SOME NONPARAMETRIC ESTIMATORS OF A LOCATION PARAMETER

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SOME NONPARAMETRIC ESTIMATORS OF A LOCATION PARAMETER

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§1. Introduction and Summary.

The purpose of this paper is to discuss some nonparametric estimators of a location parameter, especially their asymptotic relative efficiencies relative to the sample mean.

Let \( X_1, X_2, \ldots, X_n \) be a random sample from the population with cumulative distribution function \( F(x-\theta) \), where \( \theta \) is a location parameter and \( F(x) \) is assumed to belong to the family \( \mathcal{G} \) of all distribution functions that are symmetric about the origin and absolutely continuous with respect to the Lebesgue measure. Let \( \hat{0}_p \) be the median of the means of all \( p \)-tuple \( (X_{i1}, X_{i2}, \ldots, X_{ip}) \), \( \binom{N}{p} \) in number, drawn from \( X_1, X_2, \ldots, X_N \), i.e.

\[
\hat{\theta}_p = \frac{\text{Med} \sum_{i_1 < i_2 < \cdots < i_p} X_{i_1} + X_{i_2} + \cdots + X_{i_p}}{p},
\]

which we shall propose as an estimator of \( \theta \).

In the simplest case \( p=1 \), \( \hat{\theta}_1 \) is the sample median. In a recent paper \([2]\) J. L. Hodges and E. L. Lehmann derived the estimator \( \hat{\theta}_2 \) of \( \theta \) from the one sample Wilcoxon statistic. Some of their results are as follows. The asymptotic efficiency of \( \hat{\theta}_1 \) relative to the sample mean \( X \), denoted \( A.R.E. (\hat{\theta}_1 | X) \), in the sense of reciprocal ratio of asymptotic variances, is \( \frac{4f^2}{\sigma^2} \), where \( f \) denotes the density corresponding to \( F \) and \( \sigma^2 \) its variance, while \( A.R.E. (\hat{\theta}_2 | X) = \frac{12f^2}{\sigma^2} \). The infimum of these efficiencies with respect to the underlying distribution are well known to be 0 and 0.864, respectively. Our investigation is a generalization of these results.

In Section 2 we shall discuss some properties of \( \hat{\theta}_p \). In Section 3 we shall state our main results that the infimum of \( A.R.E. (\hat{\theta}_p | X) \) with respect to the population distribution is always greater than or equal to 0.864 for even \( p \), but not so for odd \( p \), even if \( p \geq 3 \). In Section 4 we shall consider the case in which \( N \) observations are divided into \( p \) groups and define alternative estimators of \( \theta \) and recommend some of them as estimators of \( \theta \).

§2. Some properties of \( \hat{\theta}_p \).

By means of a rank test statistic \( T(x), X=(X_1, \ldots, X_N) \), which satisfies the condition (1) \( T(x+a) \) is a nondecreasing function of \( a \) for all \( x \), (2)
$E_0T(x) = \mu$, where $\mu$ is independent of $F$ and $E_0$ denotes the expectation under $\theta = 0$. Hodges and Lehmann [2] defined the estimator of $\theta$ as follows.

(2. 1) $\hat{\theta} = \frac{\theta^* + \theta^{**}}{2},$

where $\theta^* = \inf_\theta; T(x-\theta) < \mu;$ and $\theta^{**} = \sup_\theta; T(x-\theta) > \mu.$

If we put

(2. 2) $T(X) = \frac{1}{(N)} \sum_{\{i_1, \ldots, i_p\}; X_{i_1} + \cdots + X_{i_p} > 0, i_1 < i_2 < \cdots < i_p},$

where $\sum$ means the number of $p$-tuble $(i_1, i_2, \ldots, i_p)$ such that $X_{i_1} + \cdots + X_{i_p} > 0$, then the estimator $\hat{\theta}_p$ and $\hat{\theta}$ defined in (1. 1) and (2. 1), respectively, are seen to be identical. Therefore all results in [2] hold for the estimator $\hat{\theta}_p$, i.e. (a) the distribution of $\hat{\theta}_p$ is absolutely continuous with respect to the Lebesgue measure, (b) the distribution of $\hat{\theta}_p$ is symmetric about $\theta$, so that $\hat{\theta}_p$ is an unbiased estimator of $\theta$, (c) $\hat{\theta}_p$ is translation invariant, (d) the asymptotic relative efficiency of the test based on the test statistic $T(x)$ defined in (2. 2) with respect to $t$-test is equal to A.R.E. ($\hat{\theta}_p, X$), (e) we shall have the lemma below (see [2] p. 607).

