SOME NONPARAMETRIC ESTIMATORS OF A LOCATION PARAMETER

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SOME NONPARAMETRIC ESTIMATORS OF
A LOCATION PARAMETER

By

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§1. Introduction and Summary.

The purpose of this paper is to discuss some nonparametric estimators
of a location parameter, especially their asymptotic relative efficiencies
relative to the sample mean.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the population with
cumulative distribution function $F(x-\theta)$, where $\theta$ is a location parameter
and $F(x)$ is assumed to belong to the family $\mathcal{F}$ of all distribution functions
that are symmetric about the origin and absolutely continuous with respect
to the Lebesgue measure. Let $\tilde{e}_p$ be the median of the means of all $p$-tuple
$(X_{i_1}, X_{i_2}, \ldots, X_{i_p})$ in number, drawn from $X_1, X_2, \ldots, X_n$, i.e.

$$
(1.1) \hat{\theta}_p = \frac{\text{Med} \frac{X_{i_1} + X_{i_2} + \cdots + X_{i_p}}{p}}{\tilde{e}_p},
$$

which we shall propose as an estimator of $\theta$.

In the simplest case $p=1$, $\tilde{e}_1$ is the sample median. In a recent paper
[2] J. L. Hodges and E. L. Lehmann derived the estimator $\hat{\theta}_1$ of $\theta$ from
the one sample Wilcoxon statistic. Some of their results are as follows.

The asymptotic efficiency of $\hat{\theta}_1$ relative to the sample mean $\bar{X}$, denoted
A.R.E. $(\hat{\theta}_1, \bar{X})$, in the sense of reciprocal ratio of asymptotic variances, is
$$
4\sigma^2 f(\theta)^2, \text{ where } f \text{ denotes the density corresponding to } F \text{ and } \sigma^2 \text{ its variance,}
$$
while $A.R.E. (\hat{\theta}_2, \bar{X}) = 12\sigma^2 (f'f) dx$. The infimum of these efficiencies with
respect to the underlying distribution are well known to be 0 and 0.864,
respectively. Our investigation is a generalization of these results.

In Section 2 we shall discuss some properties of $\hat{\theta}_p$. In Section 3 we
shall state our main results that the infimum of $A.R.E. (\hat{\theta}_p, X)$ with respect
to the population distribution is always greater than or equal to 0.864 for
even $p$, but not so for odd $p$, even if $p \geq 3$. In Section 4 we shall consider
the case in which $N$ observations are divided into $p$ groups and define
alternative estimators of $\theta$ and recomend some of them as estimators of $\theta$.

§2. Some properties of $\hat{\theta}_p$.

By means of a rank test statistic $T(x)$, $X=(X_1, \ldots, X_N)$, which satisfies
the condition (1) $T(x+a)$ is a nondecreasing function of $a$ for all $x$, (2)
$E_0T(x) = \mu$, where $\mu$ is independent of $F$ and $E_0$ denotes the expectation under $\theta = 0$. Hodges and Lehmann [2] defined the estimator of $\theta$ as follows.

\begin{equation}
\hat{\theta} = \frac{\theta^{*} + \theta^{**}}{2},
\end{equation}

where $\theta^{*} = \inf \theta; T(x - \theta) < \mu$; and $\theta^{**} = \sup \theta; T(x - \theta) > \mu$.

If we put

\begin{equation}
T(X) = \frac{1}{N} \sum_{(i_1, \ldots, i_p)} \sum_{i_1 + \cdots + i_p > 0} X_{i_1} + \cdots + X_{i_p} > 0, \quad i_1 < i_2 < \cdots < i_p,
\end{equation}

where $\sum$ means the number of p-tuple $(i_1, \ldots, i_p)$ such that $X_{i_1} + X_{i_2} + \cdots + X_{i_p} > 0$, then the estimator $\hat{\theta}_p$ and $\hat{\theta}$ defined in (1.1) and (2.1), respectively, are seen to be identical. Therefore all results in [2] hold for the estimator $\hat{\theta}_p$, i.e. (a) the distribution of $\hat{\theta}_p$ is absolutely continuous with respect to the Lebesgue measure, (b) the distribution of $\hat{\theta}_p$ is symmetric about $\theta$, so that $\hat{\theta}_p$ is an unbiased estimator of $\theta$, (c) $\hat{\theta}_p$ is translation invariant, (d) the asymptotic relative efficiency of the test based on the test statistic $T(x)$ defined in (2.2) with respect to $t$-test is equal to A.R.E. $(\hat{\theta}_p, X)$, (e) we shall have the lemma below (see [2] p. 607).

