

## A DELIVERY-LAG INVENTORY CONTROL PROCESS WITH EMERGENCY AND NON-STATIONARY STOCHASTIC DEMANDS

Kodama, Masanori  
Kumamoto University

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# A DELIVERY-LAG INVENTORY CONTROL PROCESS WITH EMERGENCY AND NON-STATIONARY STOCHASTIC DEMANDS

By

Masanori KODAMA

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## §1. Introduction.

In [1] and [2] authors investigated one-commodity inventory model which, in addition to regular ordering of arbitrary amount of stock to be delivered one period later, there is also provided emergency ordering with instantaneous delivery of stock which is naturally more expensive than regular ordering. The model presented here is the generalization of the one discussed in [1] and [2]. We consider an  $n$ -period, one-commodity dynamic inventory model with non-stationary stochastic demands, and with a  $k$  (constant) period-lag delivery of regular orders, and with  $k$  "emergency" orders, characterized by delivery lags  $\mu_1=0, \mu_2=1, \dots, \mu_k=k-1$ . We shall assume the emergency quantity  $m_{r,n-j+1}^i$  of the time lag  $r$  for the period  $j$ , at the beginning of which the demand density is given by  $\varphi_i$ , can be any positive amount not exceeding an upper bound  $m_u$ ,  $0 \leq m_{r,n-i+1}^i \leq m_u$  ( $i=1, 2, \dots, m, j=1, 2, \dots, n, r=0, 1, \dots, k-1$  for  $n \geq k, r=0, 1, \dots, n-1$  for  $n < k$ ), and the regular order is  $m_{k,n-j+1}^i \geq 0$  ( $i=1, 2, \dots, m, j=1, 2, \dots, n$ ). This upper bound will be the same for each period. The cumulative demand in each period is a non-negative random variable whose distribution may change from period to period according to a Markov transition law with matrix  $P=[p_{ij}]$  ( $i, j=1, 2, \dots, m$ ) where  $p_{ij} \geq 0$  and  $\sum_{j=1}^m p_{ij} = 1$  for each  $i$ . It is assumed that the demand density remains unchanged during one period. In other words, when the demand in a given period is  $\varphi_i$  in the following period the demand density changes to  $\varphi_j$  with probability  $p_{ij}$  ( $i, j=1, 2, \dots, m$ ). The technique utilized in this paper was partly suggested by [3], [4], [5], and [6].

The inventory periods  $I_1, I_2, \dots, I_n$  are numbered from left to right. At beginning of the  $j$ -th period ( $j=1, 2, \dots, n$ ) two actions have to be taken (i) placing  $k$  emergency orders, (ii) issuing a regular order to be delivered at the end of the  $j+k-1$  period. The delivery lag  $\lambda=k$  is constant throughout the rest of the paper. We impose the following conditions on the model.

- (1.1) There is backlogging of excess demand.
- (1.2) The known distribution function of demand is absolutely continuous

with respect to the Lebesgue measure. The density will be denote by  $\varphi_i(\xi)$  ( $i=1, 2, \dots, m$ ).

- (1.3) The holding cost function  $h(\eta)$  and the penalty cost function  $p(\eta)$  are twice differentiable, positive convex function for positive arguments.

- (1.4) There is credit function  $v(\eta)$  defined by

$$v(\eta) = \begin{cases} v\eta & \eta \geq 0 \\ 0 & \eta < 0 \end{cases}$$

The reduced penalty cost, the net penalty cost, is defined in the following way. If at the begining of some period the order of size  $z$  to be delivered at the end of the period, has been known and a demand  $D=\xi$  occurs, then the net penalty cost for this period is

$$p(\xi - y) - v[\min(z, \xi - y)],$$

where  $y$  is the starting stock level of that period.

- (1.5) There is a concave, twice differentiable salvage gain function  $w(\eta)$  that is increasing for  $\eta > 0$ , and is zero for  $\eta \leq 0$ .

- (1.6) The ordering cost function  $c_k(\eta_k)$  for regular orders to be delivered  $k$  period later is given by

$$c_k(\eta_k) = \begin{cases} c_k \eta_k & \eta_k \geq 0, \\ 0 & \eta_k < 0. \end{cases}$$

The ordering cost function  $c_j(\eta_j)$  for emergency orders to be delivered  $j$  period later is given by

$$c(\eta_j) = \begin{cases} c_j \eta_j & 0 < \eta_j \leq m_{uj}, \\ 0 & \eta_j \leq 0 \end{cases} ; \quad (j=0, 1, \dots, k-1),$$

with  $c_0 > c_1 > \dots > c_k > 0$ .