**Lemma 2.1.** For $T(X)$ and $\hat{\theta}_p$ defined by (2. 2) and (2. 1), respectively, and for all $a$

$$P\{T(X-a) < \mu\} \leq P\{\hat{\theta}_p-a\} \leq P\{T(X-a) \leq \mu\}.$$ 

Let

(2. 3) $G_p(y) = \int \cdots \int F(y-x_2-\cdots-x_p)f(x_2)\cdots f(x_p)dx_2\cdots dx_p,$

(2. 4) $\lambda_p(F) = \int f(x)G_{p-1}(\theta)dx,$

and let $g_p(y)$ be the p.d.f. of $G_p(y)$. Then we obtain the following theorem.

**Theorem 2.1.** Suppose $G_p(y)$ has the derivative $g_p(0) = 0$ at $y=0$. Then $N^{1/2}(\hat{\theta}_p-\theta)$ has a limiting normal distribution with mean 0 and variance $(\lambda_p(F)-1/4)/g_p^2(0)$.

**Proof** For any real $u$, let

(2. 5) $U_N = \frac{1}{N} \sum_{N} \phi_N(X_{i_1}, \cdots, X_{i_p}),$

where $\phi_N(x_1, \cdots, x_p) = 1$ if $x_1 + \cdots + x_p > pu/N^{1/2}$, otherwise. Note that $\mu = E_0T(X) = 1/2$ and $T(X-u/N^{1/2}) = U_N$, then from above (c) and Lemma 2. 1

$$\lim_{N \to \infty} P_{\phi}\{N^{1/2}(\hat{\theta}_p-\theta) \leq \mu\} = \lim_{N \to \infty} P_{\phi}\{\hat{\theta}_p \leq u/N^{1/2}\};$$
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\[ T(X - u/N^{1/2}) \leq \frac{1}{2} \]

\[ N^{1/2}(U_N - E_0 U_N) \leq N^{1/2}(1/2 - E_0 U_N) \]

Since \( U_N \) is a \( U \)-statistic, for which \( \varphi_N \) is uniformly bounded, it follows from the general theory of \( U \)-statistic [3] that \( N^{1/2}(U_N - E_0 U_N) \) has a limiting normal distribution with mean 0 and variance \( \phi^2([P_0|X_1 + X_2 + \cdots + X_p > 0, X_1 + X_2^* + \cdots + X_p^* > 0] - (P_0|X_1 + \cdots + X_p^* > 0)^2) = \phi^2(\varphi(F) - 1/4) \), where the \( X_i \) and \( X_i^* \) are independent and identically distributed with c.d.f. \( F(x) \). On the other hand \( N^{1/2}(1/2 - E_0 U_N) = N^{1/2}(G_\phi(pu/N^{1/2}) - 1/2) = N^{1/2}(G_\phi(pu/N^{1/2}) - G_\phi(0)) \rightarrow \mu \phi(pu(0)) \), as \( N \rightarrow \infty \), which completes the proof.

§3. Asymptotic efficiency of \( \hat{\theta}_p \)

It is well known that \( N^{1/2}(X - \theta) \) has a limiting normal distribution with mean 0 and variance \( \sigma^2_\theta \). Therefore from Theorem 2.1

(3.1) \[ A.R.E. (\hat{\theta}_p X) = \sigma^2_\theta g^2(0)/(\lambda^2(F) - 1/4) \]

(3.2) \[ A.R.E. (\hat{\theta}_p \hat{\theta}_q) = g^2(0)/(\lambda^2(F) - 1/4) \]

Especially

\[ A.R.E. (\hat{\theta}_p \hat{\theta}_q) = g^2(0)/(4f^2(0)(\lambda^2(F) - 1/4)) \]

\[ A.R.E. (\hat{\theta}_p \hat{\theta}_q) = g^2(0)/(12(f^2(x)dx)^2(\lambda^2(F) - 1/4)) \]

Now we shall evaluate the value of \( A.R.E. (\hat{\theta}_p X) \). For this purpose we require following two lemmas.

