:

(Bi) If $0<\xi\leqq\xi_{m+n}$, then we have

(3. 25) $$R_B(\delta(\xi)\,;\,\xi)=W_1\xi+C_m+C_n.$$

(Bii) If $\xi_{m+n}<\xi<1$, then we have

(3. 26) $$R_B(\delta(\xi)\,;\,\xi)=W_1\xi\int_{x_m(\xi)+r_n}^{\infty} f_{m+n}(y,\ \tau_1)dy$$
$$+W_2(1-\xi)\int_0^{x_m(\xi)+r_n} f_{m+n}(y,\ \tau_2)dy+C_m+C_n.$$

§4. Main results.

Before the enunciations of the theorems, we denote by $d^{(1)}(\xi\,;\,A,\,B)$

and $d^{(2)}(\xi; A, B)$ $d(\xi; A, B)$ in the case (I) and (II), respectively. Let us introduce the following notations:

$$(4.1)\qquad A^{(i)}_{m,n}(x_m(\xi); r_n, s_n)=exp\{a_m(x_m(\xi))\}-exp\{a_{m+n}(x_m(\xi)+r_n)+s_n\},$$

$$(4.2)\qquad B_{m,n}(x_m(\xi); r_n, s_n)=a_m(x_m(\xi))-a_{m+n}(x_m(\xi)+r_n)-s_n,\quad (i=1, 2).$$

(I) We have for each ξ in $0<\xi<1$

$$(4.3)\qquad d^{(1)}(\xi; A, B)=W_1\xi\int_{x_m(\xi)+r_n}^{\infty} f_{m+n}(y, \tau_1)dy+W_2(1-\xi)\int_{-\infty}^{x_m(\xi)+r_n} f_{m+n}(y, \tau_2)dy$$
$$-W_1\xi\int_{x_m(\xi)}^{\infty} f_m(y, \tau_1)dy-W_2(1-\xi)\int_{-\infty}^{x_m(\xi)} f_m(y, \tau_2)dy+C_n.$$

(II) Let us assume that $H(\tau_1, \tau_2)>0$, then we have the following:

(1) For each ξ in $0<\xi\leqq\xi_{m+n}$, we have

$$(4.4)\qquad d^{(2)}(\xi; A,B)=C_n.$$

(2) For each ξ in $\xi_{m+n}<\xi\leqq\xi_m$, we have

$$(4.5)\qquad d^{(2)}(\xi; A, B)=W_1\xi\int_{x_m(\xi)+r_n}^{\infty} f_{m+n}(y, \tau_1)dy+W_2(1-\xi)\int_{0}^{x_m(\xi)+r_n} f_{m+n}(y, \tau_2)dy$$
$$-W_1\xi+C_n.$$

(3) For each ξ in $\xi_m<\xi<1$, we have

$$(4.6)\qquad d^{(2)}(\xi; A, B)=W_1\xi\int_{x_m(\xi)+r_n}^{\infty} f_{m+n}(y, \tau_1)dy+W_2(1-\xi)\int_{0}^{x_m(m)+r_n} f_{m+n}(y, \tau_2)dy$$
$$-W_1\xi\int_{x_m(\xi)}^{\infty} f_m(y, \tau_1)dy-W_2(1-\xi)\int_{0}^{x_m(\xi)} f_m(y, \tau_2)dy+C_n.$$

Theorem 1. *In the case (I) the following assertions hold:*

($\mathring{1}$) *we have*

$$(4.7)\qquad A^{(1)}_{m,n}(x; r_n, s_n)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} exp\{-mb(i\tau)-i\tau x\}[1-exp\{s_n-nb(i\tau)-i\tau r_n\}]d\tau.$$

($\mathring{2}$) *We have*

$$(4.8)\qquad sign\{(\partial^2/\partial\xi^2)\{d^{(1)}(\xi; A, B)\}\}=sign\{A^{(1)}_{m,n}(x_m(\xi); r_n, s_n)\}.$$

Proof: Ad($\mathring{1}$) Since the function $f_m(x, \tau)=exp\{\tau x+mb(\tau)+a_m(x)\}$ is the probability density function, we have

$$(4.9)\qquad \int_{-\infty}^{\infty} exp\{\tau x+a_m(x)\}dx=exp\{-mb(\tau)\}.$$

Hence we have, in view of the conditions in the Case (I), (4, 10) $$exp\{a_m(x)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} exp\{-mb(i\tau)-i\tau x\}d\tau ,$$

therefore we have the right hand side of (4. 7) as an immediate consequence of (4. 10).