- (1.7) (a)  $\alpha^j \lim_{\eta \rightarrow \infty} w'(\eta) = \alpha^j \lim_{\eta \rightarrow \infty} \int_0^\eta w'(\eta - \xi) \varphi_i(\xi) d\xi < c_j < \alpha^{j-1} v$ ,  
 (b)  $w'(0) < v$ , (c)  $0 < \alpha \leq 1$ , ( $j=1, 2, \dots, k$ ;  $i=1, 2, \dots, m$ ),

- (1.8)  $L(\eta, \varphi_i) - v \int_0^\eta (\eta - \xi) \varphi_i(\xi) d\xi$  is convex,

where  $L(\eta, \varphi_i)$ , the expected one-period loss arising from penalty and holding costs, is given by

$$(1.9) \quad L(\eta, \varphi_i) = \begin{cases} \int_0^\eta h(\eta - \xi) \varphi_i(\xi) d\xi + \int_\eta^\infty p(\xi - \eta) \varphi_i(\xi) d\xi & \eta > 0, \\ \int_0^\infty p(\xi - \eta) \varphi_i(\xi) d\xi & \eta \leq 0. \end{cases}$$

We shall assume that all integrals occurring in this paper exist, and are finite, and that all integration and differentiation where needed can be interchange. This impose certain restrictions on the class of demands densities.

## §2. The Dynamic Model with a Linear Ordering Cost—general case.

Let  $f_n(x, x_1, \dots, x_{k-1}; \varphi_i)$  denote the total discounted expected loss for an  $n$ -period inventory model, where the demand density in first period is  $\varphi_i$ ,  $x$  is the inventory on hand after the receipt of prior orders at the beginning of the first period,  $x_i$  ( $i=1, 2, \dots, k-1$ ) is the amount of the stock on order that is to be received at the end of period  $i$ , and an optimal ordering policy is followed in period  $1, 2, \dots, n$ . We obtain from the principle of optimality

$$(2.1) \quad f_n(x, x_1, \dots, x_{k-1}; \varphi_i) = \min_{m_{u_0} \geq m_0 \geq 0, \dots, m_{u_{k-1}} \geq m_{k-1} \geq 0, m_k \geq 0} \left\{ \sum_{j=0}^k c_j m_j + L(x + m_0, \varphi_i) \right. \\ \left. - V(x + m_0, x_1 + m_1, \varphi_i) + \alpha \sum_{i=1}^m p_{ii1} \int_0^\infty f_{n-1}(x + x_1 + m_0 + m_1 - t, x_2 \right. \\ \left. + m_2, \dots, x_{k-1} + m_{k-1}, m_k; \varphi_i) \varphi_i(t) dt \right\} \quad n \geq k, k \geq 2.$$

$$(2.2) \quad f_n(x, x_1, \dots, x_n; \varphi_i) = \min_{m_{u_0} \geq m_0 \geq 0, \dots, m_{u_{n-1}} \geq m_{n-1} \geq 0, m_{u_n} \geq m_n \geq 0} \left\{ \sum_{j=0}^n c_j m_j + L(x + m_0, \varphi_i) \right. \\ \left. - V(x + m_0, x_1 + m_1, \varphi_i) + \alpha \sum_{i=1}^u p_{ii1} \int_0^\infty f_{n-1}(x + x_1 + m_0 + m_1 - t, x_2 + m_2, \dots, \right. \\ \left. x_n + m_n; \varphi_i) \varphi_i(t) dt \right\} \quad n < k, k \geq 2.$$

$$\int_0^\infty f_0(u - t, \varphi_i) \varphi_i(t) dt = - \int_0^u w(u - t) \varphi_i(t) dt,$$

where

$$(2.3) \quad V(x, z, \varphi_i) = \begin{cases} v \int_0^{x+z} (t-x) \varphi_i(t) dt + vz \int_{x+z}^\infty \varphi_i(t) dt & x+z > 0, x \leq 0, \\ v \int_x^{x+z} (t-x) \varphi_i(t) dt + vz \int_{x+z}^\infty \varphi_i(t) dt & x+z > 0, x > 0, \\ vx & x+z \leq 0. \end{cases}$$

From the above the relation it is evident that  $V(x, z, \varphi_i)$  represents the expected one-period credit cost when the starting stock level is given by  $x$  and the size of order to delivered at the end of the period by  $z$ . It will be convenient to introduce the function

$$(2.4) \quad V(u, \varphi_i) = \begin{cases} -vu + v \int_0^u (u-\xi) \varphi_i(\xi) d\xi & u \geq 0, \\ -vu & u < 0. \end{cases}$$