**Lemma 3.1.** Let \( X_{i1}, X_{i2}, \cdots, X_{ic} \), \( i=1, 2, \cdots, c \), be independent random samples from the population with c.d.f. \( F(x - \theta_i) \), and let

\[ U^{(i_1i_2i_3i_4)} = \frac{1}{N} \sum_{a, b=1}^{N} \varphi(Z_{i_1i_2i_3i_4}, Z_{i_1i_2i_3i_4}, \mu), \]

where \( Z_{i_1i_2i_3i_4} = X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4} \) and \( \varphi(Z_{a}, Z_{a}) = 1 \) if \( Z_{a} + Z_{a} > 0 \), =0 otherwise. Then the random vector with components \( N^{1/2}(U^{(i_1i_2i_3i_4)} - E_0 U^{(i_1i_2i_3i_4)}) \) has a normal distribution with mean 0 and covariance matrix

\[ \left[ \lambda^2(\varphi(Gr), j_1j_2j_3j_4) - 1/4 \right], \]

where

(3.3) \[ \lambda^2(\varphi(Gr), j_1j_2j_3j_4) = P_0\{Z_{i_1i_2i_3i_4} + Z_{i_1i_2i_3i_4} > 0, Z_{j_1j_2j_3} > 0 \}. \]

Proof is obvious from the general theory of generalized \( U \)-statistic (see
Lemma 3.2. For $\lambda_p(F)$ defined by (2.4) it holds that for all $F \in \mathfrak{F}$

\begin{equation}
1/4 \leq \lambda_{2m}(F) \leq m+1/12m, \quad m=1, 2, \ldots.
\end{equation}

Proof The left inequality is easy from the Schwarz' inequality; 

$$
\lambda_p(F) = \int f(x)G_{2m-1}(x)dx \geq (\int f(x)G_{2m-1}(x)dx)^2 = (P_0 | X_1 + X_2m > 0)^2 = 1/4,
$$

for the distribution of $X_1, X_2, \ldots, X_{2m}$ is symmetric about the origin. To prove the right inequality, consider the random vector $Y$ with components

\begin{equation}
Y_{i_1i_2 \ldots i_{2m}}, Y_{i_1i_2 \ldots i_{2m}}, \ldots, Y_{i_1i_2 \ldots i_{2m}}.
\end{equation}

where $Y_{i_1i_2 \ldots i_{2m}} = N_{1/2}(U_{i_1i_2 \ldots i_{2m}} - E(U_{i_1i_2 \ldots i_{2m}}))$ and $U_{i_1i_2 \ldots i_{2m}}$ are defined in Lemma 3.1. By (3.3) the asymptotic covariance of $Y_{i_1i_2 \ldots i_{2m}}$ and $Y_{j_1j_2 \ldots j_{2m}}$ is given by

\begin{equation}
4\left(\lambda_{2m}(F) - \frac{1}{4}\right) = 0; \quad \text{if } i_1, \ldots, i_m, j_1, \ldots, j_m \text{ are all different}
\end{equation}

\begin{equation}
= \frac{1}{3}; \quad \text{if } (i_1, \ldots, i_m, j_1, \ldots, j_m) = (j_1, j_2, \ldots, j_m)
\end{equation}

\begin{equation}
4\left(\lambda_{2m}(F) - \frac{1}{4}\right); \quad \text{otherwise}.
\end{equation}

Hence the asymptotic covariance matrix of $Y$, denoted by $\Sigma_m$, is written as follows.

\begin{equation}
\Sigma_m =
\begin{pmatrix}
(i_{11} \cdots i_{1m}) & \cdots & (i_{11} \cdots i_{mm}) & \cdots & (i_{11} \cdots i_{mm}) \\
(i_{m1} \cdots i_{mm}) & \cdots & (i_{m1} \cdots i_{mm}) & \cdots & (i_{m1} \cdots i_{mm}) \\
(i_{11} \cdots i_{mm}) & \cdots & (i_{m1} \cdots i_{mm}) & \cdots & (i_{11} \cdots i_{mm}) \\
(i_{m1} \cdots i_{mm}) & \cdots & (i_{11} \cdots i_{mm}) & \cdots & (i_{m1} \cdots i_{mm}) \\
(i_{11} \cdots i_{mm}) & \cdots & (i_{m1} \cdots i_{mm}) & \cdots & (i_{11} \cdots i_{mm}) \\
\end{pmatrix}
\end{equation}

\begin{equation}
= 1/3 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) \cdots 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) 0 \cdots 0
\end{equation}

\begin{equation}
= 1/3 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) \cdots 4\left(\lambda_{2m}(F) - \frac{1}{4}\right) 0 \cdots 0
\end{equation}

