SOME NONPARAMETRIC ESTIMATORS OF A LOCATION PARAMETER

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SOME NONPARAMETRIC ESTIMATORS OF A LOCATION PARAMETER

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§1. Introduction and Summary.

The purpose of this paper is to discuss some nonparametric estimators of a location parameter, especially their asymptotic relative efficiencies relative to the sample mean.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the population with cumulative distribution function $F(x-\theta)$, where $\theta$ is a location parameter and $F(x)$ is assumed to belong to the family $\mathcal{F}$ of all distribution functions that are symmetric about the origin and absolutely continuous with respect to the Lebesgue measure. Let $\hat{\theta}_p$ be the median of the means of all $p$-tuple $(X_{i_1}, X_{i_2}, \ldots, X_{i_p})$, $\binom{N}{p}$ in number, drawn from $X_1, X_2, \ldots, X_N$, i.e.

\[ \hat{\theta}_p = \text{Med} \frac{X_{i_1} + X_{i_2} + \cdots + X_{i_p}}{p}, \]

(1.1)

which we shall propose as an estimator of $\theta$.

In the simplest case $p=1$, $\hat{\theta}_1$ is the sample median. In a recent paper [2] J. L. Hodges and E. L. Lehmann derived the estimator $\hat{\theta}_2$ of $\theta$ from the one sample Wilcoxon statistic. Some of their results are as follows. The asymptotic efficiency of $\hat{\theta}_1$ relative to the sample mean $\bar{X}$, denoted A.R.E. $(\hat{\theta}_1 \mid \bar{X})$, in the sense of reciprocal ratio of asymptotic variances, is $4\sigma_f^2 f(0)$, where $f$ denotes the density corresponding to $F$ and $\sigma_f^2$ its variance, while A.R.E. $(\hat{\theta}_2 \mid \bar{X}) = 12\sigma_f^2(f f' \, dx)^2$. The infimum of these efficiencies with respect to the underlying distribution are well known to be 0 and 0.864, respectively. Our investigation is a generalization of these results.

In Section 2 we shall discuss some properties of $\hat{\theta}_p$. In Section 3 we shall state our main results that the infimum of A.R.E. $(\hat{\theta}_p \mid X)$ with respect to the population distribution is always greater than or equal to 0.864 for even $p$, but not so for odd $p$, even if $p \geq 3$. In Section 4 we shall consider the case in which $N$ observations are divided into $p$ groups and define alternative estimators of $\theta$ and recommend some of them as estimators of $\theta$.

§2. Some properties of $\hat{\theta}_p$.

By means of a rank test statistic $T(x), X = (X_1, \ldots, X_N)$, which satisfies the condition (1) $T(x+a)$ is a nondecreasing function of $a$ for all $x$, (2)
Let $E_0 T(x) = \mu$, where $\mu$ is independent of $F$ and $E_0$ denotes the expectation under $\theta = 0$. Hodges and Lehmann [2] defined the estimator of $\theta$ as follows.

\[(2.1) \quad \hat{\theta} = \frac{\theta^* + \theta^{**}}{2},\]

where $\theta^* = \inf_{\theta}; T(x - \theta) < \mu$; and $\theta^{**} = \sup_{\theta}; T(x - \theta) > \mu$.

If we put

\[(2.2) \quad T(X) = \frac{1}{\binom{N}{p}} \sum_{i_1 < i_2 < \cdots < i_p} (X_{i_1} + \cdots + X_{i_p}) > 0, \quad i_1 < i_2 < \cdots < i_p,\]

where $\binom{N}{p}$ means the number of $p$-tuble $(i_1, i_2, \ldots, i_p)$ such that $X_{i_1} + X_{i_2} + \cdots + X_{i_p} > 0$, then the estimator $\hat{\theta}_p$ and $\hat{\theta}$ defined in (1.1) and (2.1), respectively, are seen to be identical. Therefore all results in [2] hold for the estimator $\hat{\theta}_p$, i.e. (a) the distribution of $\hat{\theta}_p$ is absolutely continuous with respect to the Lebesgue measure, (b) the distribution of $\hat{\theta}_p$ is symmetric about $\theta$, so that $\hat{\theta}_p$ is an unbiased estimator of $\theta$, (c) $\hat{\theta}_p$ is translation invariant, (d) the asymptotic relative efficiency of the test based on the test statistic $T(x)$ defined in (2.2) with respect to the test is equal to A.R.E. $(\hat{\theta}_p, X)$, (e) we shall have the lemma below (see [2] p. 607).