Then  $V(x, z, \varphi_i)$  can be written

$$(2.5) \quad V(x, z, \varphi_i) = -V(x+z, \varphi_i) + V(x, \varphi_i).$$

### §3. The Dynamic Model with a Linear Ordering Cost— $k$ -period case.

We assume throughout the rest of the paper that the interval in ordering of stock is  $k$ -period. Then we have for  $n \geq k$

$$\begin{aligned}
 (3.1) \quad f_n(x; \varphi_i) &= \min_{\substack{m_{u_0} \geq m_0 \geq 0, \dots, m_{u_{k-1}} \geq m_{k-1} \geq 0, m_k \geq 0}} \left\{ \sum_{j=0}^k c_j m_j + L(x + m_0, \varphi_i) \right. \\
 &\quad + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty L(x + m_0 + m_1 - t, \varphi_{i_1}) \varphi_i(t) dt + \dots + \alpha^{k-1} \sum_{i_1=1}^m \sum_{i_2=1}^m \dots \\
 &\quad \sum_{i_{k-1}=1}^m p_{ii_1} p_{i_1 i_2} \dots p_{i_{k-2} i_{k-1}} \int_0^\infty \dots \int_0^\infty L(x + m_0 + m_1 + \dots + m_{k-1} - t - t_1 \dots \\
 &\quad - t_{k-2}, \varphi_{i_{k-1}}) \varphi_i(t) \varphi_{i_1}(t_1) \dots \varphi_{i_{k-2}}(t_{k-2}) dt dt_1 \dots dt_{k-2} + [V(x + m_0 + m_1, \varphi_i) \\
 &\quad - V(x + m_0, \varphi_i)] + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty [V(x + m_0 + m_1 + m_2 - t, \varphi_{i_1}) - V(x + m_0 \\
 &\quad + m_1 - t, \varphi_{i_1})] \varphi_i(t) dt + \dots + \alpha^{k-1} \sum_{i_1=1}^m \sum_{i_2=1}^m \dots \sum_{i_{k-1}=1}^m p_{ii_1} p_{i_1 i_2} \dots p_{i_{k-2} i_{k-1}} \int_0^\infty \dots \\
 &\quad \int_0^\infty [V(x + m_0 + m_1 + \dots + m_k - t - t_1 \dots - t_{k-2}, \varphi_{i_{k-1}}) - V(x + m_0 + m_1 + \dots \\
 &\quad + m_{k-1} - t - t_1 \dots - t_{k-2}, \varphi_{i_{k-1}})] \varphi_i(t) \varphi_{i_1}(t_1) \dots \varphi_{i_{k-2}}(t_{k-2}) dt dt_1 \dots dt_{k-2} \\
 &\quad + \alpha^k \sum_{i_1=1}^m \sum_{i_2=1}^m \dots \sum_{i_k=1}^m p_{ii_1} p_{i_1 i_2} \dots p_{i_{k-1} i_k} \int_0^\infty \dots \int_0^\infty f_{n-k}(x + m_0 + m_1 + \dots + m_k - t \\
 &\quad - t_1 \dots - t_{k-1}, \varphi_{i_k}) \varphi_i(t) \varphi_{i_1}(t_1) \dots \varphi_{i_{k-1}}(t_{k-1}) dt dt_1 \dots dt_{k-1} \Big\} \\
 &= \min_{x \leq u_0 \leq x + m_{u_0}} \left\{ [(c_0 - c_1)u_0 + L(u_0, \varphi_i) - V(u_0, \varphi_i) + \min_{u_0 \leq u_1 \leq u_0 + m_{u_1}} \{ [(c_1 - c_2)u_1 \right. \\
 &\quad + L_1(u_1, \varphi_i) + V(u_1, \varphi_i) - V_1(u_1, \varphi_i)] + \min_{u_1 \leq u_2 \leq u_1 + m_{u_2}} \{ \dots + \min_{u_{k-2} \leq u_{k-1} \leq u_{k-2} + m_{u_{k-1}}} \\
 &\quad \{ [(c_{k-1} - c_k)u_{k-1} + L_{k-1}(u_{k-1}, \varphi_i) + V_{k-2}(u_{k-1}, \varphi_i) - V_{k-1}(u_{k-1}, \varphi_i)] \\
 &\quad + \min_{u_{k-1} \leq u_k} [c_k u_k + V_{k-1}(u_k, \varphi_i) + f_{n-k,k}(u_k, \varphi_i)] \} \dots] - c_0 x \\
 &= \min_{x \leq u_0 \leq x + m_{u_0}} G_0(u_0, \varphi_i) - c_0 x, \quad i = 1, 2, \dots, m,
 \end{aligned}$$

where

$$\begin{aligned}
 G_{kn}(u_k, \varphi_i) &= c_k u_k + V_{k-1}(u_k, \varphi_i) + f_{n-k,k}(u_k, \varphi_i), \quad n \geq k \\
 G_{jn}(u_j, \varphi_i) &= (c_j - c_{j+1})u_j + L_j(u_j, \varphi_i) + V_{j-1}(u_j, \varphi_i) - V_j(u_j, \varphi_i) + H_{jn}(u_j, \varphi_i), \\
 &\quad j = 0, 1, \dots, k-1; n \geq k, \\
 H_{k-1,n}(u_{k-1}, \varphi_i) &= \min_{u_{k-1} \leq u_k} G_{kn}(u_k, \varphi_i), \quad H_{jn}(u_j, \varphi_i) = \min_{u_j \leq u_{j+1} \leq u_j + m_{u_{j+1}}} G_{j+1,n}(u_{j+1}, \varphi_i),
 \end{aligned}$$