Put $\lambda_{2m}(F) - 1/4 = \tau/12$, then the determinant of $\Sigma_m$ is
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\[ \det \Sigma_m = \left( \frac{1}{3} \right)^{2m} \begin{vmatrix} 1 & 0 & \cdots \cdot & \cdot \\ 0 & 1 & \cdots \cdot & \cdot \\ \cdot & \cdots & \ddots & \cdot \\ \cdot & \cdots & \cdot & 1 \\ \cdot & \cdots & \cdot & \cdots \\ \end{vmatrix} = \left( \frac{1}{3} \right)^{2m} (1 - \frac{m^2 \tau^2}{12}) \]

Since \( \det \Sigma_m \geq 0 \), we have \( \tau \leq 1/m \), which implies \( \lambda_{2m}(F) \leq (3m+1)/12m \), as was to be proved.

We shall denote by \( \mathfrak{F} \) the family of distributions which belong to \( \mathfrak{F} \) and satisfy the condition of the theorem 2. 1.

**Theorem 3. 1.** Suppose that \( p \) is even. Then

\[ \inf_{F \in \mathfrak{F}} \text{A.R.E.} \left( \hat{\theta}_p | X \right) \geq 0.864. \]

**Proof** We shall put \( p = 2m, m = 1, 2, \cdots \), then

\[ g_{2m}(0) = \int g_m^2(x) \, dx. \]

From (3. 1) and lemma 3. 2,

\[ \inf_{F} \text{A.R.E.} \left( \hat{\theta}_{2m} | X \right) = \inf \frac{\sigma_{2m}^2 (0)}{\lambda_{2m}(F) - 1/4} \]

\[ = \inf \frac{12 \sigma_{2m}^2 \left( \int g_m^2(x) \, dx \right)^2}{12m(\lambda_{2m}(F) - 1/4)} \geq \inf 12 \sigma_{2m}^2 \left( \int g_m^2(x) \, dx \right)^2 \sup 12m(\lambda_{2m}(F) - 1/4) \]

where \( \sigma_{2m}^2 \) is the variance of p.d.f. \( g_m \). It has been shown by Hodges and Lehmann [1] that

\[ g_m(x) = \frac{3}{20\sqrt{5}} \left( 5 - x^2 \right) \text{ if } x^2 \leq 5, = 0 \text{ otherwise} \]

attains the infimum value 0.864 of the last expression. This completes the proof.

Remark. For even \( m \) there exists no underlying distribution \( F(x) \) which satisfies (3. 8), since the characteristic function is

\[ (3/5\sqrt{5}) \left[ (1/t^3) \sin t \sqrt{5} - (t^3 \sqrt{5}/t) \cos t \sqrt{5} \right], \]

which is negative for some \( t \). The author presents a conjecture \( \text{A.R.E.} \left( \hat{\theta}_{2m} | X \right) > 0.864 \) for all \( m > 1 \).

The above theorem does not hold for odd \( p \), as is seen in Table II for \( p = 3 \). In order to give an evaluation for odd \( p \), we shall consider the random variable \( Z_{1 \cdots i-1}, a = 1, 2, \cdots, N \), given in lemma 3. 1 and the statistic \( U(1 \cdots i-1) = N^{-1} \sum_{i=1}^{N} \psi(Z_{1 \cdots i-1}, a) \), where \( \psi(Z) = 1 \) if \( Z > 0 \), = 0 otherwise. A similar procedure as lemmas 3. 1 and 3. 2 will lead us to obtain
Though the upper bound of (3.9) is somewhat larger than that of (3.4) for even \( p \), it gives an evaluation of \( \lambda_p(F) \) for odd \( p \). Therefore we shall try to evaluate the value of A.R.E. \( \langle \hat{\theta}_p X \rangle \) for odd \( p \) by means of (3.9). Let \( \mathcal{F}_n \) be the family of distributions which are unimodal and belong to \( \mathcal{F} \). Then