**Lemma 2.1.** For $T(X)$ and $\hat{\theta}_p$ defined by (2.2) and (2.1), respectively, and for all $a$

\[P|T(X - a) < \mu| \leq P|\hat{\theta}_p - a| \leq P|T(X - a) \leq \mu|.

Let

\[(2.3) \quad G_p(y) = \int \cdots \int F(y - x_2 - \cdots - x_p) f(x_2) \cdots f(x_p) dx_2 \cdots dx_p,

\[(2.4) \quad \lambda_p(F) = \int f(x) G_{p-1}(\theta) dx,

and let $g_p(y)$ be the p.d.f. of $G_p(y)$. Then we obtain the following theorem.

**Theorem 2.1.** Suppose $G_p(y)$ has the derivative $g_p(0) > 0$ at $y=0$. Then $N^{1/2} (\hat{\theta}_p - \theta)$ has a limiting normal distribution with mean 0 and variance $(\lambda_p(F) - 1/4)/g_p^2(0)$.

**Proof** For any real $u$, let

\[(2.5) \quad U_N = \frac{1}{\binom{N}{p}} \sum_{i_1 < i_2 < \cdots < i_p} \varphi_N(X_{i_1}, \ldots, X_{i_p}),

where $\varphi_N(x_1, \ldots, x_p) = 1$ if $x_1 + \cdots + x_p > pu/N^{1/2}$, 0 otherwise. Note that $\mu = E_0 T(X) = 1/2$ and $T(X - u/N^{1/2}) = U_N$, then from above (c) and Lemma 2.1

\[\lim_{N \to \infty} P_N \left( (\hat{\theta}_p - \theta) \leq \mu \right) = \lim_{N \to \infty} P_0 \left( \hat{\theta}_p \leq u/N^{1/2} \right),\]

where $P_N$ denotes the distribution of $U_N$.
Some Nonparametric Estimators of a Location Parameter

\[ \lim_{N \to \infty} P_0 \left\{ T(X - \mu / N^{1/2}) \leq \frac{1}{2} \right\} = \lim_{N \to \infty} P_0 \left\{ N^{1/2}(U_N - E_0U_N) \leq N^{1/2}(1/2 - E_0U_N) \right\}. \]

Since \( U_N \) is a \( U \)-statistic, for which \( \varphi_N \) is uniformly bounded, it follows from the general theory of \( U \)-statistic [3] that \( N^{1/2}(U_N - E_0U_N) \) has a limiting normal distribution with mean 0 and variance \( \rho^2 = \sum_{i=1}^{p} \rho_i^2 + \rho_i^3 + \cdots + \rho_i^p \geq \frac{\rho_1}{2(\rho(F) - 1/4)}, \)

where the \( X_i \) and \( Y_j \) are independent and identically distributed with c.d.f. \( F(x) \). On the other hand \( N^{1/2}(1/2 - E_0U_N) = N^{1/2}(G_\rho(\rho u / N^{1/2}) - 1/2) = N^{1/2}(G_\rho(\rho u / N^{1/2}) - G_\rho(0)) \rightarrow pug_\rho(0), \) as \( N \rightarrow \infty \), which completes the proof.

§3. Asymptotic efficiency of \( \hat{\theta}_p \)

It is well known that \( N^{1/2}(X - \theta) \) has a limiting normal distribution with mean 0 and variance \( \sigma_\theta^2 \). Therefore from Theorem 2.1

\[ (3.1) \quad \text{A.R.E.}(\hat{\theta}_p X) = \sigma_\theta^2 g^2(0) \left( \lambda_\rho(F) - \frac{1}{4} \right), \]

\[ (3.2) \quad \text{A.R.E.}(\hat{\theta}_p \hat{\theta}_q) = g^2(0) \left( \lambda_\rho(F) - \frac{1}{4} \right) / g^2(0) \left( \lambda_\rho(F) - \frac{1}{4} \right). \]

Especially

\[ \text{A.R.E.}(\hat{\theta}_p \hat{\theta}_1) = g^2(0) / 4f^2(0) \left( \lambda_\rho(F) - \frac{1}{4} \right), \]

\[ \text{A.R.E.}(\hat{\theta}_p \hat{\theta}_2) = g^2(0) / 12 \left( f^2(x) dx \right)^2 \left( \lambda_\rho(F) - \frac{1}{4} \right). \]

Now we shall evaluate the value of \( \text{A.R.E.}(\hat{\theta}_p X) \). For this purpose we require following two lemmas.