Lemma 2.1. For $T(X)$ and $\hat{\theta}_p$ defined by (2.2) and (2.1), respectively, and for all $a$\footnote{We abbreviate $X_{i_1} + \cdots + X_{i_p}$ as $X_{(i_1, \ldots, i_p)}$.}

\begin{equation}
P\{T(X - a) < \mu\} \leq P\{\hat{\theta}_p < a\} \leq P\{T(X - a) \leq \mu\}.
\end{equation}

Let

\begin{equation}
G_p(y) = \int \cdots \int F(y - x_2 - \cdots - x_p)f(x_2)\cdots f(x_p)dx_2\cdots dx_p,
\end{equation}

\begin{equation}
\lambda_p(F) = \int f(x)G_{p-1}(\theta)dx,
\end{equation}

and let $g_p(y)$ be the p.d.f. of $G_p(y)$. Then we obtain the following theorem.

Theorem 2.1. Suppose $G_p(y)$ has the derivative $g_p(0) = 0$ at $y = 0$. Then $N^{1/2}(\hat{\theta}_p - \theta)$ has a limiting normal distribution with mean 0 and variance $(\lambda_p(F) - 1/4)/g_p^2(0)$.

Proof For any real $u$, let

\begin{equation}
U_N = \frac{1}{N} \sum_{(i_1, \ldots, i_p)} \varphi_N(X_{i_1}, \ldots, X_{i_p}),
\end{equation}

where $\varphi_N(x_1, \ldots, x_p) = 1$ if $x_1 + \cdots + x_p > pu/N^{1/2}$ otherwise. Note that $\mu = E_0T(X) = 1/2$ and $T(X - u/N^{1/2}) = U_N$, then from above (c) and Lemma 2.1

\[ \lim_{N \to \infty} P_{\theta} N^{1/2}(\hat{\theta}_p - \theta) \leq \mu = \lim_{N \to \infty} P_{\theta} \hat{\theta}_p \leq u/N^{1/2}. \]
\[
\lim_{N \to \infty} P_\theta \left\{ T \left( X - \mu/N^{1/2} \right) \leq \frac{1}{2} \right\} = \lim_{N \to \infty} P_\theta \left\{ N^{1/2} (U_N - \mu) \leq N^{1/2} (1/2 - \Phi U_N) \right\}.
\]

Since \( U_N \) is a \( U \)-statistic, for which \( \varphi_N \) is uniformly bounded, it follows from the general theory of \( U \)-statistic [3] that \( N^{1/2} (U_N - \mu) \) has a limiting normal distribution with mean 0 and variance \( \mu^2 \). Therefore from Theorem 2.1

\[
(3.1) \quad \mu \cdot g_1(0) = 0^2 g_1(0) / (2 \mu^2(F) - 1/4),
\]

\[
(3.2) \quad \text{A.R.E.}(\theta, \hat{\theta}) = g_1(0) \left( \lambda_1(F) - \frac{1}{4} \right) / g_1(0) \left( \lambda_1(F) - \frac{1}{4} \right).
\]

Especially

\[
\text{A.R.E.}(\hat{\theta}, \hat{\theta}) = g_1(0) / 2 \mu^2(0) \left( \lambda_1(F) - \frac{1}{4} \right),
\]

\[
\text{A.R.E.}(\hat{\theta}, \hat{\theta}) = g_1(0) / 12 \left( \mu^2(x) dx \right)^2 \left( \lambda_1(F) - \frac{1}{4} \right).
\]

Now we shall evaluate the value of \( \text{A.R.E.}(\hat{\theta}, X) \). For this purpose we require following two lemmas.