$$j=0, 1, \dots, k-2,$$

$$i_0=i, x+m_0=u_0, x+m_0+m_1=u_1, \dots, x+m_0+m_1+\dots+m_k=u_k,$$

$$(3.2) \quad L_l(x, \varphi_i) = \alpha \sum_{j=1}^m p_{ij} \int_0^\infty L_{l-1}(x-t, \varphi_j) \varphi_i(t) dt,$$

$$V_l(x, \varphi_i) = \alpha \sum_{j=1}^m p_{ij} \int_0^\infty V_{l-1}(x-t, \varphi_j) \varphi_i(t) dt,$$

$$W_l(x, \varphi_i) = \alpha \sum_{j=1}^m p_{ij} \int_0^x W_{l-1}(x-t, \varphi_j) \varphi_i(t) dt,$$

$$f_{n-k,l}(x, \varphi_i) = \alpha \sum_{j=1}^m p_{ij} \int_0^\infty f_{n-k,l-1}(x-t, \varphi_j) \varphi_i(t) dt,$$

$$f_{n-k,0}(x, \varphi_i) = f_{n-k}(x, \varphi_i),$$

$$L_{-1}(x, \varphi_i) = 0, V_0(x, \varphi_i) = V(x, \varphi_i), L_0(x, \varphi_i) = L(x, \varphi_i),$$

$$f_0(x, \varphi_i) = -W_0(x, \varphi_i) = -w(x).$$

It is noticed that  $f_{0k}(x, \varphi_i) = -W_k(x, \varphi_i)$  for  $x > 0$ . From the method as in the  $n \geq k$

$$(3.3) \quad f_n(x; \varphi_i) = \min_{x \leq u_0 \leq x+m_{u_0}} \left\{ [(c_0 - c_1)u_0 + L(u_0, \varphi_i) - V(u_0, \varphi_i)] \right. \\ + \min_{u_0 \leq u_1 \leq u_0+m_{u_1}} \left\{ [(c_1 - c_2)u_1 + L_1(u_1, \varphi_i) + V(u_1, \varphi_i) - V_1(u_1, \varphi_i)] \right. \\ + \min_{u_1 \leq u_2 \leq u_1+m_{u_2}} \left\{ \dots + \min_{u_{n-2} \leq u_{n-1} \leq u_{n-2}+m_{u_{n-1}}} \left\{ [(c_{n-1} - c_n)u_{n-1} \right. \right. \\ + L_{n-1}(u_{n-1}, \varphi_i) + V_{n-2}(u_{n-1}, \varphi_i) - V_{n-1}(u_{n-1}, \varphi_i)] \\ + \min_{u_{n-1} \leq u_n \leq u_{n-1}+m_{u_n}} \left\{ [c_n u_n + V_{n-1}(u_n, \varphi_i) - W_n(u_n, \varphi_i)] \dots \right\} - c_0 x \\ \left. \left. \left. \right\} \right\} \right\} - c_0 x, \quad n < k, k \geq 2,$$

where  $G_{jn}(u_j, \varphi_i)$  and  $H_{jn}(u_j, \varphi_i)$  in (3.2) are given by

$$(3.4) \quad G_{nn}(u_n, \varphi_i) = c_n u_n + V_{n-1}(u_n, \varphi_i) - W_n(u_n, \varphi_i) \\ G_{jn}(u_j, \varphi_i) = (c_j - c_{j+1})u_j + L_j(u_j, \varphi_i) + V_{j-1}(u_j, \varphi_i) - V_j(u_j, \varphi_i) \\ + H_{jn}(u_j, \varphi_i) \quad j=0, 1, \dots, n-1, n < k, \\ H_{jn}(u_j, \varphi_i) = \min_{u_j \leq u_{j+1} \leq u_j+m_{u_{j+1}}} G_{j+1,n}(u_{j+1}, \varphi_i) \quad j=0, 1, \dots, n-1; n < k.$$

We cite known results in [2] that will be needed in the analysis that follows.

LEMMA A. (i) If  $g(x)$  is convex function on the real line, then for  $0 \leq m < \infty$

$$h(x) = \min_{x \leq y \leq x+m} g(y)$$

is convex. If  $g(x)$  is bounded from below, so is  $h(x)$ .

(ii) (a) If  $g(x)$  is convex and bounded from below, then

$$k(x) = \inf_{x \leq y} g(y)$$

is convex and bounded from below. (b) If  $g(x)$  is convex, unbounded from below, but non-decreasing, then

$$k(x) = \inf_{x \leq y} g(y) = \min_{x \leq y} g(y) = g(x).$$

THEOREM 3.1. Under assumption (1.1)~(1.8), the optimal expected loss function  $f_n(x, \varphi_i)$  ( $i=1, 2, \dots, m$ ) is convex.

