SOME NONPARAMETRIC ESTIMATORS OF A LOCATION PARAMETER

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https://doi.org/10.5109/13022
SOME NONPARAMETRIC ESTIMATORS OF
A LOCATION PARAMETER

By

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(Received December 1, 1966)

§1. Introduction and Summary.

The purpose of this paper is to discuss some nonparametric estimators of a location parameter, especially their asymptotic relative efficiencies relative to the sample mean.

Let \( X_1, X_2, \ldots, X_n \) be a random sample from the population with cumulative distribution function \( F(x - \theta) \), where \( \theta \) is a location parameter and \( F(x) \) is assumed to belong to the family \( \mathcal{F} \) of all distribution functions that are symmetric about the origin and absolutely continuous with respect to the Lebesgue measure. Let \( \hat{\theta}_p \) be the median of the means of all \( p \)-tuple \( (X_{i_1}, X_{i_2}, \ldots, X_{i_p}) \), \( \binom{N}{p} \) in number, drawn from \( X_1, X_2, \ldots, X_N \), i.e.

\[
\hat{\theta}_p = \text{Med}_{i_1 < i_2 < \cdots < i_p} \frac{X_{i_1} + X_{i_2} + \cdots + X_{i_p}}{p},
\]

which we shall propose as an estimator of \( \theta \).

In the simplest case \( p = 1 \), \( \hat{\theta}_1 \) is the sample median. In a recent paper [2] J. L. Hodges and E. L. Lehmann derived the estimator \( \hat{\theta}_2 \) of \( \theta \) from the one sample Wilcoxon statistic. Some of their results are as follows. The asymptotic efficiency of \( \hat{\theta}_1 \) relative to the sample mean \( \bar{X} \), denoted \( A.R.E. (\hat{\theta}_1, \bar{X}) \), in the sense of reciprocal ratio of asymptotic variances, is \( 4f^2 \sigma^2 \), where \( f \) denotes the density corresponding to \( F \) and \( \sigma^2 \) its variance, while \( A.R.E. (\hat{\theta}_2, \bar{X}) = 12f^2 \left( \int f(x)^2 \, dx \right)^2 \). The infimum of these efficiencies with respect to the underlying distribution are well known to be 0 and 0.864, respectively. Our investigation is a generalization of these results.

In Section 2 we shall discuss some properties of \( \hat{\theta}_p \). In Section 3 we shall state our main results that the infimum of \( A.R.E. (\hat{\theta}_p, \bar{X}) \) with respect to the population distribution is always greater than or equal to 0.864 for even \( p \), but not so for odd \( p \), even if \( p \geq 3 \). In Section 4 we shall consider the case in which \( N \) observations are divided into \( p \) groups and define alternative estimators of \( \theta \) and recommend some of them as estimators of \( \theta \).

§2. Some properties of \( \hat{\theta}_p \).

By means of a rank test statistic \( T(x), X = (X_1, \ldots, X_N) \), which satisfies the condition (1) \( T(x+a) \) is a nondecreasing function of \( a \) for all \( x \), (2)
$E_\theta T(x) = \mu$, where $\mu$ is independent of $F$ and $E_\theta$ denotes the expectation under $\theta = 0$. Hodges and Lehmann [2] defined the estimator of $\theta$ as follows.

\[ (2.1) \quad \hat{\theta} = \frac{\theta^{*} + \theta^{**}}{2}, \]

where $\theta^{*} = \inf \tilde{\theta}; T(x-\tilde{\theta}) < \mu$; and $\theta^{**} = \sup \tilde{\theta}; T(x-\tilde{\theta}) > \mu$.