**Lemma 3.1.** Let \( X_{i,1}, X_{1,2}, \ldots, X_{i,N} \) be independent random samples from the population with c.d.f. \( F(x-\theta) \), \( i = 1, 2, \ldots, c \), and let

\[
U^{(i_{i_{1} \cdots i_{r}})} = \frac{1}{N} \sum_{a < b}^N \varphi(Z_{i_{1}i_{2} \cdots i_{r}}, a, Z_{i_{1}j_{2} \cdots j_{r}}, b),
\]

where \( Z_{i_{1}i_{2} \cdots i_{r}}, a = X_{i_{1}, a + \cdots + X_{i_{r}, a} \text{ and } \varphi(Z_a, Z_b) = 1 \text{ if } Z_a + Z_b > 0, = 0 \text{ otherwise. Then the random vector with components } N^{1/2}(U^{(i_{i_{1} \cdots i_{r}})} - E^0 U^{(i_{i_{1} \cdots i_{r}})}) \text{ has a normal distribution with mean 0 and covariance matrix}
\]

\[
\begin{bmatrix}
\lambda_1^{(i_{i_{1} \cdots i_{r}})} - \frac{1}{4} \\
\lambda_2^{(i_{i_{1} \cdots i_{r}}, j_{1} \cdots j_{r})} - \frac{1}{4}
\end{bmatrix},
\]

where

\[
\lambda_2^{(i_{i_{1} \cdots i_{r}}, j_{1} \cdots j_{r})} = P_\theta \left\{ Z_{i_{1}i_{2} \cdots i_{r}} + Z_{j_{1} \cdots j_{r}} > 0, Z_{j_{1} \cdots j_{r}} > 0 \right\}.
\]

Proof is obvious from the general theory of generalized \( U \)-statistic (see
Lemma 3.2. For $\lambda_F(F)$ defined by (2.4) it holds that for all $F \in \mathfrak{F}$

(3.4) \[ 1/4 \leq \lambda_{2m}(F) \leq \frac{3m+1}{12m}, \quad m = 1, 2, \ldots. \]

Proof The left inequality is easy from the Schwarz' inequality; $\lambda_F(F) = \int f(x)G_{2m-1}(x)dx \geq (\int f(x)G_{2m-1}(x)dx)^2 = (P_0 | X_1 + X_{2m} > 0 |)^2 = 1/4$, for the distribution of $X_1$, $X_2$, ..., $X_{2m}$ is symmetric about the origin. To prove the right inequality, consider the random vector $Y$ with components

(3.5)

\[
Y_{i_1,i_2,\ldots,i_m}, \quad Y_{i_1,i_2,\ldots,j_m}, \quad \ldots, \quad Y_{i_1,i_2,\ldots,i_{2m}}, \quad Y_{j_1,j_2,\ldots,i_{2m}}, \quad \ldots, \quad Y_{j_1,j_2,\ldots,j_{2m}},
\]

where $Y_{i_1,i_2,\ldots,i_m} \sim N^{1/2}(U^{(i_1-m)} - E_{m}U^{(i_1-m)})$ and $U^{(i_1-m)}$ are defined in Lemma 3.1. By (3.3) the asymptotic covariance of $Y_{i_1,\ldots,i_m}$ and $Y_{j_1,\ldots,j_m}$ is given by

\[
4\left[ \lambda_{2m}(F) \right] = 0; \text{ if } i_1, \ldots, i_m, j_1, \ldots, j_m \text{ are all different}
\]

\[
= \frac{1}{3}; \text{ if } (i_1,i_2,\ldots,i_m) = (j_1,j_2,\ldots,j_m)
\]

\[
= 4\left( \lambda_{2m}(F) - \frac{1}{4} \right); \text{ otherwise.}
\]

Hence the asymptotic covariance matrix of $Y$, denoted by $\Sigma_m$, is written as follows.

(3.6)

\[
\Sigma_m = \begin{pmatrix}
(i_{11} \cdots i_{1m}) & \cdots & (i_{m1} \cdots i_{mm}) & \cdots & (i_{11} \cdots i_{mm}) & \cdots & (i_{m1} \cdots i_{mm}) \\
(i_{11} \cdots i_{1m}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(i_{m1} \cdots i_{mm}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(i_{11} \cdots i_{1m}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(i_{m1} \cdots i_{mm}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(i_{11} \cdots i_{1m}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(i_{m1} \cdots i_{mm}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

Put $\lambda_{2m}(F) - 1/4 = \tau/12$, then the determinant of $\Sigma_m$ is
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(3. 7) \[ \det \Sigma_m = \left( \frac{1}{3} \right)^{2m} \begin{vmatrix} 1 & 0 & \cdots & \cdots & \cdots \\ \cdots & 1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdot & \cdot & \cdots \\ \cdots & \cdots & \cdot & \cdot & \cdots \\ \cdots & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \left( \frac{1}{3} \right)^{2m} \left( 1 - m^2 \tau^2 \right) \]

Since \( \det \Sigma_m \geq 0 \), we have \( \tau \leq 1/m \), which implies \( \lambda_{2m}(F) \leq (3m+1)/12m \), as was to be proved.

