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SOME NONPARAMETRIC ESTIMATORS OF A LOCATION PARAMETER

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§1. Introduction and Summary.

The purpose of this paper is to discuss some nonparametric estimators of a location parameter, especially their asymptotic relative efficiencies relative to the sample mean.

Let $X_1, X_2, \cdots, X_n$ be a random sample from the population with cumulative distribution function $F(x-\theta)$, where θ is a location parameter and $F(x)$ is assumed to belong to the family $\mathfrak{F}$ of all distribution functions that are symmetric about the origin and absolutely continuous with respect to the Lebesgue measure. Let $\hat{\theta}_p$ be the median of the means of all p-tuple $(X_{i_1}, X_{i_2}, \cdots, X_{i_p})$, $\binom{N}{p}$ in number, drawn from $X_1, X_2, \cdots, X_N$, i.e.

$$\hat{\theta}_p = \underset{i_1<i_2<\cdots<i_p}{Med} \frac{X_{i_1}+X_{i_2}+\cdots+X_{i_p}}{p}, \tag{1. 1}$$

which we shall propose as an estimator of θ.

In the simplest case $p=1$, $\hat{\theta}_1$ is the sample median. In a recent paper [2] J. L. Hodges and E. L. Lehmann derived the estimator $\hat{\theta}_2$ of θ from the one sample Wilcoxon statistic. Some of their results are as follows. The asymptotic efficiency of $\hat{\theta}_1$ relative to the sample mean $\bar{X}$, denoted $A.R.E.$ $(\hat{\theta}_1|\bar{X})$, in the sence of reciprocal ratio of asymptotic variances, is $4\sigma_f^2 f_{(0)}^2$, where f denotes the density corresponding to F and σ_f^2 its variance, while $A.R.E.$ $(\hat{\theta}_2|\bar{X})=12\sigma_f^2(\int f_{(x)}^2 dx)^2$. The infimum of these efficiencies with respect to the underlying distribution are well known to be 0 and 0.864, respectively. Our investigation is a generalization of these results.

In Section 2 we shall discuss some properties of $\hat{\theta}_p$. In Section 3 we shall state our main results that the infimum of $A.R.E.$ $(\hat{\theta}_p|\bar{X})$ with respect to the population distribution is always greater than or equal to 0.864 for even p, but not so for odd p, even if $p\geq 3$. In Section 4 we shall consider the case in which N observations are divided into p groups and define alternative estimators of θ and recomend some of them as estimators of θ.

§2. Some properties of $\hat{\theta}_p$.

By means of a rank test statistic $T(x)$, $X=(X_1,\cdots,X_N)$, which satisfies the condition (1) $T(x+a)$ is a nondecreasing function of a for all x, (2)

$E_0T(x)=\mu$, where μ is independent of F and E_0 denotes the expectation under $\theta=0$, Hodges and Lehmann [2] defined the estimator of θ as follows.

$$\hat{\theta}=\frac{\theta^*+\theta^{**}}{2}, \tag{2. 1}$$

where $\theta^*=inf\{\theta\,;\ T(x-\theta)<\mu\}$ and $\theta^{**}=sup\{\theta\,;\ T(x-\theta)>\mu\}$.

If we put

$$T(X)=\frac{1}{\binom{N}{p}}\#\{(i_1\cdots i_p)\,;\ X_{i_1}+\cdots+X_{i_p}>0,\ i_1<i_2<\cdots<i_p\}, \tag{2. 2}$$

where $\#$ means the number of p-tuble $(i_1i_2\cdots i_p)$ such that $X_{i_1}+X_{i_2}+\cdots+X_{i_p}>0$, then the estimator $\hat{\theta}_p$ and $\hat{\theta}$ defined in (1. 1) and (2. 1), respectively, are seen to be identical. Therefore all results in [2] hold for the estimator $\hat{\theta}_p$, i.e. (a) the distribution of $\hat{\theta}_p$ is absolutely continuous with respect to the Lebesgue measure, (b) the distribution of $\hat{\theta}_p$ is symmetric about θ, so that $\hat{\theta}_p$ is an unbiased estimator of θ, (c) $\hat{\theta}_p$ is translation invariant, (d) the asymptotic relative efficiency of the test based on the test statistic $T(x)$ defined in (2. 2) with respect to t-test is equal to $A.R.E.\ (\hat{\theta}_p|\bar{X})$, (e) we shall have the lemma below (see [2] p. 607).