If we put

\[ (2.2) \quad T(X) = \frac{1}{\binom{N}{p}} \left( \frac{i_1 \cdots i_p}{p} \right); X_{i_1} + \cdots + X_{i_p} > 0, \quad i_1 < i_2 < \cdots < i_p, \]

where $\#$ means the number of $p$-tuple $(i_1 i_2 \cdots i_p)$ such that $X_{i_1} + \cdots + X_{i_p} > 0$, then the estimator $\hat{\theta}_p$ and $\hat{\theta}$ defined in (1.1) and (2.1), respectively, are seen to be identical. Therefore all results in [2] hold for the estimator $\hat{\theta}_p$, i.e. (a) the distribution of $\hat{\theta}_p$ is absolutely continuous with respect to the Lebesgue measure, (b) the distribution of $\hat{\theta}_p$ is symmetric about 0, so that $\hat{\theta}_p$ is an unbiased estimator of $\theta$, (c) $\hat{\theta}_p$ is translation invariant, (d) the asymptotic relative efficiency of the test based on the test statistic $T(x)$ defined in (2.2) with respect to t-test is equal to $A.R.E. (\hat{\theta}_p, X)$, (e) we shall have the lemma below (see [2] p. 607).

**Lemma 2.1.** For $T(X)$ and $\hat{\theta}_p$ defined by (2.2) and (2.1), respectively, and for all $a$

\[ P|T(X-a) < \mu| \leq P|\hat{\theta}_p < a| \leq P|T(X-a) \leq \mu|. \]

Let

\[ (2.3) \quad G_p(y) = \int \cdots \int F(y - x_{i_2} - \cdots - x_{i_p})f(x_{i_2}) \cdots f(x_{i_p})dx_{i_2} \cdots dx_{i_p}, \]

\[ (2.4) \quad \lambda_p(F) = \int f(x)G_{p-1}(\theta)dx, \]

and let $g_p(y)$ be the p.d.f. of $G_p(y)$. Then we obtain the following theorem.

**Theorem 2.1.** Suppose $G_p(y)$ has the derivative $g_p(0) \neq 0$ at $y=0$. Then $N^{1/2} (\hat{\theta}_p - \theta)$ has a limiting normal distribution with mean 0 and variance $(\lambda_p(F) - 1/4)/g_p^2(0)$.

**Proof** For any real $u$, let

\[ (2.5) \quad U_N = \frac{1}{\binom{N}{p}} \sum_{i_1 < i_2 < \cdots < i_p} \varphi_N(X_{i_1}, \cdots, X_{i_p}), \]

where $\varphi_N(x_1, \cdots, x_p) = 1$ if $x_1 + \cdots + x_p > pu/N^{1/2}$, 0 otherwise. Note that $\mu = E_\theta T(X) = 1/2$ and $T(X-u/N^{1/2}) = U_N$, then from above (c) and Lemma 2.1

\[ \lim_{N \to \infty} P_\theta|N^{1/2}(\hat{\theta}_p - \theta) \leq \mu| = \lim_{N \to \infty} P_\theta|\hat{\theta}_p \leq u/N^{1/2}|. \]
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\[ \lim_{N \to \infty} P_0 \left\{ \frac{1}{N^{1/2}} \left| T(X - \theta) \right| \leq \frac{1}{2} \right\} = \lim_{N \to \infty} P_0 \left\{ \frac{1}{N^{1/2}} \left| U_N - E_0 U_N \right| \leq \frac{1}{2} - E_0 U_N \right\}. \]

Since \( U_N \) is a \( U \)-statistic, for which \( \varphi_N \) is uniformly bounded, it follows from the general theory of \( U \)-statistic [3] that \( \frac{1}{N^{1/2}}(U_N - E_0 U_N) \) has a limiting normal distribution with mean 0 and variance \( \sigma^2 \). Therefore from the general theory of \( U \)-statistic \( \varphi \) with mean 0 and variance \( \sigma^2 \). Therefore from Theorem 2.1

\[ \text{A.R.E.}(\hat{\theta}_p X) = \frac{\sigma^2_{\varphi}(0)}{\left( \lambda_\varphi(F) - \frac{1}{4} \right)} , \]

\[ \text{A.R.E.}(\hat{\theta}_p \hat{\theta}_q) = \frac{g^2_{\varphi}(0)}{\left( \lambda_\varphi(F) - \frac{1}{4} \right)} , \]

Especially

\[ \text{A.R.E.}(\hat{\theta}_p \hat{\theta}_1) = \frac{g^2_{\varphi}(0)}{4f^2(0)} \left( \lambda_\varphi(F) - \frac{1}{4} \right) , \]

\[ \text{A.R.E.}(\hat{\theta}_p \hat{\theta}_2) = \frac{g^2_{\varphi}(0)}{12 \left( \int f^2(x) \, dx \right)} \left( \lambda_\varphi(F) - \frac{1}{4} \right) . \]

Now we shall evaluate the value of \( \text{A.R.E.}(\hat{\theta}_p X) \). For this purpose we require following two lemmas.