We shall denote by \( \mathcal{F} \) the family of distributions which belong to \( \mathcal{F} \) and satisfy the condition of the theorem 2. 1.

**Theorem 3. 1.** Suppose that \( p \) is even. Then

\[ \inf_{F \in \mathcal{F}^*} A.R.E. \left( \hat{\theta}_p/X \right) \geq 0.864. \]

**Proof** We shall put \( p=2m, m=1, 2, \cdots, \), then

\[ g_{2m}(0) = \int g_{2m}^2(x) \, dx. \]

From (3. 1) and lemma 3. 2,

\[ \inf_{F \in \mathcal{F}^*} A.R.E. \left( \hat{\theta}_{2m} / X \right) = \inf \frac{\sigma_{g_{2m}}^2(0)}{\lambda_{2m}(F) - 1/4} \]

\[ = \inf \frac{12 \sigma_{g_{2m}}^2(\int g_{2m}^2(x) \, dx)^2}{12m(\lambda_{2m}(F) - 1/4)} \geq \inf 12 \sigma_{g_{2m}}^2(\int g_{2m}^2(x) \, dx)^2, \]

where \( \sigma_{g_{2m}}^2 \) is the variance of p.d.f. \( g_{2m} \). It has been shown by Hodges and Lehmann [1] that

(3. 8) \[ g_{2m}(x) = \frac{3}{20\sqrt{5}} (5 - x^2) \text{ if } x^2 \leq 5, = 0 \text{ otherwise} \]

attains the infimum value 0.864 of the last expression. This completes the proof.

**Remark.** For even \( m \) there exists no underlying distribution \( F(x) \) which satisfies (3. 8), since the characteristic function is

\[ (3/5 \sqrt{5}) [(1/t^4) \sin t \sqrt{5} - (t^2 / \sqrt{5}) \cos t \sqrt{5}], \]

which is negative for some \( t \). The author presents a conjecture \( A.R.E. \left( \hat{\theta}_{2m} / X \right) > 0.864 \) for all \( m \geq 1 \).

The above theorem does not hold for odd \( p \), as is seen in Table II for \( p=3 \). In order to give an evaluation for odd \( p \), we shall consider the random variable \( Z_{\alpha \cdot i_1-i_2}, \alpha=1, 2, \cdots, N \), given in lemma 3. 1 and the statistic \( \hat{U}_{\alpha \cdot i_1-i_2} = N^{-1} \sum_{i=1}^N \psi(Z_{\alpha \cdot i_1-i_2}), \) where \( \psi(Z)=1 \) if \( Z>0 \), = 0 otherwise. A similar procedure as lemmas 3. 1 and 3. 2 will lead us to obtain
Though the upper bound of (3. 9) is somewhat larger than that of (3. 4) for even $p$, it gives an evaluation of $\lambda_p(F)$ for odd $p$. Therefore we shall try to evaluate the value of $A.R.E. (\hat{\theta}_p X)$ for odd $p$ by means of (3. 9). Let $\mathcal{F}_n$ be the family of distributions which are unimodal and belong to $\mathcal{G}$. Then

**Lemma 3. 3.** If $F(x) \in \mathcal{F}_n$, then $G_p(y) \in \mathcal{F}_n$.

**Proof** It is sufficient to show that if $X$ and $Y$ are independent random variables with c.d.f. $F(x) \in \mathcal{F}_n$ and $G(y) \in \mathcal{F}_n$, respectively, then the c.d.f. $H(z)$ of the random variable $Z=X+Y$ belongs to $\mathcal{F}_n$. Since $H(z) \in \mathcal{G}$ is obvious, we shall show the unimodality of $H(z)$. Let the p.d.f. of $F$, $G$ and $H$ be $f$, $g$ and $h$, respectively. Then for arbitrary $z_2 > z_1 > 0$,