Ad(2)° For each assigned value of ξ in $0<\xi<1$ we have the following

(4. 11) $$(\partial/\partial\xi)\{d^{(1)}(\xi;\ A,\ B)\}$$
$$=W_1\int_{x_m(\xi)+rn}^{\infty} f_{m+n}(y,\ \tau_1)dy - W_2\int_{-\infty}^{x_m(\xi)+rn} (f_{m+n}(y,\ \tau_2)dy$$
$$-W_1\int_{x_m(\xi)}^{\infty} f_m(y,\ \tau_1)dy + W_2\int_{-\infty}^{x_m(\xi)} f_m(y,\ \tau_2)dy$$

(4. 12) $$(\partial^2/\partial\xi^2)\{d^{(1)}(\xi;\ AB)\}$$
$$=[(\tau_2-\tau_1)\xi(1-\xi)]^{-1}exp\{s_m\}[W_1exp\{\tau_1(\tau_2-\tau_1)^{-1}log\delta(\xi)\}$$
$$+W_2exp\{\tau_2(\tau_2-\tau_1)^{-1}log\delta(\xi)\}]A^{(1)}_{m,\ n}(x_m(\xi);\ r_n,\ s_n).$$

Since the first, the second and the third factors in (4. 12) are positive, we have the result of (2) immediately.

Theorem 2. *In the case* (I), *if there exists one and only one value* ξ^* *in* $0<\xi^*<1$ *such that* $(\partial/\partial\xi)\{B_{m,n}(x_m(\xi);\ r_n,\ s_n)\}\gtreqless 0$, *according to* $\xi\lesseqgtr\xi^*$, *then all possibilities concerning with* $d^{(1)}(\xi;\ A,\ B)$ *can be classified into the following three types*:

(1) *For all* ξ *in* $0<\xi<1$, $d^{(1)}(\xi;\ A\ B)>0$.

(2) *For all* ξ *in* $0<\xi<1$, $d^{(1)}(\xi;\ A,\ B)>0$, *except for a certain value* ξ_0 *such that* $d^{(1)}(\xi_0;\ A,\ B)=0$.

(3) *There exist* ξ_1 *and* ξ_2, $0<\xi_1<\xi_2<1$, *such that*

(i) *For all* ξ *in* $0<\xi<\xi_1$, $d^{(1)}(\xi;\ A,\ B)>0$.

(ii) *For* $\xi=\xi_i$, $d^{(1)}(\xi_i;\ A,\ B)=0$, $i=1,\ 2$.

(iii) *For all* ξ *in* $\xi_1<\xi<\xi_2$, $d^{(1)}(\xi;\ A,\ B)<0$.

(iv) *For all* ξ *in* $\xi_2<\xi<1$, $d^{(1)}(\xi;\ A,\ B)>0$.

Proof: We have

(4. 13) $$\lim_{\xi\to 0} d^{(1)}(\xi;\ A,\ B)=\lim_{\xi\to 1} d^{(1)}(\xi;\ A,B)=C_n,$$

(4. 14) $$\lim_{\xi\to 0}(\partial/\partial\xi)\{d^{(1)}(\xi;\ A,\ B)\}=\lim_{\xi\to 1}(\partial/\partial\xi)\{d^{(1)}(\xi;\ A,\ B)\}=0.$$

For each assigned value of ξ in $0<\xi<1$, we have

(4. 15) $$(\partial/\partial\xi)\ \{d^{(1)}(\xi;\ A,\ B)\}$$
$$=W_1\int_{x_m(\xi)+rn}^{\infty} f_{m+n}(y,\ \tau_1)dy-W_2\int_{-\infty}^{x_m(\xi)+rn} f_{m+n}(y,\ \tau_2)dy$$

$$-W_1\int_{x_m(\xi)}^{\infty} f_m(y,\ \tau_1)\,dy+W_2\int_{-\infty}^{x_m(\xi)} f_m(y,\ \tau_2)\,dy,$$

(4. 16) $$(\partial^2/\partial\xi^2)\{d^{(1)}(\xi;\ A,\ B)\}$$
$$=[(\tau_2-\tau_1)\xi(1-\xi)]^{-1}\exp\{s_m\}[W_1\exp\{\tau_1(\tau_2-\tau_1)^{-1}\log\delta(\xi)\}$$
$$+W_2\exp\{\tau_2(\tau_2-\tau_1)^{-1}\log\delta(\xi)\}]A^{(1)}_{m,n}(x_m(\xi);\ r_n,\ s_n).$$