**Lemma 3.1.** Let \( X_{ir}, X_{ir'} \), \( i = 1, 2, \ldots, c \), and let

\[ \frac{1}{N} \sum_{\alpha \leq \beta} \varphi(Z_{i,1} \ldots \alpha, a, Z_{i,1} \ldots \beta, Z_{i,1} \ldots \beta) , \]

where \( Z_{i,x} = X_i, X_i + X_i + \ldots, Z_{i,x} = 1 \) if \( Z_1 + Z_2 > 0 \), \( = 0 \) otherwise. Then the random vector with components \( \frac{1}{N^{1/2}}(U_{i,1} \ldots \alpha, \beta) = E_0 U_{i,1} \ldots \alpha, \beta \) has a normal distribution with mean 0 and covariance matrix

\[ (4 \left( \lambda_2(\varphi(G)) - \frac{1}{4} \right)) , \]

where

\[ \lambda_2(\varphi(G)) = 0 \left( \begin{array}{cccc} Z_{i,1} \ldots \alpha, Z_{i,1} \ldots \beta, Z_{i,1} \ldots \beta, & Z_{i,1} \ldots \beta, Z_{i,1} \ldots \beta, Z_{i,1} \ldots \beta, & Z_{i,1} \ldots \beta, Z_{i,1} \ldots \beta, Z_{i,1} \ldots \beta \end{array} \right) > 0 . \]

Proof is obvious from the general theory of generalized \( U \)-statistic (see
Lemma 3. 2. For \( \lambda_p(F) \) defined by (2. 4) it holds that for all \( F \in \mathcal{F} \)

\[
(3. 4) \quad \frac{1}{4} \leq \lambda_{2m}(F) \leq \frac{3m + 1}{12m}, \ m = 1, 2, \ldots.
\]

Proof The left inequality is easy from the Schwarz' inequality; \( \lambda_p(F) = \int f(x) G_{2m-1}(x) \, dx \geq (\int f(x) G_{2m-1}(x) \, dx)^2 = (P_0 \psi X_1 + X_{2m} > 0)^2 = 1/4 \), for the distribution of \( X_1, X_2, \ldots, X_{2m} \) is symmetric about the origin. To prove the right inequality, consider the random vector \( Y \) with components

\[
Y_{i_1 ; i_2 \ldots i_{2m}}, \ Y_{i_1 ; i_2 \ldots i_{2m}}', \ Y_{i_{1m}; i_{2m} \ldots i_{2m}}', \ Y_{i_{1m}; i_{2m} \ldots i_{2m}}', \ Y_{i_{1m}; i_{2m} \ldots i_{2m}}', \ Y_{i_{1m}; i_{2m} \ldots i_{2m}}'.
\]

where \( Y_{i_1 \ldots i_m} = N^{1/2}(U^{(i_1 \ldots i_m)} - E_0 U^{(i_1 \ldots i_m)}) \) and \( U^{(i_1 \ldots i_m)} \) are defined in Lemma 3. 1. By (3. 3) the asymptotic covariance of \( Y_{i_1 \ldots i_m} \) and \( Y_{j_1 \ldots j_m} \) is given by

\[
4\left[ \lambda_{2m}(F) - \frac{1}{4} \right] = 0; \text{ if } i_1, \ldots, i_m, j_1, \ldots, j_m \text{ are all different}
\]

\[
= \frac{1}{3}; \text{ if } (i_1 \ldots i_m) = (j_1 \ldots j_m)
\]

\[
= 4\left( \frac{1}{4} \right); \text{ otherwise.}
\]

Hence the asymptotic covariance matrix of \( Y \), denoted by \( \Sigma_m \), is written as follows.