**Lemma 3.1.** Let \( X_{i_1}, X_{i_2}, \ldots, X_{i_N} \) be independent random samples from the population with c.d.f. \( F(x - \theta_i) \), \( i = 1, 2, \ldots, c \), and let

\[ U^{(i_1i_2\ldots i_r)} = \frac{1}{N} \sum_{a,b=1}^{N} \varphi(Z_{i_1i_2\ldots i_r, a}, Z_{i_1i_2\ldots i_r, b}), \]

where \( Z_{i_1i_2\ldots i_r, a} = X_{i_1} + \cdots + X_{i_r}, a \) and \( \varphi(Z_a, Z_b) = 1 \) if \( Z_a + Z_b > 0 \), = 0 otherwise. Then the random vector with components \( N^{1/2}(U^{(i_1i_2\ldots i_r)} - E_0U^{(i_1i_2\ldots i_r)}) \) has a normal distribution with mean 0 and covariance matrix

\[ \left( \lambda^2 \left[ \left( G_{\rho} \right)^{(i_1i_2\ldots i_r)} - \frac{1}{4} \right) \right), \]

where

\[ (3.3) \quad \lambda^2 \left[ \left( G_{\rho} \right)^{(i_1i_2\ldots i_r)} - \frac{1}{4} \right) = P_0 \{ Z_{i_1i_2\ldots i_r, 1} + Z_{i_1i_2\ldots i_r, 2} > 0, Z_{i_1i_2\ldots i_r, 1} + Z_{i_1i_2\ldots i_r, 3} > 0 \}. \]

Proof is obvious from the general theory of generalized \( U \)-statistic (see...
Lemma 3.2. For \( \lambda_p(F) \) defined by (2.4) it holds that for all \( F \in \mathcal{F} \)

(3.4)

\[
\frac{1}{4} \leq \lambda_{2m}(F) \leq \frac{3m+1}{12m}, \quad m = 1, 2, \ldots.
\]

Proof The left inequality is easy from the Schwarz’ inequality; 
\( \lambda_p(F) = \int f(x)G_{2m-1}(x)dx \geq (\int f(x)G_{2m-1}(x)dx)^2 = (P_0|X_1 + X_{2m}>0)^2 = 1/4 \), for the distribution of \( X_1, X_2, \ldots, X_{2m} \) is symmetric about the origin. To prove the right inequality, consider the random vector \( Y \) with components

(3.5)

\[
Y_{i_1;j_1}, Y_{i_2;j_2}, \ldots, Y_{i_{2m};j_{2m}},
\]

where \( Y_{i_1;i_2;\ldots;i_{2m}} = N^{1/2}(U^{(i_1;i_2;\ldots;i_{2m})} - E_{U^{(i_1;i_2;\ldots;i_{2m})}}) \) and \( U^{(i_1;i_2;\ldots;i_{2m})} \) are defined in Lemma 3.1. By (3.3) the asymptotic covariance of \( Y_{i_1;i_2;\ldots;i_{2m}} \) and \( Y_{j_1;j_2;\ldots;j_{2m}} \) is given by

\[
4 \left[ \lambda_{2m}(G_{2m})^{(i_1;i_2;\ldots;i_{2m};j_1;j_2;\ldots;j_{2m})} - \frac{1}{4} \right] = 0; \quad \text{if } i_1, i_2, \ldots, i_{2m}, j_1, \ldots, j_{2m} \text{ are all different}
\]

\[
= \frac{1}{3}; \quad \text{if } (i_1, i_2, \ldots, i_{2m}) = (j_1, j_2, \ldots, j_{2m})
\]

\[
= 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right); \quad \text{otherwise}.
\]

Hence the asymptotic covariance matrix of \( Y \), denoted by \( \Sigma_m \), is written as follows.