Lemma 2.1. *For $T(X)$ and $\hat{\theta}_p$ defined by (2. 2) and (2. 1), respectively, and for all a*

$$P\{T(X-a)<\mu\}\leq P\{\hat{\theta}_p\leq a\}\leq P\{T(X-a)\leq\mu\}.$$

Let

$$G_p(y)=\int\cdots\int F(y-x_2-\cdots-x_p)f(x_2)\cdots f(x_p)\,dx_2\cdots dx_p, \tag{2. 3}$$

$$\lambda_p(F)=\int f(x)G_{p-1}^2(\theta)\,dx, \tag{2. 4}$$

and let $g_p(y)$ be the p.d.f. of $G_p(y)$. Then we obtain the following theorem.

Theorem 2.1. *Suppose $G_p(y)$ has the derivative $g_p(o)\neq 0$ at $y=0$. Then $N^{1/2}(\hat{\theta}_p-\theta)$ has a limiting normal distribution with mean 0 and variance $(\lambda_p(F)-1/4)/g_p^2(o)$.*

Proof For any real u, let

$$U_N=\frac{1}{\binom{N}{p}}\sum_{i_1<i_2<\cdots<i_p}\varphi_N(X_{i_1},\cdots,X_{i_p}), \tag{2. 5}$$

where $\varphi_N(x_1,\cdots,x_p)=1$ if $x_1+\cdots+x_p>pu/N^{1/2},=0$ otherwise. Note that $\mu=E_0T(X)=1/2$ and $T(X-u/N^{1/2})=U_N$, then from above (c) and Lemma 2. 1

$$\lim_{N\to\infty}P_\theta\{N^{1/2}(\hat{\theta}_p-\theta)\leq u\}=\lim_{N\to\infty}P_0\{\hat{\theta}_p\leq u/N^{1/2}\}$$

$$=\lim_{N\to\infty} P_0\left\{T(X-u/N^{1/2})\leq\frac{1}{2}\right\}$$

$$=\lim_{N\to\infty} P_0\{N^{1/2}(U_N-E_0U_N)\leq N^{1/2}(1/2-E_0U_N)\}.$$

Since U_N is a U-statistic, for which φ_N is uniformly bounded, it follows from the general theory of U-statistic [3] that $N^{1/2}(U_N-E_0U_N)$ has a limiting normal distribution with mean 0 and variance $p^2[P_0\{X_1+X_2+\cdots+X_p>0,\ X_1+X_2'+\ \ +X_p'>0\}-(P_0\{X_1+\cdots+X_p>0\})^2]=p^2(\lambda_p(F)-1/4)$, where the X_i' and X_j are independent and identically distributed with c.d.f. $F(x)$. On the other hand $N^{1/2}(1/2-E_0U_N)=N^{1/2}(G_p(pu/N^{1/2})-1/2)=N^{1/2}(G_p(pu/N^{1/2})-G_p(0))\to pug_p(0)$, as $N\to\infty$, which completes the proof.

§3. Asymptotic efficiency of $\hat{\theta}_p$

It is well known that $N^{1/2}(\bar{X}-\theta)$ has a limiting normal distribution with mean 0 and variance σ_f^2. Therefore from Theorem 2. 1

$$\text{(3. 1)}\qquad A.R.E.(\hat{\theta}_p\ \bar{X})=\sigma_f^2g_p^2(0)/\left(\lambda_p(F)-\frac{1}{4}\right),$$

$$\text{(3. 2)}\qquad A.R.E.(\hat{\theta}_p\ \hat{\theta}_q)=g_p^2(0)\left(\lambda_q(F)-\frac{1}{4}\right)/g_q^2(0)\left(\lambda_p(F)-\frac{1}{4}\right).$$

Especially

$$A.R.E.(\hat{\theta}_p\ \hat{\theta}_1)=g_p^2(0)/4f^2(0)\left(\lambda_p(F)-\frac{1}{4}\right),$$

$$A.R.E.(\hat{\theta}_p\ \hat{\theta}_2)=g_p^2(0)/12\left(\int f^2(x)dx\right)^2\left(\lambda_p(F)-\frac{1}{4}\right).$$

Now we shall evaluate the value of $A.R.E.\ (\hat{\theta}_p\ \bar{X})$. For this purpose we require following two lemmas.