\[
\Sigma_m = \begin{pmatrix}
(i_{11} \ldots i_{1m}) & \ldots & (i_{1m} \ldots i_{1m}) & (i_{11} \ldots i_{1m}) & \ldots & (i_{1m} \ldots i_{1m}) \\
1/3 & \ddots & 0 & 4\left( \lambda_{2m}(F) - \frac{1}{4} \right) \ldots 4\left( \lambda_{2m}(F) - \frac{1}{4} \right) \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(i_{1m} \ldots i_{1m}) & \ddots & \ddots & \ddots & \ddots & \ddots \\
4\left( \lambda_{2m}(F) - \frac{1}{4} \right) \ldots 4\left( \lambda_{2m}(F) - \frac{1}{4} \right) & \ddots & \ddots & \ddots & \ddots & 1/3 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(i_{1m} \ldots i_{1m}) & \ddots & \ddots & \ddots & \ddots & 0 \\
4\left( \lambda_{2m}(F) - \frac{1}{4} \right) \ldots 4\left( \lambda_{2m}(F) - \frac{1}{4} \right) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(i_{1m} \ldots i_{1m}) & \ddots & \ddots & \ddots & \ddots & 1/3 \\
\end{pmatrix}
\]

Put \( \lambda_{2m}(F) - 1/4 = \pi/12 \), then the determinant of \( \Sigma_m \) is
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(3.7) \[ \det \Sigma_m = \left( \frac{1}{3} \right)^{2m} \begin{vmatrix} 1 & 0 & r & \cdots & r \\ 0 & 1 & r & & r \\ & & \ddots & \ddots & \ddots \\ r & & \cdots & 1 & 0 \\ & & & & 0 & 1 \end{vmatrix} = \left( \frac{1}{3} \right)^{2m} \left( 1 - m^2 \tau^2 \right) \]

Since \( \det \Sigma_m \geq 0 \), we have \( \tau \leq 1/m \), which implies \( \lambda_{2m}(F) \leq (3m+1)/12m \), as was to be proved.

We shall denote by \( \mathcal{F} \) the family of distributions which belong to \( \mathcal{F} \) and satisfy the condition of the theorem 2.1.

**Theorem 3.1.** Suppose that \( p \) is even. Then

\[ \inf_{F \in \mathcal{F}} A.R.E. \left( \hat{\theta}_p, X \right) \geq 0.864. \]

**Proof** We shall put \( p = 2m, m = 1, 2, \cdots, \) then

\[ g_{2m}(0) = \int g^2_m(x) \, dx. \]

From (3.1) and lemma 3.2,

\[ \inf_{F \in \mathcal{F}} A.R.E. \left( \hat{\theta}_{2m}, X \right) = \inf_{F \in \mathcal{F}} \frac{\sigma_{g_{2m}(0)}}{\lambda_{2m}(F) - 1/4} \]

\[ = \inf_{F \in \mathcal{F}} \frac{12 \sigma_{g_{2m}}^2 \left( \int g^2_m(x) \, dx \right)^2}{12m \left( \lambda_{2m}(F) - 1/4 \right)} \geq \sup_{F \in \mathcal{F}} \frac{12 \sigma_{g_{2m}}^2 \left( \int g^2_m(x) \, dx \right)^2}{12m \left( \lambda_{2m}(F) - 1/4 \right)} \]

\[ \geq \inf_{F \in \mathcal{F}} 12 \sigma_{g_{2m}}^2 \left( \int g^2_m(x) \, dx \right)^2, \]

where \( \sigma_{g_{2m}}^2 \) is the variance of p.d.f. \( g_m \). It has been shown by Hodges and Lehmann [1] that

(3.8) \[ g_m(x) = \frac{3}{20\sqrt{5}} (5 - x^2) \text{ if } x^2 \leq 5, = 0 \text{ otherwise} \]

attains the infimum value 0.864 of the last expression. This completes the proof.

Remark. For even \( m \) there exists no underlying distribution \( F(x) \) which satisfies (3.8), since the characteristic function is

\[ (3/5\sqrt{5}) \left[ (1/t^5) \sin t\sqrt{5} - (t/5/t^5) \cos t\sqrt{5} \right], \]

which is negative for some \( t \). The author presents a conjecture \( A.R.E. \left( \hat{\theta}_{2m}, X \right) > 0.864 \) for all \( m > 1 \).