**Lemma 3.3.** (1) If \( F(x) \in \mathcal{F}_n \), then \( G_p(y) \in \mathcal{F}_n \).

**Proof** It is sufficient to show that if \( X \) and \( Y \) are independent random variables with c.d.f. \( F(x) \in \mathcal{F}_n \) and \( G(y) \in \mathcal{F}_n \), respectively, then the c.d.f. \( H(z) \) of the random variable \( Z = X + Y \) belongs to \( \mathcal{F}_n \). Since \( H(z) \in \mathcal{F} \) is obvious, we shall show the unimodality of \( H(z) \). Let the p.d.f. of \( F \), \( G \) and \( H \) be \( f \), \( g \) and \( h \), respectively. Then for arbitrary \( z > z_1 > 0 \),

\[
h(z_2) - h(z_1) = \int_{-\infty}^{\infty} f(z_2 - y) - f(z_1 - y) \cdot g(y) \, dy
\]

\[
= \int_{-\infty}^{(z_1 + z_2)/2} f(z_2 - y) - f(z_1 - y) \cdot g(y) \, dy + \int_{(z_1 + z_2)/2}^{\infty} f(z_2 - y) - f(z_1 - y) \cdot g(y) \, dy
\]

\[
= \int_{(z_1 + z_2)/2}^{\infty} f(z_2 - y) - f(z_1 - y) \cdot g(y) - g(z_1 + z_2 - y) \, dy.
\]

Now \( z_2 - y \leq z_1 - y \) and \( y \geq z_1 + z_2 - y \) for \( y \geq (z_1 + z_2)/2 \), so that from symmetry and unimodality of \( F \), \( G \), it follows that \( f(z_2 - y) \geq f(z_1 - y) \), \( g(y) \leq g(z_1 + z_2 - y) \) for \( y > (z_1 + z_2)/2 \). Hence \( h(z_2) \leq h(z_1) \), as was to be proved.

Let \( \mathcal{F}_n^* \) be the family of distributions which are unimodal and belong to \( \mathcal{F}_n^0 \). From lemma 3.3 \( g_{2m}^* (0) \geq g_{2m-1}^* (x) \) for any \( F \in \mathcal{F}_n^* \). Therefore \( g_{2m}^* (0) = \int f(x) g_{2m-1}^* (x) \, dx \leq g_{2m-1}^* (0) \). Hence from theorem 3.1,

\[
\inf_{F \in \mathcal{F}_n^*} \sigma^2 g_{2m-1}^* (0) \geq \inf_{F \in \mathcal{F}_n^*} \sigma^2 g_{2m}^* (0) \geq 0.864 \quad 12m, \text{ for } m = 1, 2, \ldots.
\]

Combining this with (3.9), we obtain the theorem below.

**Theorem 3.2.** For odd \( p \) it holds that

\[
\inf_{F \in \mathcal{F}_n^*} \text{A.R.E.} \langle \hat{\theta}_p X \rangle \geq 0.288 \frac{2p}{p+1}
\]

Some numerical values of \( g_p(0) \), \( \lambda_p(F) \) and A.R.E. \( \langle \hat{\theta}_p \bar{X} \rangle \) for normal, uniform and double exponential distributions are given in the following tables.

---

(1) The lemma and the proof was given in more generalised form by professor K. Isii, Osaka University.
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Table I  \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p (0) )</td>
<td>0.3389</td>
<td>0.2829</td>
<td>0.1995</td>
<td>0.1784</td>
<td>0.1262</td>
<td>0.0982</td>
</tr>
<tr>
<td>( \lambda_p (F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.2902</td>
<td>0.2820</td>
<td>0.2659</td>
<td>0.2579</td>
</tr>
<tr>
<td>A.R.E. (( \hat{\theta}_p, X ) )</td>
<td>0.6366</td>
<td>0.9500</td>
<td>0.9894</td>
<td>0.9933</td>
<td>0.9983</td>
<td>0.9996</td>
</tr>
</tbody>
</table>