\[
\begin{align*}
    h(z_2) - h(z_1) &= \int_{-\infty}^{(z_1+z_2)/2} \{f(z_2-y) - f(z_1-y)\} g(y)\,dy \\
    &= \int_{-\infty}^{(z_1+z_2)/2} \{f(z_2-y) - f(z_1-y)\} g(y)\,dy + \int_{(z_1+z_2)/2}^{\infty} \{f(z_2-y) - f(z_1-y)\} g(y)\,dy \\
    &= \int_{(z_1+z_2)/2}^{\infty} \{f(z_2-y) - f(z_1-y)\} g(y)\,dy - \int_{(z_1+z_2)/2}^{\infty} \{f(z_2-y) - f(z_1-y)\} g(z_1+z_2-y)\,dy.
\end{align*}
\]

Now $|z_2 - y| \leq |z_1 - y|$ and $|y| \geq |z_1 + z_2 - y|$ for $y \geq (z_1 + z_2)/2$, so that from symmetry and unimodality of $F$, $G$, it follows that $f(z_2-y) \geq f(z_1-y)$, $g(y) \leq g(z_1+z_2-y)$ for $y > (z_1 + z_2)/2$. Hence $h(z_2) \leq h(z_1)$, as was to be proved.

Let $\mathcal{F}_n^+$ be the family of distributions which are unimodal and belong to $\mathcal{G}^+$. From lemma 3. 3 $g_{2m}(0) \geq g_{2m-1}(x)$ for any $F \in \mathcal{F}_n$. Therefore $g_{2m}(0) = \int f(x) g_{2m-1}(x)\,dx \leq g_{2m-1}(0)$. Hence from theorem 3. 1,

\[
\inf_{F \in \mathcal{F}_n^+} \sigma^2 \geq \inf_{F \in \mathcal{F}_n^+} \frac{g_{2m}(0)}{g_{2m-1}(0)} \geq 0.864, \quad \text{for } m=1, 2, \ldots.
\]

Combining this with (3. 9), we obtain the theorem below.

**Theorem 3. 2.** For odd $p$ it holds that

\[
\text{Inf}_{F \in \mathcal{F}_n^+} A.R.E. (\hat{\theta}_p X) \geq 0.288 \frac{2p}{p+1}
\]

Some numerical values of $g_p(0)$, $\lambda_p(F)$ and $A.R.E. (\hat{\theta}_p X)$ for normal, uniform and double exponential distributions are given in the following tables.

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(1) The lemma and the proof was given in more generaliged form by professor K. Isii, Osaka University.
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Table I \( f(x) = \left(\frac{1}{\sqrt{2\pi}}\right) \exp \left(-\frac{x^2}{2}\right) \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p(0) )</td>
<td>0.3389</td>
<td>0.2829</td>
<td>0.1995</td>
<td>0.1784</td>
<td>0.1262</td>
<td>0.0982</td>
</tr>
<tr>
<td>( \hat{\theta}_p(F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.2902</td>
<td>0.2820</td>
<td>0.2659</td>
<td>0.2579</td>
</tr>
<tr>
<td>A.R.E. ( \langle \hat{\theta}_p, X \rangle )</td>
<td>0.6366</td>
<td>0.9500</td>
<td>0.9894</td>
<td>0.9933</td>
<td>0.9983</td>
<td>0.9996</td>
</tr>
</tbody>
</table>

Table II \( f(x) = x \left(\frac{1}{2}, \frac{1}{2}\right) \), = 0 otherwise

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p(0) )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.7500</td>
<td>0.6667</td>
<td>0.5990</td>
<td>0.5500</td>
</tr>
<tr>
<td>( \hat{\theta}_p(F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.3052</td>
<td>0.2909</td>
<td>0.2825</td>
<td>0.2771</td>
</tr>
<tr>
<td>A.R.E. ( \langle \hat{\theta}_p, X \rangle )</td>
<td>0.3333</td>
<td>1.0000</td>
<td>0.8490</td>
<td>0.9061</td>
<td>0.9192</td>
<td>0.9296</td>
</tr>
</tbody>
</table>