The above theorem does not hold for odd \( p \), as is seen in Table II for \( p = 3 \). In order to give an evaluation for odd \( p \), we shall consider the random variable \( Z_{\alpha \tau_1 \tau_2 \cdots \tau_n} \), \( \alpha = 1, 2, \cdots, N \), given in lemma 3.1 and the statistic \( U_{\alpha \tau_1 \tau_2 \cdots \tau_n} = N^{-1} \sum_{i=1}^N \psi(Z_{\alpha \tau_1 \tau_2 \cdots \tau_n}), \) where \( \psi(Z) = 1 \) if \( Z > 0, = 0 \) otherwise. A similar procedure as lemmas 3.1 and 3.2 will lead us to obtain
Though the upper bound of (3.9) is somewhat larger than that of (3.4) for even \( p \), it gives an evaluation of \( \lambda_p(F) \) for odd \( p \). Therefore we shall try to evaluate the value of A.R.E. \( \langle \hat{\theta}_p, X \rangle \) for odd \( p \) by means of (3.9). Let \( \mathcal{F}_n \) be the family of distributions which are unimodal and belong to \( \mathcal{F} \). Then

**Lemma 3.3.** If \( F(x) \in \mathcal{F}_n \), then \( G_p(y) \in \mathcal{F}_n \).

**Proof** It is sufficient to show that if \( X \) and \( Y \) are independent random variables with c.d.f. \( F(x) \in \mathcal{F}_n \) and \( G(y) \in \mathcal{F}_n \), respectively, then the c.d.f. \( H(z) \) of the random variable \( Z = X + Y \) belongs to \( \mathcal{F}_n \). Since \( H(z) \in \mathcal{F} \) is obvious, we shall show the unimodality of \( H(z) \). Let the p.d.f. of \( F, G \) and \( H \) be \( f, g \) and \( h \), respectively. Then for arbitrary \( z_i > z_j > 0 \),

\[
h(z_2) - h(z_1) = \int_{z_1}^{z_2} |f(z_2 - y) - f(z_1 - y)| g(y) \, dy
\]

Now \( z_2 - y \leq z_1 - y \) and \( y \geq z_1 + z_2 - y \) for \( y \geq (z_1 + z_2)/2 \), so that from symmetry and unimodality of \( F, G \), it follows that \( f(z_2 - y) \geq f(z_1 - y), g(y) \leq g(z_1 + z_2 - y) \) for \( y > (z_1 + z_2)/2 \). Hence \( h(z_2) \leq h(z_1) \), as was to be proved.

Let \( \mathcal{F}_n^* \) be the family of distributions which are unimodal and belong to \( \mathcal{F}^* \). From lemma 3.3 \( g_{2m}(0) \geq g_{2m-1}(x) \) for any \( F \in \mathcal{F}_n \). Therefore \( g_{2m}(0) = \int f(x) g_{2m-1}(x) \, dx \leq g_{2m-1}(0) \). Hence from theorem 3.1,

\[
\inf_{F \in \mathcal{F}_n^*} \sigma^2 g^2_{2m-1}(0) \geq \inf_{F \in \mathcal{F}_n^*} \frac{\sigma^2}{m} g^2_{2m}(0) \\
\geq \frac{0.864}{12m}, \text{ for } m = 1, 2, \ldots.
\]

Combining this with (3.9), we obtain the theorem below.

**Theorem 3.2.** For odd \( p \) it holds that

\[
(3.10) \quad \inf_{F \in \mathcal{F}_n^*} \text{A.R.E. } \langle \hat{\theta}_p, X \rangle \geq \frac{0.288 \cdot 2p}{p+1}
\]

Some numerical values of \( g_p(0), \lambda_p(F) \) and A.R.E. \( \langle \hat{\theta}_p, X \rangle \) for normal, uniform and double exponential distributions are given in the following tables.