The proof is given by induction on the number  $n$ .

Case (a)  $n \leq k$ . Then from (3.4)

$$(3.5) \quad \begin{cases} G'_{11}(0, \varphi_i) = c_1 + V'(0, \varphi_i) - W'_1(0, \varphi_i) = c_1 - v < 0, \\ \lim_{u_1 \rightarrow \infty} G'_{11}(u_1, \varphi_i) = c_1 + \lim_{u_1 \rightarrow \infty} V'(u_1, \varphi_i) - \lim_{u_1 \rightarrow \infty} W'_1(u_1, \varphi_i) = c_1 - \alpha \lim_{u_1 \rightarrow \infty} w'(u_1) > 0 \end{cases}$$

$$(3.6) \quad G_{11}(u_1, \varphi_i) = \begin{cases} (c_1 - v)u_1 + v \int_0^{u_1} (u_1 - t) \varphi_i(t) dt - \alpha \int_0^{u_1} w(u_1 - t) \varphi_i(t) dt & u_1 > 0, \\ (c_1 - v)u_1 & u_1 < 0. \end{cases}$$

The function  $G_{11}(u_1, \varphi_i)$  is convex by (1.5) and (1.7b), attains a minimum at some positive  $\bar{x}_{11}(\varphi_i)$ , and increases for  $u_1$  large enough. Hence  $H_{01}(u_0, \varphi_i)$  is convex, bounded from below and increasing for  $u_0$  large enough by Lemma A. Condition (1.8) establishes convexity of  $L(u_0, \varphi_i) - V(u_0, \varphi_i)$ . Hence  $G_{01}(u_0, \varphi_i)$  is convex, since it is the sum of convex functions. By Lemma A  $f_1(x, \varphi_i)$  is convex. Although  $G_{01}(u_0, \varphi_i)$  is increasing, at least for  $u_0$  large enough,  $f_1(x, \varphi_i)$  can be convex decreasing, if

$$(3.7) \quad \lim_{u_0 \rightarrow \infty} G'_{01}(u_0, \varphi_i) - c_0 = \lim_{u_0 \rightarrow \infty} L'(u_0, \varphi_i) - \alpha \lim_{u_0 \rightarrow \infty} \int_0^{u_0} w'(u_0 - t) \varphi_i(t) dt \leq 0$$

Assume that the theorem is true for  $n-1$ , and that  $V_{n-2}(u, \varphi_i) - W_{n-1}(u, \varphi_i)$ ,  $L_{n-2}(u, \varphi_i) - V_{n-2}(u, \varphi_i)$  and  $V_{n-2}(u, \varphi_i)$  is convex. Then from (3.2) and (3.4)

$$(3.8) \quad \begin{cases} G'_{nn}(0, \varphi_i) = c_n + V'_{n-1}(0, \varphi_i) - W'_n(0, \varphi_i) = c_n - \alpha^{n-1}v < 0, \\ \lim_{u_n \rightarrow \infty} G'_{nn}(u_n, \varphi_i) = c_n + \lim_{u_n \rightarrow \infty} V'_{n-1}(u_n, \varphi_i) - \lim_{u_n \rightarrow \infty} W'_n(u_n, \varphi_i) \\ \hspace{15em} = c_n - \alpha^n \lim_{u_n \rightarrow \infty} w'(u_n) > 0, \\ W_j(0, \varphi_i) = W'_j(0, \varphi_i) = 0, \hspace{10em} j=1, 2, \dots, n. \end{cases}$$

$$(3.9) \quad \begin{aligned} G_{nn}(u_n, \varphi_i) &= c_n u_n + V_{n-1}(u_n, \varphi_i) - W_n(u_n, \varphi_i) \\ &= c_n u_n + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^{u_n} [V_{n-2}(u_n - t, \varphi_{i_1}) - W_{n-1}(u_n - t, \varphi_{i_1})] \varphi_i(t) dt. \end{aligned}$$

The function  $G_{nn}(u_n, \varphi_i)$  is convex, since it is the sum of convex function, attains a minimum at some positive  $\bar{x}_{nn}(\varphi_i)$ , and increase for  $u_n$  large enough.