Table II \( f(x) = \frac{1}{2} x e \left( -\frac{1}{2} x, \frac{1}{2} x \right), =0 \text{ otherwise} \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p (0) )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.7500</td>
<td>0.6967</td>
<td>0.5990</td>
<td>0.5500</td>
</tr>
<tr>
<td>( \lambda_p (F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.3052</td>
<td>0.2909</td>
<td>0.2825</td>
<td>0.2771</td>
</tr>
<tr>
<td>A.R.E. (( \hat{\theta}_p, X ) )</td>
<td>0.3333</td>
<td>1.0000</td>
<td>0.8490</td>
<td>0.9061</td>
<td>0.9192</td>
<td>0.9296</td>
</tr>
</tbody>
</table>

Table III \( f(x) = \frac{1}{2} e^{-x} \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_p (0) )</td>
<td>0.5000</td>
<td>0.2500</td>
<td>0.1875</td>
<td>0.1563</td>
<td>0.1367</td>
<td>0.1230</td>
</tr>
<tr>
<td>( \lambda_p (F) )</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.3032</td>
<td>0.2908</td>
<td>0.2809</td>
<td>0.2761</td>
</tr>
<tr>
<td>A.R.E. (( \hat{\theta}_p, X ) )</td>
<td>2.0000</td>
<td>1.5000</td>
<td>1.3207</td>
<td>1.2439</td>
<td>1.2118</td>
<td>1.1582</td>
</tr>
</tbody>
</table>

It would be interesting to compute the numerical values of A.R.E. (\( \hat{\theta}_p, X \) ) with respect to the following distributions.

\[
f(x) = \frac{\xi}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \frac{1-\xi}{2} e^{-x}, \quad 0 \leq \xi \leq 1
\]

\[
f(x) = \frac{1}{\left(1 + \frac{1+\alpha}{2}\right)^2 + (1+\alpha)/2} e^{\left(-\frac{1}{2} x^{\xi+\alpha}\right)}, \quad -1 < \alpha \leq 1.
\]

These two families include a normal distribution (\( \xi = 1, \alpha = 0 \)) as well as a double exponential distribution (\( \xi = 0, \alpha = 1 \)). It is expected that for any \( p = 3, 4, \ldots \) there exists a value of \( \xi \) or \( \alpha \) for which A.R.E. (\( \hat{\theta}_p, X \) ) attains its maximum value \( \geq 1 \) at \( p \).

§4. Alternative estimators of \( \theta \)

Suppose that \( N \) observations \( X_1, X_2, \ldots, X_N \) are divided in some way into \( p \) groups, which denoted by \( (X_1^{(1)}, \ldots, X_n^{(1)}), (X_1^{(2)}, \ldots, X_n^{(2)}), \ldots, (X_1^{(p)}, \ldots, X_n^{(p)}) \) where \( n_i = \rho_i N, \ i = 1, 2, \ldots, p \) and \( \rho_1 + \rho_2 + \cdots + \rho_p = 1 \). Then we can construct several alternative estimators of \( \theta \) such as

\[
\hat{\theta}^*_a = \text{med} \left\{ \sum_{i=1}^{n_a} X_{i+a} / \rho \right\},
\]

\[
i_a = 1, 2, \ldots, n_a,
\]

\[
\alpha = 1, 2, \ldots, p
\]
\[ \hat{\theta}_p^{**} = \frac{1}{p} \sum_{\alpha=1}^{p} \hat{\theta}_p^{(\alpha)}, \quad \text{where} \quad \hat{\theta}_p^{(\alpha)} = \text{med} \frac{X_i^{(\alpha)} + X_j^{(\alpha)}}{2}, \]
\[ i < j, \quad i, j = 1, 2, \ldots, n, \]

(4.3)  \[ \hat{\theta}_p^{**} = \text{med} \frac{X_i + X_j}{2}, \quad \text{where} \quad X_i = \frac{1}{p} \sum_{\alpha=1}^{p} X_i^{(\alpha)}, \]
\[ i, j = 1, 2, \ldots, n, \]
provided \( n_1 = n_2 = \cdots = n_p = n \).