Lemma 3. 1. *Let $X_{i,1}, X_{i,2},\cdots, X_{i,N}$ be independent random samples from the population with c.d.f. $F(x-\theta_i)$, $i=1, 2, \cdots, c$, and let*

$$U^{(i_1i_2\cdots i_r)}=\frac{1}{\binom{N}{2}}\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{N}\varphi(Z_{i_1i_2\cdots i_r,\alpha},\ Z_{j_1j_2\cdots j_r,\beta}),$$

where $Z_{i_1i_2\cdots i_r,\alpha}=X_{i_1,\alpha}+\cdots+X_{i_r,\alpha}$ and $\varphi(Z_\alpha, Z_\beta)=1$ if $Z_\alpha+Z_\beta>0$, $=0$ otherwise. Then the random vector with components $N^{1/2}(U^{(i_1\cdots i_r)}-E_0U^{(i_1\cdots i_r)})$ has a normal distribution with mean 0 and covariance matrix $\left(4\left[\lambda_{2\,(Gr)}^{(i_1\cdots i_r;\ j_1\cdots j_r)}-\frac{1}{4}\right]\right)$, *where*

$$\text{(3. 3)}\qquad \lambda_{2\,(Gr)}^{(i_1\cdots i_r;\ j_1\cdots j_r)}=P_0\{Z_{i_1\cdots i_r,1}+Z_{i_1\cdots i_r,2}>0,\ Z_{j_1\cdots j_r,1}+Z_{j_1\cdots j_r,3}>0\}.$$

Proof is obuious from the general theory of generalized U-statistic (see

[3] P. 964.).

Lemma 3. 2. *For $\lambda_p(F)$ defined by (2. 4) it holds that for all $F \in \mathfrak{F}$*

$$\frac{1}{4} \leq \lambda_{2m}(F) \leq \frac{3m+1}{12m}, \quad m=1,\ 2, \cdots. \tag{3. 4}$$

Proof The left inequality is easy from the Schwarz′ inequality; $\lambda_p(F) = \int f(x) G_{2m-1}^2(x)\,dx \geq (\int f(x) G_{2m-1}(x)\,dx)^2 = (P_0\{X_1 + \quad + X_{2m} > 0\})^2 = 1/4$, for the distribution of X_1, X_2, $\cdots$, X_{2m} is symmetric about the origin. To prove the right inequality, consider the random vector Y with components

$$\begin{array}{l} Y_{i_{11}i_{12}\cdots i_{1m}},\ Y_{i_{21}i_{22}\cdots i_{2m}}, \cdots,\ Y_{i_{m1}i_{m2}\cdots i_{mm}}, \\ Y_{i_{11}i_{21}\cdots i_{m1}},\ Y_{i_{12}i_{22}\cdots i_{m2}}, \cdots,\ Y_{j_{1m}i_{2m}\cdots i_{mm}}, \end{array} \tag{3. 5}$$

where $Y_{i_1 i_2 \cdots i_m} = N^{1/2}(U^{(i_1 \cdots i_m)} - E_0 U^{(i_1 \cdots i_m)})$ and $U^{(i_1 \cdots i_m)}$ are defined in Lemma 3. 1. By (3. 3) the asymptotic covariance of $Y_{i_1 \cdots i_m}$ and $Y_{j_1 \cdots j_m}$ is given by

$$4\left[\lambda_{2\,(G_m)}^{(i_1 \cdots i_m : j_1 \cdots j_m)} - \frac{1}{4}\right] = 0\,; \text{ if } i_1, \cdots, i_m, j_1, \cdots, j_m \text{ are all different}$$

$$, = \frac{1}{3}\,; \text{ if } (i_1 i_2 \cdots i_m) = (j_1 j_2 \cdots j_m)$$

$$, = 4\left(\lambda_{2m}(F) - \frac{1}{4}\right)\,; \text{ otherwise.}$$

Hence the asymptotic convariance matrix of Y, denoted by Σ_m, is written as follows.