By Lemma A,  $H_{n-1,n}(u_{n-1}, \varphi_i)$  is convex and increasing for  $u_{n-1}$  large enough. By (3.2), (3.4), and the induction hypothesis,  $L_{n-1}(u, \varphi_i) - V_{n-1}(u, \varphi_i)$  and  $V_{n-2}(u, \varphi_i)$  are convex. Hence  $G_{n-1,n}(u_{n-1}, \varphi_i)$  is convex and increasing for  $u_{n-1}$  large enough. By Lemma A, (3.4), it is proved inductively that  $H_{j,n}(u_j, \varphi_i)$  and  $G_{j,n}(u_j, \varphi_i)$  ( $j=0, 1, \dots, n-2, n \leq k, i=1, 2, \dots, m$ ) are convex, and increasing for  $u_j$  large enough. Hence  $f_n(x, \varphi_i)$  is convex by Lemma A. Although  $G_{0,n}(u_0, \varphi_i)$  is increasing, at least for  $u_0$  large enough,  $f_n(x, \varphi_i)$  can be convex decreasing, if

$$\begin{aligned}
 (3.10) \quad \lim_{u_0 \rightarrow \infty} G'_{0,n}(u_0, \varphi_i) - c_0 &= -c_1 + \lim_{u_0 \rightarrow \infty} L'(u_0, \varphi_i) + \lim_{u_0 \rightarrow \infty} G'_{1,n}(u_0, \varphi_i) \\
 &= -c_n + \lim_{u_0 \rightarrow \infty} L'(u_0, \varphi_i) + \lim_{u_0 \rightarrow \infty} L'_1(u_0, \varphi_i) + \dots \\
 &\quad + \lim_{u_0 \rightarrow \infty} L'_{n-1}(u_0, \varphi_i) + \lim_{u \rightarrow \infty} G'_{nn}(u_0, \varphi_i) \\
 &= \sum_{j=0}^{n-1} \lim_{u_0 \rightarrow \infty} L'_j(u, \varphi_i) - \alpha^n \lim_{u_0 \rightarrow \infty} w'(u_0) \leq 0.
 \end{aligned}$$

Case (b)  $n \geq k+1$ .  $n=k+1$ . Then from (3.2)

$$\begin{aligned}
 (3.11) \quad G_{k,k+1}(u_k, \varphi_i) &= c_k u_k + V_{k-1}(u_k, \varphi_i) + f_{1k}(u_k, \varphi_i) \\
 &= c_k u_k + V_{k-1}(u_k, \varphi_i) + \alpha^k \sum_{i_1=1}^m \dots \sum_{i_k=1}^m p_{ii_1} \dots p_{i_{k-1}i_k} \int_0^\infty \dots \\
 &\quad \int_0^\infty f_1(u_k - t \dots - t_{k-1}, \varphi_i) \varphi_i(t) \dots \varphi_{i_{k-1}}(t_{k-1}) dt \dots dt_{k-1}
 \end{aligned}$$

is convex, since it is the sum of convex function by case (a). By Lemma A and case (a),  $H_{j,k+1}(u_j, \varphi_i)$  and  $G_{j,k+1}(u_j, \varphi_i)$  ( $j=0, 1, \dots, k-1$ ) are convex. Hence  $f_{k+1}(x, \varphi_i)$  is convex.

Assume that the theorem is true for  $n-1$ . Then from (3.2)

$$\begin{aligned}
 (3.12) \quad G_{kn}(u_k, \varphi_i) &= c_k u_k + V_{k-1}(u_k, \varphi_i) + f_{n-k,k}(u_k, \varphi_i) \\
 &= c_k u_k + V_{k-1}(u_k, \varphi_i) + \alpha^k \sum_{i_1=1}^m \dots \sum_{i_k=1}^m p_{ii_1} \dots p_{i_{k-1}i_k} \int_0^\infty \dots \\
 &\quad \int_0^\infty f_{n-k}(u_k - t \dots - t_{k-1}, \varphi_i) \varphi_i(t) \dots \varphi_{i_{k-1}}(t_{k-1}) dt \dots dt_{k-1}
 \end{aligned}$$

is convex, since it is the sum of three convex functions, By Lemma A,  $H_{k-1,n}(u_{k-1}, \varphi_i)$  is convex and so is  $G_{k-1,n}(u_{k-1}, \varphi_i)$ . By Lemma A and case (a),  $H_{j,n}(u_j, \varphi_i)$  and  $G_{j,n}(u_j, \varphi_i)$  ( $j=0, 1, \dots, n-1$ ) are convex. Hence  $f_n(x, \varphi_i)$  is convex.

Remark. There exists the positive  $\bar{x}_{nn}(\varphi_i)$  determined by

$$(3.13) \quad G_{nn}(\bar{x}_{nn}(\varphi_i), \varphi_i) = 0 \quad n=1, 2, \dots, k; i=1, 2, \dots, m.$$

If  $\varphi_i(t) > 0$  for  $t > 0$  ( $i=1, 2, \dots, m$ ) then  $\bar{x}_{nn}(\varphi_i)$  is uniquely determined for  $i=1, 2, \dots, m$ . In this case  $G_{nn}(u_n, \varphi_i)$  is strictly convex for  $u_n > 0$ .