(3.6)

\[
\Sigma_m = \begin{pmatrix}
(i_{11} \cdots i_{1m}) & (i_{m1} \cdots i_{mm}) & (i_{11} \cdots i_{1m}) & (i_{m1} \cdots i_{mm}) \\
(i_{11} \cdots i_{1m}) & 1/3 & 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right) & 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right) \\
(i_{m1} \cdots i_{mm}) & 0 & 1/3 & 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right) \\
(i_{11} \cdots i_{1m}) & 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right) & 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right) & 1/3 \\
(i_{m1} \cdots i_{mm}) & 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right) & 4 \left( \lambda_{2m}(F) - \frac{1}{4} \right) & 0 \\
(i_{11} \cdots i_{1m}) & \cdots & \cdots & \cdots \\
(i_{m1} \cdots i_{mm}) & \cdots & \cdots & \cdots \\
(i_{11} \cdots i_{1m}) & \cdots & \cdots & \cdots \\
(i_{m1} \cdots i_{mm}) & \cdots & \cdots & \cdots \\
(i_{11} \cdots i_{1m}) & \cdots & \cdots & \cdots \\
(i_{m1} \cdots i_{mm}) & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

Put \( \lambda_{2m}(F) - 1/4 = r/12 \), then the determinant of \( \Sigma_m \) is
Some Nonparametric Estimators of a Location Parameter

(3.7) \[
\det \Sigma_m = \left( \frac{1}{3} \right)^{2m} \begin{vmatrix}
1 & 0 & \cdots & \cdots \\
0 & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
\cdots & \cdots & \cdots & 1 \\
\end{vmatrix} = \left( \frac{1}{3} \right)^{2m} \left( 1 - m^2 \tau^2 \right)
\]

Since \( \det \Sigma_m \geq 0 \), we have \( \tau \leq 1/m \), which implies \( \lambda_{2m}(F) \leq (3m+1)/12m \), as was to be proved.

We shall denote by \( \mathcal{F}^* \) the family of distributions which belong to \( \mathcal{F} \) and satisfy the condition of the theorem 2.1.

**Theorem 3.1.** Suppose that \( p \) is even. Then

\[
\inf_{F \in \mathcal{F}^*} \text{A.R.E.} \left( \hat{\theta}_p, X \right) \geq 0.864.
\]

**Proof** We shall put \( p = 2m, m = 1, 2, \cdots \), then

\[
g_m(x) = \int g_m^*(x) \, dx. \quad \text{From (3.1) and lemma 3.2,}
\]

\[
\inf_{F} \text{A.R.E.} \left( \hat{\theta}_{2m}, X \right) = \inf \frac{\sigma^2 g_m^2(0)}{\lambda_{2m}(F) - 1/4}
\]

\[
= \inf \frac{12 \sigma^2 g_m^2 \left( \int g_m^*(x) \, dx \right)^2}{12m(\lambda_{2m}(F) - 1/4)} \geq \sup 12m(\lambda_{2m}(F) - 1/4)
\]

\[
\geq \inf \frac{12 \sigma^2 g_m^2 \left( \int g_m^*(x) \, dx \right)^2}{12m(\lambda_{2m}(F) - 1/4)}
\]

where \( \sigma^2 \) is the variance of p.d.f. \( g_m \). It has been shown by Hodges and Lehmann [1] that

\[
g_m(x) = \frac{3}{20v \sqrt{5} (5 - x^2)} \quad \text{if} \quad x^2 \leq 5, = 0 \quad \text{otherwise}
\]

attains the infimum value 0.864 of the last expression. This completes the proof.

Remark. For even \( m \) there exists no underlying distribution \( F(x) \) which satisfies (3.8), since the characteristic function is

\[
(3/5 \sqrt{v} \sqrt{5}) [ (1/t^4) \sin tv \sqrt{5} - (v \sqrt{5}/t^2) \cos tv \sqrt{5} ],
\]

which is negative for some \( t \). The author presents a conjecture \( \text{A.R.E.} \left( \hat{\theta}_{2m}, X \right) > 0.864 \) for all \( m > 1 \).