$$\Sigma_m = \begin{array}{c} \\ (i_{11} \cdots i_{1m}) \\ \vdots \\ (i_{m1} \cdots i_{mm}) \\ (i_{11} \cdots i_{m1}) \\ \vdots \\ (i_{1m} \cdots i_{mm}) \end{array} \begin{array}{c} \begin{array}{cccccc} (i_{11} \cdots i_{1m}) & \cdots\cdots & (i_{m1} \cdots i_{mm}) & (i_{11} \cdots i_{m1}) & \cdots\cdots & (i_{1m} \cdots i_{mm}) \end{array} \\ \left(\begin{array}{ccc|ccc} 1/3 & & 0 & 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) & \cdots & 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1/3 & 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) & \cdots & 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) \\ \hline 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) & \cdots & 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) & 1/3 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) & \cdots & 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) & 0 & & 1/3 \end{array}\right) \end{array}. \tag{3. 6}$$

Put $\lambda_{2m}(F) - 1/4 = \gamma/12$, then the determinant of Σ_m is

(3. 7)
$$\det \Sigma_m=\left(\frac{1}{3}\right)^{2m}\begin{vmatrix} 1 & & 0 & r\cdots r \\ & \ddots & & \vdots \quad \vdots \\ 0 & & 1 & r\cdots r \\ r\cdots r & & & 1 \quad\quad 0 \\ \vdots \quad \vdots & & & \ddots \\ r\cdots r & & & 0 \quad\quad 1 \end{vmatrix}=\left(\frac{1}{3}\right)^{2m}(1-m^2r^2)$$

Since $\det \Sigma_m \geq 0$, we have $r\leq 1/m$, which implies $\lambda_{2m}(F)\leq(3m+1)/12m$, as was to be proved.

We shall denote by $\mathfrak{F}^*$ the family of distributions which belong to $\mathfrak{F}$ and satisfy the condition of the theorem 2. 1.

Theorem 3. 1. *Suppose that p is even. Then*

$$\inf_{F\in\mathfrak{F}^*} A.R.E.\ (\hat{\theta}_p|\bar{X})\geq 0.864.$$

Proof We shall put $p=2m$, $m=1, 2,\cdots$, then

$g_{2m}(0) = \int g_m^2(x)\ dx$. From (3. 1) and lemma 3. 2.,

$$\begin{aligned} \inf\ A.R.E.(\hat{\theta}_{2m}|\bar{X}) &= \inf\frac{\sigma_f^2 g_{2m}^2(0)}{\lambda_{2m}(F)-1/4} \\ &= \inf\frac{12\sigma_{g_m}^2\left(\int g_m^2(x)dx\right)^2}{12m(\lambda_{2m}(F)-1/4)} \geq \frac{\inf\ 12\sigma_{g_m}^2\left(\int g_m^2(x)dx\right)^2}{\sup 12m(\lambda_{2m}(F)-1/4)} \\ &\geq \inf\ 12\ \sigma_{g_m}^2\left(\int g_m^2(x)dx\right)^2, \end{aligned}$$

where $\sigma_{g_m}^2$ is the variance of p.d.f. g_m. It has been shown by Hodges and Lehmann [1] that

(3. 8)
$$g_m(x)=\frac{3}{20\sqrt{5}}(5-x^2) \text{ if } x^2\leq 5, \ =0 \text{ otherwise}$$

attains the infimum value 0.864 of the last expression. This completes the proof.

Remark. For even m there exists no underlying distribution $F(x)$ which satisfies (3. 8), since the characteristic function is

$$(3/5\sqrt{5})[(1/t^3)\ \sin t\sqrt{5}-(\sqrt{5}/t^2)\ \cos t\sqrt{5}],$$

which is negative for some t. The author presents a conjecture *A.R.E.* $(\hat{\theta}_{2m}|\bar{X})>0.864$ for all $m>1$.