---

(1) The lemma and the proof was given in more generaliged form by professor K. Isii, Osaka University.
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Table I \( f(x) = \left( \frac{1}{\sqrt{2\pi}} \right) \exp \left( -\frac{x^2}{2} \right) \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p(0) )</td>
<td>0.3389</td>
<td>0.2329</td>
<td>0.1995</td>
<td>0.1784</td>
<td>0.1262</td>
<td>0.0982</td>
</tr>
<tr>
<td>( \lambda_p(F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.2902</td>
<td>0.2820</td>
<td>0.2659</td>
<td>0.2579</td>
</tr>
<tr>
<td>A.R.E. (( \hat{\theta}_p</td>
<td>X ))</td>
<td>0.6366</td>
<td>0.5000</td>
<td>0.4894</td>
<td>0.4933</td>
<td>0.9983</td>
</tr>
</tbody>
</table>

Table II \( f(x) = \frac{1}{2} x \left( \frac{1}{2}, \frac{1}{2} \right), =0 \) otherwise

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p(0) )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.7500</td>
<td>0.6667</td>
<td>0.5990</td>
<td>0.5500</td>
</tr>
<tr>
<td>( \lambda_p(F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.3052</td>
<td>0.2909</td>
<td>0.2825</td>
<td>0.2771</td>
</tr>
<tr>
<td>A.R.E. (( \hat{\theta}_p</td>
<td>X ))</td>
<td>0.3333</td>
<td>1.0000</td>
<td>0.8490</td>
<td>0.9061</td>
<td>0.9192</td>
</tr>
</tbody>
</table>

Table III \( f(x) = \frac{1}{2} e^{-x} \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p(0) )</td>
<td>0.5000</td>
<td>0.2500</td>
<td>0.1875</td>
<td>0.1563</td>
<td>0.1367</td>
<td>0.1230</td>
</tr>
<tr>
<td>( \lambda_p(F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.3032</td>
<td>0.2908</td>
<td>0.2809</td>
<td>0.2761</td>
</tr>
<tr>
<td>A.R.E. (( \hat{\theta}_p</td>
<td>X ))</td>
<td>2.0000</td>
<td>1.5000</td>
<td>1.3207</td>
<td>1.2439</td>
<td>1.2118</td>
</tr>
</tbody>
</table>

It would be interesting to compute the numerical values of A.R.E. (\( \hat{\theta}_p | X \)) with respect to the following distributions.

\[
(3.11) \quad f(x) = \frac{\epsilon}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \frac{(1-\epsilon)}{2} e^{-x}, \quad 0 \leq \epsilon \leq 1
\]

\[
(3.12) \quad f(x) = \frac{1}{\left(1+\frac{1+\alpha}{2}\right)^{\frac{x^{1+\alpha}}{1+\alpha}}} e^{-\frac{x^{\frac{1}{1+\alpha}}}{2}}, \quad -1 < \alpha \leq 1.
\]

These two families include a normal distribution (\( \epsilon = 1, \alpha = 0 \)) as well as a double exponential distribution (\( \epsilon = 0, \alpha = 1 \)). It is expected that for any \( p = 3, 4, \ldots \) there exists a value of \( \epsilon \) or \( \alpha \) for which A.R.E. (\( \hat{\theta}_p | X \)) attains its maximum value \( \geq 1 \) at \( p \).

§4. Alternative estimators of \( \theta \)

Suppose that \( N \) observations \( X_1, X_2, \ldots, X_N \) are divided in some way into \( p \) groups, which denoted by \( (X_1^{(i)}, \ldots, X_i^{(i)}), (X_2^{(i)}, \ldots, X_n^{(2)}), \ldots, (X_p^{(i)}, \ldots, X_N^{(p)}) \) where \( n_i = \rho_i N, i = 1, 2, \ldots, p \) and \( \rho_1 + \rho_2 + \ldots + \rho_p = 1 \). Then we can construct several alternative estimators of \( \theta \) such as

\[
(4.1) \quad \hat{\theta}_p = \text{med} X_{i_1} + X_{i_2} + \cdots + X_{i_p}, \quad i_1 = 1, 2, \ldots, n_a, \quad \alpha = 1, 2, \ldots, p.
\]
Theorem 4. 1.

(1) Under the same condition as in theorem 3. 1., $N^{1/2} (\hat{\theta}_p - \theta)$ has a limiting normal distribution with mean 0 and variance $p^{-2} (\rho_1^{-1} + \cdots + \rho_p^{-1}) (\lambda_p (F) - 1/4) g_p^2 (0)$. 