Table III \( f(x) = \frac{1}{2} e^{-x} \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p(0) )</td>
<td>0.5000</td>
<td>0.2500</td>
<td>0.1875</td>
<td>0.1563</td>
<td>0.1367</td>
<td>0.1230</td>
</tr>
<tr>
<td>( \hat{\theta}_p(F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.3032</td>
<td>0.2908</td>
<td>0.2869</td>
<td>0.2761</td>
</tr>
<tr>
<td>A.R.E. ( \langle \hat{\theta}_p, X \rangle )</td>
<td>2.0000</td>
<td>1.5000</td>
<td>1.3207</td>
<td>1.2439</td>
<td>1.2118</td>
<td>1.1582</td>
</tr>
</tbody>
</table>

It would be interesting to compute the numerical values of A.R.E. \( \langle \hat{\theta}_p, X \rangle \) with respect to the following distributions.

\[
(3.11) \quad f(x) = \frac{\varepsilon}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \frac{(1-\varepsilon)}{2} e^{-\frac{\varepsilon x}{2}}, \quad 0 \leq \varepsilon \leq 1
\]

\[
(3.12) \quad f(x) = \frac{1}{\left(1 + \frac{1+\alpha}{2}\right)^{2+(1+\alpha)/2}} \exp \left\{-\frac{1}{2} x^{\frac{\varepsilon}{1+\alpha}}\right\}, \quad -1 < \alpha \leq 1.
\]

These two families include a normal distribution \((\varepsilon = 1, \alpha = 0)\) as well as a double exponential distribution \((\varepsilon = 0, \alpha = 1)\). It is expected that for any \( p = 3, 4, \cdots \) there exists a value of \( \varepsilon \) or \( \alpha \) for which A.R.E. \( \langle \hat{\theta}_p, X \rangle \) attains its maximum value \( \geq 1 \) at \( \hat{\theta}_p \).

§4. Alternative estimators of \( \theta \)

Suppose that \( N \) observations \( X_1, X_2, \cdots, X_N \) are divided in some way into \( p \) groups, which denoted by \((X_1^{(1)}, \cdots, X_n^{(1)}), (X_1^{(2)}, \cdots, X_n^{(2)}), \cdots, (X_1^{(p)}, \cdots, X_n^{(p)})\) where \( n_i = \rho_i N, i = 1, 2, \cdots, p \) and \( \rho_1 + \rho_2 + \cdots + \rho_p = 1 \). Then we can construct several alternative estimators of \( \theta \) such as

\[
(4.1) \quad \hat{\theta}_p^* = \text{med} \left\{ \frac{1}{p} \sum_{i=1}^{n_a} X_{i}^{(i)} \right\}, \\
\quad i_a = 1, 2, \cdots, n_a \\
\quad \alpha = 1, 2, \cdots, p
\]
\[
\hat{\theta}_p^{\bullet \bullet \bullet} = \frac{1}{p} \sum_{a=1}^{p} \hat{\theta}_{(a)}, \quad \text{where} \quad \hat{\theta}_{(a)} = \med_{i < j} \frac{X_i^{(a)} + X_j^{(a)}}{2}, \quad i, j = 1, 2, \ldots, n_a
\]

(4.3) \[
\hat{\theta}_p^{\bullet \bullet \bullet} = \med_{i > j} \frac{X_i + X_j}{2}, \quad \text{where} \quad X_i = \frac{1}{p} \sum_{a=1}^{p} X_i^{(a)}, \quad i, j = 1, 2, \ldots, n
\]

provided \(n_1 = n_2 = \cdots = n_p = n\).

**Theorem 4.1.**

1. **Under the same condition as in theorem 3.1.,** \(N^{1/2} (\hat{\theta}_p^{\bullet \bullet} - \theta)\) has a limiting normal distribution with mean 0 and variance \(p^{-2} (\rho_1^{-1} + \cdots + \rho_p^{-1}) (\lambda_p (F) - 1/4) g^2_p (0)\).

2. **Suppose that** \(G_2(y)\) **has the derivative** \(g_2(0) \neq 0\) **at** \(y = 0\). Then \(N^{1/2} (\hat{\theta}_p^{\bullet \bullet \bullet} - \theta)\) **has a limiting normal distribution with mean 0 and variance** \(p^{-2} (\rho_1^{-1} + \rho_2^{-1} + \cdots + \rho_p^{-1}) \left[12g^2_p(0)\right]^{-1}\).