The above theorem does not hold for odd p, as is seen in Table II for $p=3$. In order to give an evaluation for odd p, we shall consider the random variable $Z_{i_1i_2\cdots i_r,\alpha}$, $\alpha=1, 2, \cdots, N$, given in lemma 3. 1 and the statistic $U_{(i_1i_2\cdots i_r)} = N^{-1}\sum_{i=1}^{N}\psi(Z_{i_1i_2\cdots i_r,\alpha})$, where $\psi(Z)=1$ if $Z>0$, $=0$ otherwise. A similar procedure as lemmas 3. 1 and 3. 2 will lead us to obtain

(3. 9) $$\frac{1}{4}\leq\lambda_p(F)\leq\frac{1+p}{4p},\quad p=1,\ 2,\ \cdots$$

Though the upper bound of (3. 9) is somewhat larger than that of (3. 4) for even p, it gives an evaluation of $\lambda_p(F)$ for odd p. Therefore we shall try to evaluate the value of $A.R.E.\ (\hat{\theta}_p, \bar{X})$ for odd p by means of (3. 9). Let $\mathfrak{F}_u$ be the family of distributions which are unimodal and belong to $\mathfrak{F}$. Then

Lemma 3. 3.[(1)] *If $F(x)\in\mathfrak{F}_u$, then $G_p(y)\in\mathfrak{F}_u$.*

Proof It is sufficient to show that if X and Y are independent random variables with c.d.f. $F(x)\in\mathfrak{F}_u$ and $G(y)\in\mathfrak{F}_u$, respectively, then the c.d.f. $H(z)$ of the random variable $Z=X+Y$ belongs to $\mathfrak{F}_u$. Since $H(z)\in\mathfrak{F}$ is obvious, we shall show the unimodality of $H(z)$. Let the p.d.f. of F, G and H be f, g and h, respectively. Then for arbitrary $z_2>z_1>0$,

$$h(z_2)-h(z_1)=\int_{-\infty}^{\infty}\{f(z_2-y)-f(z_1-y)\}g(y)dy$$
$$=\int_{-\infty}^{(z_1+z_2)/2}\{f(z_2-y)-f(z_1-y)\}g(y)dy+\int_{(z_1+z_2)/2}^{\infty}\{f(z_2-y)-f(z_1-y)\}g(y)dy$$
$$=\int_{(z_1+z_2)/2}^{\infty}\{f(z_2-y)-f(z_1-y)\}\{g(y)-g(z_1+z_2-y)\}dy.$$

Now $|z_2-y|\leq|z_1-y|$ and $|y|\geq|z_1+z_2-y|$ for $y\geq(z_1+z_2)/2$, so that from symmetry and unimodality of F, G, it follows that $f(z_2-y)\geq f(z_1-y)$, $g(y)\leq g(z_1+z_2-y)$ for $y>(z_1+z_2)/2$. Hence $h(z_2)\leq h(z_1)$, as was to be proved.

Let $\mathfrak{F}_u^*$ be the family of distributions which are unimodal and belong to $\mathfrak{F}^*$. From lemma 3. 3 $g_{2m}(0)\geq g_{2m-1}(x)$ for any $F\in\mathfrak{F}_u$. Therefore $g_{2m}(0)=\int f(x)g_{2m-1}(x)dx\leq g_{2m-1}(0)$. Hence from theorem 3. 1,

$$\inf_{F\in\mathfrak{F}_u^*}\sigma_f^2 g^2{}_{2m-1}(0)\geq\inf_{F\in\mathfrak{F}_u^*}\frac{\sigma_{g_m}^2}{m}g_{2m}^2(0)$$
$$\geq\frac{0.864}{12m},\quad\text{for } m=1,\ 2,\ \cdots.$$

Combining this with (3. 9), we obtain the theorem below.

Theorem 3. 2. For odd p it holds that

(3. 10) $$\inf_{F\in\mathfrak{F}_u^*}A.R.E.\ (\hat{\theta}_p|\bar{X})\geq0.288\frac{2p}{p+1}$$

Some numerical values of $g_p(0)$, $\lambda_p(F)$ and $A.R.E.\ (\hat{\theta}_p|\bar{X})$ for normal, uniform and double exponential distributions are given in the following tables.

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(1) The lemma and the proof was given in more generaliged form by professor K. Isii, Osaka University.