(2) Suppose that $G_2 (y)$ has the derivative $g_2 (0) \neq 0$ at $y = 0$. Then $N^{1/2} (\hat{\theta}_{p**} - \theta)$ has a limiting normal distribution with mean 0 and variance $p^{-2} (\rho_1^{-1} + \rho_2^{-1} + \cdots + \rho_p^{-1}) [12 g_2^2 (0)]^{-1}$. 

(3) Under the same condition as in (1) $N^{1/2} (\hat{\theta}_{p***} - \theta)$ has a limiting normal distribution with mean 0 and variance $12 [p g_{p^2} (0)]^{-1}$. 

Proof (1) Since $\hat{\theta}_p^*$ can be represented by a U-statistic $T^* (X) = \left( \left( n_1 \right)_1 \cdots \left( n_p \right)_1 \right)^{-1} \hat{\mathbb{I}} (i_1, \ldots, i_p); X_{i_1} + X_{i_2} + \cdots + X_{i_p} > 0$, $i_a = 1, 2, \ldots, n$, $\alpha = 1, 2, \ldots, p$, in the same way as (2. 1), the proof is analogous to that of theorem 3. 1. (2) follows from the relation $N^{1/2} (\hat{\theta}_{p**} - \theta) = p^{-1} \sum_{a=1}^p \rho_a^{-1/2} n_a^{1/2} (\hat{\theta}^{(a)} - \theta)$, where $n_a^{1/2} (\hat{\theta}^{(a)} - \theta)$, $\alpha = 1, 2, \ldots, p$, are independent and asymptotically normally distributed with mean 0 and variance $[12 g_2^2 (0)]^{-1}$.

(3) $\lim_{N \to \infty} P_0 \{ N^{1/2} (\hat{\theta}_{p***} - \theta) \leq u \} = \lim_{N \to \infty} P_0 \{ n^{1/2} \hat{\theta}_{p***} \leq p^{-1/2} u \}$. Since $X_i, i = 1, 2, \ldots, n$, are independent and identically distributed with p.d.f. $p g_p (x)$ when $\theta = 0$, from the theorem 3. 1. $n^{1/2} \hat{\theta}_{p***}$ has a limiting normal distribution with mean 0 and variance $[12 g_p^2 (0)]^{-1}$, as was to be proved.

It is seen by the theorem that for $N$ fixed $n_1 = n_2 = \cdots = n_p$ is the best choice of the group sizes in order to make the asymptotic variance of $\hat{\theta}_p^*$ or $\hat{\theta}_{p**}$ minimum. In this case the estimator $\hat{\theta}_p^*$ has the same asymptotic distribution as $\hat{\theta}_p$. Now since $\hat{\theta}_p^*$ as well as $\hat{\theta}_p$ has the same asymptotic distribution as $\hat{\theta}_{p***}$, considering a trouble involved in computing $\hat{\theta}_p$ and $\hat{\theta}_{p***}$, we might as well recommend $\hat{\theta}_{p***}$ as an estimator of $\theta$ when $N$ is large and $n_1 = n_2 = \cdots = n_p$.

On the other hand for arbitrary $n_1, n_2, \ldots, n_p$ it will be preferable to use $\hat{\theta}_p, p = 2m, m = 1, 2, \ldots$, as an estimator of $\theta$, for $\hat{\theta}_p$ or $\hat{\theta}_{p**}$ has a large loss of efficiency in this case.

Since $A.R.E. (\hat{\theta}_p** X) = 12 p^2 g_{p^2} (0) = 12 \sigma_p^2 \left( \int g_p^2 (x) dx \right)^2$, the infimum of $A.R.E. (\hat{\theta}_{p***} X)$ never falls below 0.864.

Therefore $\hat{\theta}_{p***}$ will also be recommended for a practical use as an estimator.
of $\theta$ when sample size is large and $n_1 = \cdots = n_p$.

§5. Acknowledgement.

The author should like to express his deepest gratitude to Prof. T. Kitagawa Kyushu University, who suggested the problem and gave kind criticism and encouragement. The author also wishes his hearty thanks to Profs. M. Okamoto and K. Isii, Osaka University, for their generous help and guidance during the course of the entire work.

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