3. **Under the same condition as in (1),** \(N^{1/2} (\hat{\theta}_p^{\bullet \bullet \bullet} - \theta)\) **has a limiting normal distribution with mean 0 and variance** \(12[p g^2_p (0)]^{-1}\).

**Proof** (1) Since \(\hat{\theta}_p^{\bullet \bullet}\) **can be represented by a U-statistic** \(T^p(X) = \left(\begin{array}{c}
\ell_1 \\
n_1 \end{array}\right) \cdots \left(\begin{array}{c}
n_p \\
n_p \end{array}\right)^{-1} \sum_{i_1, \ldots, i_p} X_{i_1} + \cdots + X_{i_p}, \quad i_\alpha = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, p\), **in the same way as** (2.1), **the proof is analogous to that of theorem 3.1.** (2) **follows from the relation** \(N^{1/2} (\hat{\theta}_p^{\bullet \bullet \bullet} - \theta) = p^{-1} \sum_{\alpha=1}^{p} \rho_\alpha^{-1/2} N_{n_\alpha}^{1/2} (\hat{\theta}_p^{(\alpha)} - \theta), \quad n_\alpha^{1/2} (\hat{\theta}_p^{(\alpha)} - \theta), \quad \alpha = 1, 2, \ldots, p; \) **are independent and asymptotically normally distributed with mean 0 and variance** \(12 [g^2_p(0)]^{-1}\).

(3) \(\lim_{N \to \infty} P_N \left| N^{1/2} (\hat{\theta}_p^{\bullet \bullet \bullet} - \theta) - \theta \right| \leq u = \lim_{N \to \infty} P_N \left| n^{1/2} \hat{\theta}_p^{\bullet \bullet \bullet} - \theta \right| \leq p^{-1/2} u.\) **Since** \(X_i, \quad i = 1, 2, \ldots, n;\) **are independent and identically distributed with p.d.f.** \(p g_p (p)\) **when** \(\theta = 0,\) **from the theorem 3.1,** \(n^{1/2} \hat{\theta}_p^{\bullet \bullet \bullet}\) **has a limiting normal distribution with mean 0 and variance** \(12 [g^2_p (0)]^{-1},\) **as was to be proved.**

It is seen by the theorem that for **\(N\) fixed** \(n_1 = n_2 = \cdots = n_p\) **is the best choice of the group sizes in order to make the asymptotic variance of** \(\hat{\theta}_p^{\bullet \bullet}\) **or** \(\hat{\theta}_p^{\bullet \bullet \bullet}\) **minimum.** **In this case the estimator** \(\hat{\theta}_p^{\bullet \bullet}\) **has the same asymptotic distribution as** \(\hat{\theta}_p.\) **Now since** \(\hat{\theta}_p^{\bullet \bullet}\) **as well as** \(\hat{\theta}_p^{\bullet \bullet \bullet}\) **has the same asymptotic distribution as** \(\hat{\theta}_p^{\bullet \bullet}\), **considering a trouble involved in computing** \(\hat{\theta}_p\) **and** \(\hat{\theta}_p^{\bullet \bullet}\), **we might as well recommend** \(\hat{\theta}_p^{\bullet \bullet \bullet}\) **as an estimator of** \(\theta\) **when** \(N\) **is large and** \(n_1 = n_2 = \cdots = n_p.\)**

On the other hand for arbitrary \(n_1, n_2, \ldots, n_p\) **it will be preferable to use** \(\hat{\theta}_p, \quad p = 2m, \quad m = 1, 2, \ldots,\)** **as an estimator of** \(\theta,\) **for** \(\hat{\theta}_p^{\bullet \bullet}\) **or** \(\hat{\theta}_p^{\bullet \bullet \bullet}\) **has a large loss of efficiency in this case.**

Since \(A.R.E. (\hat{\theta}_p^{\bullet \bullet \bullet} X) = 12 p \sigma^2 p g^2_p (0) = 12 \sigma_p^2 \left(\int g^2_p (x) \, dx\right)^2,\)** the infimum of \(A.R.E. (\hat{\theta}_p^{\bullet \bullet \bullet} X)\) **never falls below** 0.864. **Therefore** \(\hat{\theta}_p^{\bullet \bullet \bullet}\) **will also be recommended for a practical use as an estimator**
of $\theta$ when sample size is large and $n_1 = \cdots = n_p$.

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Osaka University