The above theorem does not hold for odd \( p \), as is seen in Table II for \( p = 3 \). In order to give an evaluation for odd \( p \), we shall consider the random variable \( Z_{1i_1-i_{12}} \), \( \alpha = 1, 2, \cdots, N \), given in lemma 3.1 and the statistic

\[
U_{(i_1i_2-ir, \alpha)} = N^{-1} \sum_{i=1}^{N} \psi(Z_{1i-i_{12}}, \alpha), \quad \text{where} \quad \psi(Z) = 1 \quad \text{if} \quad Z > 0, = 0 \quad \text{otherwise}.
\]

A similar procedure as lemmas 3.1 and 3.2 will lead us to obtain...
Though the upper bound of (3. 9) is somewhat larger than that of (3. 4) for even \( p \), it gives an evaluation of \( \lambda_p(F) \) for odd \( p \). Therefore we shall try to evaluate the value of A.R.E. \( (\hat{\theta}_p X) \) for odd \( p \) by means of (3. 9). Let \( \mathcal{F}_n \) be the family of distributions which are unimodal and belong to \( \mathcal{F} \). Then

**Lemma 3. 3.** (1) If \( F(x) \in \mathcal{F}_n \), then \( G_p(y) \in \mathcal{F}_n \).

**Proof** It is sufficient to show that if \( X \) and \( Y \) are independent random variables with c.d.f. \( F(x) \in \mathcal{F}_n \) and \( G(y) \in \mathcal{F}_n \), respectively, then the c.d.f. \( H(z) \) of the random variable \( Z=X+Y \) belongs to \( \mathcal{F}_n \). Since \( H(z) \in \mathcal{F} \) is obvious, we shall show the unimodality of \( H(z) \). Let the p.d.f. of \( F, G \) and \( H \) be \( f, g \) and \( h \), respectively. Then for arbitrary \( z_2, z_1 > 0 \),

\[
h(z_2) - h(z_1) = \int_{z_1+z_2}^{z_2} [f(z_2 - y) - f(z_1 - y)] g(y) \, dy \\
= \int_{z_1}^{z_2} [f(z_2 - y) - f(z_1 - y)] g(y) \, dy + \int_{z_1+z_2}^{z_2} [f(z_2 - y) - f(z_1 - y)] g(y) \, dy \\
= \int_{z_1}^{z_2} [f(z_2 - y) - f(z_1 - y)] g(y) \, dy - \int_{z_1+z_2}^{z_2} [f(z_2 - y) - f(z_1 - y)] g(y) \, dy.
\]

Now \( |z_2 - y| \leq |z_1 - y| \) and \( y \geq |z_1 + z_2 - y| \) for \( y \geq (z_1 + z_2)/2 \), so that from symmetry and unimodality of \( F, G \), it follows that \( f(z_2 - y) \geq f(z_1 - y) \), \( g(y) \leq g(z_1 + z_2 - y) \) for \( y > (z_1 + z_2)/2 \). Hence \( h(z_2) \leq h(z_1) \), as was to be proved.

Let \( \mathcal{F}^* \) be the family of distributions which are unimodal and belong to \( \mathcal{F} \). From lemma 3. 3, \( g^2_m(0) \geq g^2_{m-1}(x) \) for any \( F \in \mathcal{F}^* \). Therefore \( g^2_m(0) = \int f(x) g^2_{m-1}(x) \, dx \leq g^2_{m-1}(0) \). Hence from theorem 3. 1,

\[
\inf_{F \in \mathcal{F}^*} \sigma^2 g^2_{2m-1}(0) \geq \inf_{F \in \mathcal{F}^*} \sigma^2 g^2_{2m-1}(0) \\
\geq \frac{0.864}{12m}, \text{ for } m=1, 2, \ldots.
\]

Combining this with (3. 9), we obtain the theorem below.

**Theorem 3. 2.** For odd \( p \) it holds that

\[
(3. 10) \quad \inf_{F \in \mathcal{F}^*} \text{A.R.E. } (\hat{\theta}_p X) \geq 0.288 \frac{2p}{p+1}
\]

Some numerical values of \( g_p(0), \lambda_p(F) \) and A.R.E. \( (\hat{\theta}_p X) \) for normal, uniform and double exponential distributions are given in the following tables.