Table I $f(x)=(1/\sqrt{2\pi})\ \exp\ (-x^2/2)$

p	1	2	4	5	10	20
$g_p\ (0)$	0.3989	0.2829	0.1995	0.1784	0.1262	0.0892
$\lambda_p\ (F)$	0.5000	0.3333	0.2902	0.2820	0.2659	0.2579
$A.R.E.\ (\hat{\theta}_p \mid \bar{X})$	0.6366	0.9500	0.9894	0.9933	0.9983	0.9996

Table II $f(x)=1\ x\in\left(-\frac{1}{2},\ \frac{1}{2}\right),\ =0$ otherwise

p	1	2	3	4	5	6
$g_p\ (0)$	1.0000	1.0000	0.7500	0.6667	0.5990	0.5500
$\lambda_p\ (F)$	0.5000	0.3333	0.3052	0.2909	0.2825	0.2771
$A.R.E.\ (\hat{\theta}_p \mid \bar{X})$	0.3333	1.0000	0.8490	0.9061	0.9192	0.9296

Table III $f(x)=\frac{1}{2}e^{-|x|}$

p	1	2	3	4	5	6
$g_p\ (0)$	0.5000	0.2500	0.1875	0.1563	0.1367	0.1230
$\lambda_p\ (F)$	0.5000	0.3333	0.3032	0.2908	0.2809	0.2761
$A.R.E.\ (\hat{\theta}_p \mid \bar{X})$	2.0000	1.5000	1.3207	1.2439	1.2118	1.1582

It would be interesting to compute the numerical values of $A.R.E.$ $(\hat{\theta}_p \mid \bar{X})$ with respect to the following distributions.

(3. 11) $$f(x)=\frac{\varepsilon}{\sqrt{2\pi}}e^{-\frac{x}{2}}+\frac{(1-\varepsilon)}{2}e^{-|x|},\ 0\leq\varepsilon\leq 1$$

(3. 12) $$f(x)=\frac{1}{\left(1+\frac{1+\alpha}{2}\right)2^{1+(1+\alpha)/2}}\,exp\left\{-\frac{1}{2}\ |x|^{\frac{2}{1+\alpha}}\right\},\ -1<\alpha\leq 1.$$

These two families include a normal distribution ($\varepsilon=1$, $\alpha=0$) as well as a double exponential distribution ($\varepsilon=0$, $\alpha=1$). It is expected that for any $p=3, 4,\cdots$ there exists a value of ε or α for which $A.R.E.$ $(\hat{\theta}_p \mid \bar{X})$ attains its maximum value ≥ 1 at p.

§4. Alternative estimators of θ

Suppose that N observations $X_1, X_2,\cdots, X_N$ are divided in some way into p groups, which denoted by $(X_1^{(1)},\cdots, X_{n_1}^{(1)})$, $(X_1^{(2)},\cdots, X_{n_2}^{(2)}),\cdots$, $(X_1^{(p)},\cdots, X_{n_p}^{(p)})$ where $n_i=\rho_i N$, $i=1, 2, \cdots, p$ and $\rho_1+\rho_2+\cdots+\rho_p=1$. Then we can construct several alternative estimators of θ such as

(4. 1) $$\hat{\theta}_p^{*}=med\frac{X_{i_1}+X_{i_2}+\cdots+X_{i_p}}{p},\quad i_\alpha=1,\ 2,\cdots,n_\alpha,\quad \alpha=1,\ 2,\cdots,p$$

(4. 2) $$\hat{\theta}_p^{**}=\frac{1}{p}\sum_{\alpha=1}^{p}\hat{\theta}^{(\alpha)},\ \text{where}\ \hat{\theta}^{(\alpha)}=\underset{i<j}{med}\frac{X_i^{(\alpha)}+X_j^{(\alpha)}}{2},\quad i,j=1,\ 2,\cdots,\ n_\alpha$$

(4. 3) $$\hat{\theta}_p^{***}=\underset{i>j}{med}\frac{X_i+X_j}{2},\ \text{where}\ X_i=\frac{1}{p}\sum_{\alpha=1}^{p}X_i^{(\alpha)}\quad i,\ j=1,\ 2,\cdots,n$$

provided $n_1=n_2=\cdots=n_p=n$.

Theorem 4. 1.