**Theorem 4.1.**

1. *Under the same condition as in theorem 3.1, \( N^{1/2} (\hat{\theta}_p^g - \theta) \) has a limiting normal distribution with mean 0 and variance \( p^{-1} (\rho_1^{-1} + \cdots + \rho_p^{-1}) \)
\((\lambda_p(F) - 1/4) g^{2}_p(0)\).*

2. *Suppose that \( G_2(y) \) has the derivative \( g_2(z) \neq 0 \) at \( z = 0 \). Then \( N^{1/2} (\hat{\theta}_p^{**} - \theta) \) has a limiting normal distribution with mean 0 and variance \( p^{-1} (\rho_1^{-1} + \cdots + \rho_p^{-1}) \)
\((12 g^{2}_2(0))^{-1} \).*

3. *Under the same condition as in (1) \( N^{1/2} (\hat{\theta}_p^{**} - \theta) \) has a limiting normal distribution with mean 0 and variance \( 12 [p g^{2}_2(0)]^{-1} \).*

**Proof.**

(1) Since \( \hat{\theta}_p^g \) can be represented by a U-statistic \( T^*(X) = \left[ \left( \frac{n_1}{p} \right)^{-1/2} \right] \left( \int \cdots \right) \) \( X_1 + X_2 + \cdots + X_p \geq 0, \) \( i = 1, 2, \ldots, n; \alpha = 1, 2, \ldots, p \), in the same way as (2.1), the proof is analogous to that of theorem 3.1. (2) follows from the relation \( N^{1/2} (\hat{\theta}_p^{**} - \theta) = p^{-1} \sum_{\alpha=1}^{p} \rho^{-1/2}_\alpha n^{1/2}_\alpha (\hat{\theta}_p^{(\alpha)} - \theta), \) where \( n^{1/2}_\alpha (\hat{\theta}_p^{(\alpha)} - \theta), \alpha = 1, 2, \ldots, p, \) are independent and asymptotically normally distributed with mean 0 and variance \( 12 g^2_2(0) \).

(3) \( \lim \frac{N}{n^{1/2} (\hat{\theta}_p^{**} - \theta)} \leq u \Rightarrow \lim P_n \frac{n^{1/2} (\hat{\theta}_p^{**} - \theta)}{p^{-1/2} u} \leq 0. \) Since \( X_i, i = 1, 2, \ldots, n, \) are independent and identically distributed with p.d.f. \( pg_p(x) \) when \( \theta = 0, \) from the theorem 3.1. \( n^{1/2} \hat{\theta}_p^{**} \) has a limiting normal distribution with mean 0 and variance \( 12 p g^2_p(0) \), as was to be proved.

It is seen by the theorem that for \( N \) fixed \( n_1 = n_2 = \cdots = n_p \) is the best choice of the group sizes in order to make the asymptotic variance of \( \hat{\theta}_p^g \) or \( \hat{\theta}_p^{**} \) minimum. In this case the estimator \( \hat{\theta}_p^g \) has the same asymptotic distribution as \( \hat{\theta}_p \). Now since \( \hat{\theta}_p^g \) as well as \( \hat{\theta}_p \) has the same asymptotic distribution as \( \hat{\theta}_p^{**} \), considering a trouble involved in computing \( \hat{\theta}_p \) and \( \hat{\theta}_p^{**} \), we might as well recommend \( \hat{\theta}_p^{**} \) as an estimator of \( \theta \) when \( N \) is large and \( n_1 = n_2 = \cdots = n_p \).

On the other hand for arbitrary \( n_1, n_2, \ldots, n_p \) it will be preferable to use \( \hat{\theta}_p \), \( p = 2m, m = 1, 2, \ldots, \) as an estimator of \( \theta \), for \( \hat{\theta}_p^{**} \) has a large loss of efficiency in this case.

Since \( A.R.E. \left( \hat{\theta}_p^{**} \right) X = 12 p \sigma_p^2 g^2_p(0) = 12 \sigma_p^2 \left( \int g^2_p(x) dx \right)^2, \) the infimum of \( A.R.E. \left( \hat{\theta}_p^{**} \right) X \) never falls below 0.864. Therefore \( \hat{\theta}_p^{**} \) will also be recommended for a practical use as an estimator.
of $\theta$ when sample size is large and $n_1 = \cdots = n_p$.

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Osaka University