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(1) The lemma and the proof was given in more generalized form by professor K. Isii, Osaka University.
Some Nonparametric Estimators of a Location Parameter

Table I \( f(x) = \left(\frac{1}{\sqrt{2\pi}}\right) \exp\left(-\frac{x^2}{2}\right) \)

\[\begin{array}{cccccc}
p & 1 & 2 & 4 & 5 & 10 & 20 \\
g_p(0) & 0.3989 & 0.2829 & 0.1995 & 0.1784 & 0.1262 & 0.0892 \\
\lambda_p (F) & 0.5000 & 0.3333 & 0.2902 & 0.2820 & 0.2659 & 0.2579 \\
A.R.E. (\tilde{\theta}_p, X) & 0.6386 & 0.9500 & 0.9894 & 0.9933 & 0.9983 & 0.9996 \\
\end{array}\]

Table II \( f(x) = x e^{-\left(\frac{1}{2}, \frac{1}{2}\right)} \), \( = 0 \) otherwise

\[\begin{array}{cccccc}
p & 2 & 3 & 4 & 5 & 6 \\
g_p(0) & 1.0000 & 1.0000 & 0.7500 & 0.6667 & 0.5990 & 0.5500 \\
\lambda_p (F) & 0.5000 & 0.3333 & 0.3052 & 0.2909 & 0.2825 & 0.2771 \\
A.R.E. (\tilde{\theta}_p, X) & 0.3333 & 1.0000 & 0.8490 & 0.9061 & 0.9192 & 0.9296 \\
\end{array}\]

Table III \( f(x) = \frac{1}{2} e^{-x} \)

\[\begin{array}{cccccc}
p & 1 & 2 & 3 & 4 & 5 & 6 \\
g_p(0) & 0.5000 & 0.2500 & 0.1875 & 0.1563 & 0.1367 & 0.1230 \\
\lambda_p (F) & 0.5000 & 0.3333 & 0.3032 & 0.2908 & 0.2809 & 0.2761 \\
A.R.E. (\tilde{\theta}_p, X) & 2.0000 & 1.5000 & 1.3207 & 1.2439 & 1.2118 & 1.1582 \\
\end{array}\]

It would be interesting to compute the numerical values of A.R.E. \( (\tilde{\theta}_p, X) \) with respect to the following distributions.

\[
(3.11) \quad f(x) = \frac{\xi}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \frac{1-\xi}{2} e^{-x}, \quad 0 \leq \xi \leq 1
\]

\[
(3.12) \quad f(x) = \frac{1}{\left(1 + \frac{1+\alpha}{2}\right)^{2(1+\alpha)/2}} \exp\left\{ -\frac{1}{2} x^{\frac{2}{1+\alpha}} \right\}, \quad -1 < \alpha \leq 1.
\]

These two families include a normal distribution (\( \xi = 1, \alpha = 0 \)) as well as a double exponential distribution (\( \xi = 0, \alpha = 1 \)). It is expected that for any \( p = 3, 4, \ldots \) there exists a value of \( \xi \) or \( \alpha \) for which A.R.E. \( (\tilde{\theta}_p, X) \) attains its maximum value \( \geq 1 \) at \( p \).

\[\text{§4. Alternative estimators of } \theta\]

Suppose that \( N \) observations \( X_1, X_2, \ldots, X_N \) are divided in some way into \( p \) groups, which denoted by \( (X_1^{(1)}, \ldots, X_n^{(1)}), (X_1^{(2)}, \ldots, X_n^{(2)}), \ldots, (X_1^{(p)}, \ldots, X_n^{(p)}) \) where \( n_i = \rho_i N, \quad i = 1, 2, \ldots, p \) and \( \rho_1 + \rho_2 + \cdots + \rho_p = 1 \). Then we can construct several alternative estimators of \( \theta \) such as

\[
(4.1) \quad \tilde{\theta}^* = \text{med}_{i_0} \sum_{i=1}^{n} X_{i, i_0}, \quad i_0 = 1, 2, \ldots, n_a, \quad a = 1, 2, \ldots, p
\]
Theorem 4.1.