(1) *Under the same condition as in theorem 3. 1., $N^{1/2}\ (\hat{\theta}_p^{*}-\theta)$ has a limiting normal distribution with mean 0 and variance $p^{-2}(\rho_1^{-1}+\cdots+\rho_p^{-1})(\lambda_p(F)-1/4)g_p^{-2}(0)$.*

(2) *Suppose that $G_2(y)$ has the derivative $g_2(0)\neq 0$ at $y=0$. Then $N^{1/2}\ (\hat{\theta}_p^{**}-\theta)$ has a limiting normal distribution with mean 0 and variance $p^{-2}\ (\rho_1^{-1}+\rho_2^{-1}+\cdots+\rho_p^{-1})\ [12g_2^2(0)]^{-1}$.*

(3) *Under the same condition as in (1) $N^{1/2}(\hat{\theta}_p^{***}-\theta)$ has a limiting normal distribution with mean 0 and variance $12[pg_{2p}^2(0)]^{-1}$.*

Proof (1) Since $\hat{\theta}_p^{*}$ can be represented by a U-statistic $T^*(X)=\left[\binom{n_1}{1}\cdots\binom{n_p}{1}\right]^{-1}\#\{(i_1\cdots i_p)\ ;\ X_{i_1}+X_{i_2}+\cdots+X_{i_p}>0,\ i_\alpha=1,\ 2,\cdots,n\ ;\ \alpha=1,\ 2,\cdots,\ p\}$ in the same way as (2. 1), the proof is analogous to that of theorem3. 1. (2) follows from the relation $N^{1/2}(\hat{\theta}_p^{**}-\theta)=p^{-1}\sum_{\alpha=1}^{p}\rho_\alpha^{-1/2}n_\alpha^{1/2}(\hat{\theta}^{(\alpha)}-\theta)$, where $n_\alpha^{1/2}(\hat{\theta}^{(\alpha)}-\theta)$, $\alpha=1,2,\cdots,\ p$, are independent and asymptotically normally distributed with mean 0 and variance $[12\ g_2(0)^2]^{-1}$.

(3) $\lim_{N\to\infty} P_\theta\{N^{1/2}(\hat{\theta}_p^{***}-\theta)\leq u\}=\lim_{n\to\infty} P_0\{n^{1/2}\hat{\theta}_p^{***}\leq p^{-1/2}u\}$. Since $\bar{X}_i$, $i=1,2,\cdots,n$, are independent and identically distriduted with p.d.f. $pg_p(px)$ when $\theta=0$, from the theorem 3. 1. $n^{1/2}\ \hat{\theta}_p^{***}$ has a limiting normal distribution with mean 0 and variance $[12p^2\ g_{2p}^2(0)]^{-1}$, as was to be proved.

It is seen by the theorem that for N fixed $n_1=n_2=\cdots=n_p$ is the best choice of the group sizes in order to make the asymptotic variance of $\hat{\theta}_p^{*}$ or $\hat{\theta}_p^{**}$ minimum. In this case the estimator $\hat{\theta}_p^{*}$ has the same asymptotic distribution as $\hat{\theta}_p$. Now since $\hat{\theta}_2^{*}$ as well as $\hat{\theta}_2$ has the same asymptotic distribution as $\hat{\theta}_p^{**}$, considering a trouble involved in computing $\hat{\theta}_p$ and $\hat{\theta}_p^{*}$, we might as well recomend $\hat{\theta}_p^{**}$ as an estimator of θ when N is large and $n_1=n_2\cdots=n_p$.

On the other hand for arbitrary $n_1,\ n_2,\cdots,n_p$ it will be preferable to use $\hat{\theta}_p$, $p=2m$, $m=1,\ 2,\cdots$, as an estimator of θ, for $\hat{\theta}_p^{*}$ or $\hat{\theta}_p^{**}$ has a large loss of efficiency in this case.

Since $A.R.E.\ (\hat{\theta}_p^{***}\ \bar{X})=12p\sigma_f^2 g_{2p}^2(0)=12\ \sigma_{g_p}^2\left(\int g_p^2(x)dx\right)^2$, the infimum of $A.R.E.\ (\hat{\theta}_p^{***}\ \bar{X})$ never falls below 0.864.

Therefore $\hat{\theta}_p^{***}$ will also be recommended for a practical use as an estimator of θ when sample size is large and $n_1 = \cdots = n_p$.

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