**THEOREM 3.2.** *Let assumption (1.1)~(1.8) hold. If*



$$\begin{aligned}
 (3.14) \quad & (a) \lim_{t \rightarrow -\infty} L'(t, \varphi_i) + c_0 - c_1 + v < 0, \quad (b) \lim_{t \rightarrow \infty} L'(t, \varphi_i) - \alpha \lim_{t \rightarrow \infty} w'(t) > 0, \\
 & i = 1, 2, \dots, m, \\
 & (c) v < p'(0), \quad p(0) = h(0), \quad (d) \varphi_i(t) > 0 \text{ for } t > 0, \\
 & i = 1, 2, \dots, m.
 \end{aligned}$$

are satisfied, then (i) there exists a unique positive  $\bar{x}_{jn}(\varphi_i)$  determined by

$$\begin{aligned}
 (3.15) \quad & G'_{jn}(\bar{x}_{jn}(\varphi_i), \varphi_i) = 0 \quad j = 1, 2, \dots, n; \quad n = 1, 2, \dots, k-1; \quad i = 1, 2, \dots, m, \\
 & G'_{jn}(\bar{x}_{jn}(\varphi_i), \varphi_i) = 0 \quad j = 1, 2, \dots, k; \quad n \geq k; \quad i = 1, 2, \dots, m.
 \end{aligned}$$

(ii) there exists a unique finite  $\bar{x}_{0n}(\varphi_i)$  determined by

$$(3.16) \quad G'_{0n}(\bar{x}_{0n}(\varphi_i), \varphi_i) = 0 \quad i = 1, 2, \dots, m.$$

(iii)  $f_n(x, \varphi_i)$  is convex, decreasing for  $x$  small enough, increasing for  $x$  large enough.

Proof (by induction).  $n \leq k$ . The existence of  $\bar{x}_{nn}(\varphi_i) > 0$  has been shown before under less stringent conditions.

Since

$$\begin{aligned}
 \lim_{u_0 \rightarrow \infty} G'_{01}(u_0, \varphi_i) &= (c_0 - c_1) + \lim_{u_0 \rightarrow \infty} L'(u_0, \varphi_i) + \lim_{u_0 \rightarrow \infty} H'_{01}(u_0, \varphi_i) \\
 &= (c_0 - c_1) + \lim_{u_0 \rightarrow \infty} L'(u_0, \varphi_i) + \lim_{u_0 \rightarrow \infty} G'_{11}(u_0, \varphi_i) > 0
 \end{aligned}$$

and

$$\lim_{u_0 \rightarrow -\infty} G'_{01}(u_0, \varphi_i) = (c_0 - c_1) + v + \lim_{u_0 \rightarrow -\infty} L'(u_0, \varphi_i) < 0$$

it follows that  $\bar{x}_{01}(\varphi_i)$  is finite. The uniqueness assertions are consequences of (3.14c, d).

(it is noticed that  $G'_{01}(u, \varphi_i) < 0$  is not assured). If  $p(\eta)$  is linear,  $\bar{x}_{01}(\varphi_i)$  is even positive. Thus we have

$$(3.17) \quad f'_1(x, \varphi_i) = \begin{cases} G'_{01}(x + m_{u_0}, \varphi_i) - c_0 & x + m_{u_0} < \bar{x}_{01}(\varphi_i), \\ -c_0 & x < \bar{x}_{01}(\varphi_i) < x + m_{u_0}, \\ G'_{01}(x, \varphi_i) - c_0 & \bar{x}_{01}(\varphi_i) < x, \end{cases}$$

hence,  $\lim_{x \rightarrow -\infty} f'_1(x, \varphi_i) < 0$ , moreover by (3.14 b),  $\lim_{x \rightarrow \infty} f'_1(x, \varphi_i) > 0$ , and also by Theorem 3.1  $f_1(x, \varphi_i)$  is convex.

That  $\bar{x}_{11}(\varphi_i) > 0$  implies

$$\begin{aligned}
 (3.13) \quad & G'_{01}(0, \varphi_i) - c_0 = -c_1 + L'(0, \varphi_i) - V'(0, \varphi_i) + H'_{01}(0, \varphi_i) \\
 & = v - c_1 - \int_0^\infty p'(t) \varphi_i(t) dt < v - c_1 - p'(0) < 0.
 \end{aligned}$$

Hence  $f'_1(0, \varphi_i) < 0$ .

We assume that the theorem is true for  $n-1$  and that is true for  $n$  and  $j = n, n-1, \dots, n-r$  ( $r \leq n-2$ ). That  $\bar{x}_{n-r,n}(\varphi_i) > 0$  implies

$$(3.19) \quad H'_{n-(r+1),n}(0, \varphi_i) = \begin{cases} G'_{n-r,n}(m_{u_{n-r}}, \varphi_i) < 0 & 0 < \bar{x}_{n-r,n}(\varphi_i) - m_{u_{n-r}} \\ 0 & \bar{x}_{n-r,n}(\varphi_i) - m_{u_{n-r}} < 0 < \bar{x}_{n-r,n}(\varphi_i) \end{cases}$$

hence

$$(3.20) \quad \begin{aligned} G'_{n-(r+1),n}(0, \varphi_i) &= (c_{n-(r+1)} - c_{n-r}) + L'_{n-(r+1)}(0, \varphi_i) + V'_{n-(r+2)}(0, \varphi_i) \\ &\quad - V'_{n-(r+1)}(0, \varphi_i) + H'_{n-(r+1),n}(0, \varphi_i) \\ &\leq (c_{n-(r+1)} - \alpha^{n-(r+2)}v) - c_{n-r} - \alpha^{n-(r+1)}(p'(0) - v) < 0. \end{aligned}$$