Hence we have

(4. 17) $$sign\{(\partial^2/\partial\xi^2)\{d^{(1)}(\xi;\ A,\ B)\}\}=sign\{A^{(1)}_{m,n}(x_m(\xi);\ r_n,\ s_n)\}$$
$$=sign\{B_{m,n}(x_m(\xi);\ r_n,\ s_n)\}.$$

By means of the assumption to the theorem, we have the following six cases concerning with the types of $A^{(1)}_{m,n}(x_m(\xi);\ r_n,\ s_n)$.

(1) $A^{(1)}_{m,n}(x_m(\xi);\ r_n,\ s_n)>0$ for all ξ in $0<\xi<1$.

(2) There exists one and only one value $\xi^{(2)}_0$ such that

$$A^{(1)}_{m,n}(x_m(\xi);\ r_n,\ s_n)\ \begin{cases}>0 \text{ for all } \xi \text{ in } 0<\xi<\xi^{(2)}_0,\\ =0 \text{ for } \xi=\xi^{(2)}_0,\\ <0 \text{ for all } \xi \text{ in } \xi^{(2)}_0<\xi<1.\end{cases}$$

(3) There exists one and only one value $\xi^{(3)}_0$ such that

$$A^{(1)}_{m,n}(x_m(\xi);\ r_n,\ s_n)\ \begin{cases}<0 \text{ for all } \xi \text{ in } 0<\xi<\xi^{(3)}_0,\\ =0 \text{ for } \xi=\xi^{(3)}_0,\\ >0 \text{ for all } \xi \text{ in } \xi^{(3)}_0<\xi<1.\end{cases}$$

(4) There exists exactly two values $\xi^{(4)}_1$ and $\xi^{(4)}_2$ in $0<\xi^{(4)}_1<\xi^{(4)}_2<1$, such that

$$A^{(1)}_{m,n}(x_m(\xi)\ ;\ r_n,s_n)\ \begin{cases}<0 \text{ for all } \xi \text{ in } 0<\xi<\xi^{(4)}_1 \text{ for in } \xi^{(4)}_2<\xi<1,\\ =0 \text{ for } \xi=\xi^{(4)}_1 \text{ or } \xi=\xi^{(4)}_2,\\ >0 \text{ for all } \xi \text{ in } \xi^{(4)}_1<\xi<\xi^{(4)}_2.\end{cases}$$

(5) $A^{(1)}_{m,n}(x_m(\xi)\ ;\ r_n,\ s_n)<0$ for all ξ in $0<\xi<1$.

(6) $A^{(1)}_{m,n}(x_m(\xi)\ ;\ r_n,\ s_n)\ \begin{cases}<0 \text{ for all } \xi \text{ in } 0<\xi<1 \text{ except for } \xi^*,\\ =0 \text{ for } \xi=\xi^*.\end{cases}$

Now the case (1) and the case (5) contradict to (4. 14). In the case (2) by means of (4. 14) we have $(\partial/\partial\xi)\ \{d^{(1)}(\xi;\ A,\ B)\}>0$ for all ξ in $0<\xi<1$. This contradicts to (4. 13). By means of the similar discussions to that in the case (2), the case (3) does not hold true. The discussions in the case (6) reduce to that in the case (5). In the case (4) in virtue of (4. 14) there exists one and only one value $\tilde{\xi}$ such that

$$\text{sign}\ \{(\partial/\partial\xi)\{d^{(1)}(\xi\ ;\ A,\ B)\}\}\ \begin{cases}<0 \text{ for all } \xi \text{ in } 0<\xi<\tilde{\xi},\\ =0 \text{ for } \xi=\tilde{\xi},\\ >0 \text{ for all } \xi \text{ in } \tilde{\xi}<\xi<1.\end{cases}$$

Hence in virtue of (4. 13) we can obtain the resnlt of the theorem.

Theorem. 3. *In the case* (II), *let us assume that* $H(\tau_1,\ \tau_2)>0$, *then the following assertions hold*:

($\mathring{1}$) *we have*

(4. 18) $$A_{m,n}^{(2)}(x\,;\,r_n,\ s_n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} exp\{-mb(-\sigma)\}exp\{\sigma x\} \times[1-exp\{s_n-nb(-\sigma)+\sigma r_n\}]d\sigma,$$

where c is an arbitrary positive constant.