(1) **Under the same condition as in theorem 3.1, \( N^{1/2} (\hat{\theta}^* - \theta) \) has a limiting normal distribution with mean 0 and variance \( p^{-2}(\rho_1^{-1} + \cdots + \rho_p^{-1}) (\lambda_p(F) - 1/4)g_p^{-2}(0) \).

(2) Suppose that \( G_2(y) \) has the derivative \( g_2(0) \neq 0 \) at \( y=0 \). Then \( N^{1/2} (\hat{\theta}^{***} - \theta) \) has a limiting normal distribution with mean 0 and variance \( p^{-2}(\rho_1^{-1} + \rho_2^{-1} + \cdots + \rho_p^{-1}) [12g_2^2(0)]^{-1} \).

(3) **Under the same condition as in (1) \( N^{1/2} (\hat{\theta}^{***} - \theta) \) has a limiting normal distribution with mean 0 and variance \( 12[p_g^2(0)]^{-1} \).

Proof (1) Since \( \hat{\theta}^* \) can be represented by a U-statistic \( T^*(X) = \left( \begin{array}{c} n_1 \\ \vdots \\ n_p \end{array} \right), \) \( X_{i_1} + X_{i_2} + \cdots + X_{i_p} > 0, i_a = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, p \), in the same way as (2.1), the proof is analogous to that of theorem 3.1. (2) follows from the relation \( N^{1/2} (\hat{\theta}^{***} - \theta) = p^{-1} \sum_{a=1}^{p} \rho_a^{-1/2} n_a^{1/2} (\hat{\theta}^{(a)} - \theta) \), where \( n_a^{1/2} (\hat{\theta}^{(a)} - \theta) \), \( \alpha = 1, 2, \cdots, p \), are independent and asymptotically normally distributed with mean 0 and variance \( [12g_2^2(0)]^{-1} \).

(3) \( \lim_{N \to \infty} P_0 \{ N^{1/2} (\hat{\theta}^{***} - \theta) \leq u \} = \lim_{n \to \infty} P_0 \{ n^{1/2} \hat{\theta}^{***} \leq p^{-1/2} u \} \). Since \( X_i, i=1, 2, \cdots, n \), are independent and identically distributed with p.d.f. \( pg_p(p,x) \) when \( \theta = 0 \), from theorem 3.1, \( n^{1/2} \hat{\theta}^{***} \) has a limiting normal distribution with mean 0 and variance \( [12g_2^2(0)]^{-1} \), as was to be proved.

It is seen by the theorem that for \( N \) fixed \( n_1 = n_2 = \cdots = n_p \) is the best choice of the group sizes in order to make the asymptotic variance of \( \hat{\theta}^* \) or \( \hat{\theta}^{***} \) minimum. In this case the estimator \( \hat{\theta}^* \) has the same asymptotic distribution as \( \hat{\theta}^p \). Now since \( \hat{\theta}^* \) as well as \( \hat{\theta}^p \) has the same asymptotic distribution as \( \hat{\theta}^{***} \), considering a trouble involved in computing \( \hat{\theta}^p \) and \( \hat{\theta}^{***} \), we might as well recommend \( \hat{\theta}^{***} \) as an estimator of \( \theta \) when \( N \) is large and \( n_1 = n_2 = \cdots = n_p \).

On the other hand for arbitrary \( n_1, n_2, \cdots, n_p \) it will be preferable to use \( \hat{\theta}^p, p=2m, m=1, 2, \cdots \), as an estimator of \( \theta \), for \( \hat{\theta}^* \) or \( \hat{\theta}^{***} \) has a large loss of efficiency in this case.

Since \( A.R.E. (\hat{\theta}^{***} X) = 12p\sigma_p^2g_p^2(0) = 12 \sigma_p^2 \left( \int g_p^2(x)dx \right)^2 \), the infimum of \( A.R.E. (\hat{\theta}^{***} X) \) never falls below 0.864.

Therefore \( \hat{\theta}^{***} \) will also be recommended for a practical use as an estimator...
of $\theta$ when sample size is large and $n_1 = \cdots = n_p$.

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Osaka University