($\mathring{2}$) *we have*

(i) $sign\{(\partial^2/\partial\xi^2)\{d^{(2)}(\xi\,;\,A,\ B)\}\}=0$ *for all* ξ *in* $0<\xi<\xi_{m+n}$,

(ii) $sign\{(\partial^2/\partial\xi^2)\{d^{(2)}(\xi\,;\,A,\ B)\}\}<0$ *for all* ξ *in* $\xi_{m+n}\leqq\xi<\xi_m$,

(iii) $sign\{(\partial^2/\partial\xi^2)\{d^{(2)}(\xi\,;\,A,\ B)\}\}=sign\{A_{m,n}^{(2)}(x\,;\,r_n,\ s_n)\}$

for all ξ *in* $\xi_m\leqq\xi<1$.

Proof: Ad($\mathring{1}$) Since $f_m(x,\ \tau)=exp\{\tau x+mb(\tau)+a_m(x)\}$ is a probability density function, we have, for any τ in $0<\tau<\infty$,

(4. 19) $$\int_0^\infty exp\{\tau x+a_m(x)\}dx=exp\{-mb(\tau)\}.$$

By means of the Laplace inversion formula and in view of the conditions of Case (II), we have

(4. 20) $$exp\{a_m(x)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\{-mb(-\sigma)+\sigma x\}d\sigma,\ (x>0),$$

where c is an arbitrary positive constant. The proof of ($\mathring{1}$) follows form (4. 20) immediately.

Ad($\mathring{2}$) We have the following:

(1) For all ξ in $0<\xi\leqq\xi_{m+n}$ we have

(4. 21) $$(\partial/\partial\xi)\{d^{(2)}(\xi\,;\,A,\ B)\}=0.$$

(2) For all ξ in $\xi_{m+n}<\xi\leqq\xi_m$, we have

(4. 22) $$(\partial/\partial\xi)\{d^{(2)}(\xi\,;\,A,\ B)\} =W_1\int_{x_m(\xi)+rn}^{\infty} f_{m+n}(y,\ \tau_1)dy-W_2\int_0^{x_m(\xi)+rn} f_{m+n}(y,\ \tau_2)dy\ -W_1.$$

(3) For each ξ in $\xi_m<\xi<1$ we have

(4. 23) $$(\partial/\partial\xi)\{d^{(2)}(\xi\,;\,A,\ B)\} =W_1\int_{x_m(\xi)+rn}^{\infty} f_{m+n}(y,\ \tau_1)dy-W_2\int_0^{x_m(\xi)+rn} f_{m+n}(y,\ \tau_2)dy$$

$$-W_1\int_{x_m(\xi)}^{\infty} f_m(x,\ \tau_1)\,dy + W_2\int_0^{x_m(\xi)} f_m(x,\ \tau_2)\,dx.$$

Hence we have

(1) For all ξ in $0<\xi\leqq\xi_{m+n}$, we have

(4. 24) $$(\partial^2/\partial\xi^2)\{d^{(2)}(\xi;\ A,\ B)\}=0.$$

(2) For all ξ in $\xi_{m+n}<\xi\leqq\xi_m$, we have

(4. 25) $$(\partial^2/\partial\xi^2)\{d^{(2)}(\xi;\ A,\ B)\}$$
$$=[W_1 f_{m+n}(x_m(\xi)+r_n,\ \tau_1)-W_2 f_{m+n}(x_m(\xi)+r_n,\ \tau_2)]$$
$$/\{\xi(1-\xi)(\tau_2-\tau_1)\}.$$

(3) For all ξ in $\xi_m<\xi<1$, we have

(4. 26) $$(\partial^2/\partial\xi^2)\{d^{(2)}(\xi;\ A,\ B)\}$$
$$=exp\{s_m\}[(\tau_2-\tau_1)\xi(1-\xi)]^{-1}[W_1 exp\{\tau_1(\tau_2-\tau_1)^{-1}log\delta(\xi)\}$$
$$+W_2 exp\{\tau_2(\tau_2-\tau_1)^{-1}log\delta(\xi)\}]A_{m,n}^{(2)}(x_m(\xi);\ r_n,\ s_n).$$

Since the first, the second and the third factors in (4. 26) are positive the result of $(\mathring{2})$(iii) immediately.

§5. Examples.

The purpose of this section is to give each one example to each of two cases (I) and (II) for which Theorem 2 and 3 hold true respectively

Example 1. (Normal population $N(\tau,\ 1)$). Let us put

(5. 1) $$b(\tau) = -\tau^2/2,$$

then in virtue of Theorem 1 we have

(5. 2) $$A_{m,n}^{(1)}(x_m(\xi);\ r_n,\ s_n)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} exp\{-m\tau^2/2-ix\tau\}[1-exp\{s_n-n\tau^2/2-ir_n\tau\}]d\tau.$$

In order calculate the value in the right hand side of (5. 2), let us put

(5. 3) $$A_l^{(1)}\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty} exp\{-l\tau^2/2-ix\tau\}d\tau,\ (l=m,\ m+n),$$

then we have

(5. 4) $$A_l^{(1)}=\frac{\sqrt{2/l}}{\pi}\int_0^{\infty} exp\{-\sigma^2\}cos(\sqrt{2/l}\,x\sigma)\,d\sigma$$
$$=(2\pi l)^{-1/2}exp\{-x^2/(2l)\},\ (l=m,\ m+n).$$

Therefore we have

(5. 5)
$$A^{(1)}_{m,n}(x(\xi)\ ;\ r_n,\ s_n)$$
$$=(2\pi m)^{-1/2}exp\{-x^2/(2m)\}-(2(m+n)\pi)^{-1/2}$$
$$\times exp\{-[x+r_n]^2/(2(m+n))+s_n\},$$

where we have put

(5. 6)
$$r_n=n\tau_1\tau_2/2,$$

(5. 7)
$$s_n=n(\tau_1+\tau_2)/2.$$

Moreover we have

(5. 8)
$$(\partial/\partial\xi)\{B_{m,n}(x_m(\xi)\ ;\ r_n,\ s_n)\}$$
$$=-\{nx_m(\xi)-(m-1)H(\tau_1,\ \tau_2)\}/\{m(m+n)(\tau_2-\tau_1)\xi(1-\xi)\}.$$

Since the function $x_m(\xi)$ defined by (3. 10) is a monotone increasing function with a continuous first derivative with regard to ξ in $0<\xi<1$, there exists one and only one value ξ^* in $0<\xi^*<1$ such that sign$\{(\partial/\partial\xi)\{B_{m,n}(\xi\ ;\ r_n,\ s_n)\}\}\gtreqless 0$, according to $\xi\lesseqgtr\xi^*$. Hence the assumption to Theorem 2 is satisfied and the type (3) holds true in this case.

Example 2. (The Pearson distribution of type III with unknown τ).

(5. 8)
$$b(\tau)=\log(-\tau),\ (\tau<0),$$

then we have

(5. 9)
$$A^{(2)}_{m,n}(x\ ;\ r_n,\ s_n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[1-\sigma^{-n}exp\{s_n+\sigma r_n\}]exp\{\sigma x\}\sigma^{-m}d\sigma,$$

Now we have

(5. 10)
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}exp\{\sigma x\}\sigma^{-l}d\sigma=x^{l-1}/\Gamma(l),\ (l=m,\ m+n).$$

Hence we have

(5. 11)
$$A^{(2)}_{m,n}(x\ ;r_n,\ s_n)=x^{m-1}/\Gamma(m)-exp\{s_n\}(x+r_n)^{m+n-1}/\Gamma(m+n),$$

where we have put

(5. 12)
$$r_n=n(\sigma_1-\sigma_2)^{-1}log(\sigma_1/\sigma_2),$$

(5. 13)
$$s_n=n(\sigma_1-\sigma_2)^{-1})log(\sigma_2{}^{\sigma_1}/\sigma_1{}^{\sigma_2}).$$

Consequently we have, in the case (2)(iii) in Theorem 3 for all ξ in $\xi_m\leqq\xi<1$,

(5. 8)
$$sign\left\{\frac{\partial^2}{\partial\xi^2}d^{(2)}(\xi\ ;\ A,\ B)\right\}\gtreqless 0$$

according to

(5. 9)
$$x^{m-1}/\Gamma(m)\gtreqless\exp\{s_n\}(x+r_n)^{m+n-1}/\Gamma(m+n).$$

Acknowledgement.

The author would like to express his hearty thanks to Professor T. Kitagawa of Mathematical institute, Kyushu University for his encouragements and criticisms during this paper was prepared.

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