On the other side

$$(3.21) \quad \begin{aligned} \lim_{u_{n-(r+1)} \rightarrow \infty} G'_{n-(r+1),n}(u_{n-(r+1)}, \varphi_i) &= (c_{n-(r+1)} - c_{n-r}) \\ &\quad + \lim_{u_{n-(r+1)} \rightarrow \infty} L'_{n-(r+1)}(u_{n-(r+1)}, \varphi_i) + \lim_{u_{n-(r+1)} \rightarrow \infty} G'_{n-r,n}(u_{n-(r+1)}, \varphi_i) > 0. \end{aligned}$$

There exists  $\bar{x}_{n-(r+1),n}(\varphi_i) > 0$  such that  $G'_{n-(r+1),n}(\bar{x}_{n-(r+1),n}, \varphi_i) = 0$ . Uniqueness follows from (3.14 c, d).

Since

$$\begin{aligned} \lim_{u_0 \rightarrow -\infty} G'_0(u_0, \varphi_i) &= (c_0 - c_1) + \lim_{u_0 \rightarrow -\infty} L'(u_0, \varphi_i) - \lim_{u_0 \rightarrow -\infty} V'(u_0, \varphi_i) + \lim_{u_0 \rightarrow -\infty} H'_{0n}(u_0, \varphi_i) \\ &< (c_0 - c_1) + \lim_{u_0 \rightarrow -\infty} L'(u_0, \varphi_i) + v < 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{u_0 \rightarrow \infty} G'_0(u_0, \varphi_i) &= (c_0 - c_1) + \lim_{u_0 \rightarrow \infty} L'(u_0, \varphi_i) - \lim_{u_0 \rightarrow \infty} V'(u_0, \varphi_i) + \lim_{u_0 \rightarrow \infty} H'_{0n}(u_0, \varphi_i) \\ &= (c_0 - c_1) + \lim_{u_0 \rightarrow \infty} L'(u_0, \varphi_i) + \lim_{u_0 \rightarrow \infty} G'_{1n}(u_0, \varphi_i) > 0. \end{aligned}$$

it follows that there is finite  $\bar{x}_{0n}(\varphi_i)$  such that  $G'_{0n}(\bar{x}, \varphi_i) = 0$ . Uniqueness is shown as before. Since

$$(3.22) \quad f'_n(x, \varphi_i) = \begin{cases} G'_{0n}(x + m_{u_0}, \varphi_i) - c_0 & x + m_{u_0} < \bar{x}_{0n}(\varphi_i), \\ -c_0 & x < \bar{x}_{0n}(\varphi_i) < x + m_{u_0}, \\ G'_{0n}(x, \varphi_i) - c_0 & \bar{x}_{0n}(\varphi_i) < x. \end{cases}$$

it follows that

$$\begin{aligned} (3.23) \quad \lim_{x \rightarrow \infty} f'_n(x, \varphi_i) &= \lim_{x \rightarrow \infty} G'_{0n}(x, \varphi_i) - c_0 = -c_1 + \lim_{x \rightarrow \infty} L'(x, \varphi_i) + \lim_{x \rightarrow \infty} G'_{1n}(x, \varphi_i) \\ &= \sum_{j=0}^{n-1} \lim_{x \rightarrow \infty} L'_j(x, \varphi_i) - \alpha^n \lim_{x \rightarrow \infty} w'(x) \\ &> \lim_{x \rightarrow \infty} L'(x, \varphi_i) - \alpha \lim_{x \rightarrow \infty} w'(x) > 0. \end{aligned}$$

by (3.14 b), and  $\lim_{x \rightarrow -\infty} f'_n(x, \varphi_i) < -c_0 < 0$ .

By Theorem 3.1  $f_n(x, \varphi_i)$  is convex. Hence the last assertion is true. That  $\bar{x}_{1n}(\varphi_i) > 0$  implies

$$\begin{aligned} (3.24) \quad G'_{0n}(0, \varphi_i) - c_0 &= -c_1 + L'(0, \varphi_i) - V'(0, \varphi_i) + H'_{0n}(0, \varphi_i) \\ &= v - c_1 - \int_0^\infty p'(t) \varphi_i(t) dt < v - c_1 - p'(0) < 0. \end{aligned}$$

Hence  $f'_n(0, \varphi_i) < 0$ .

Case (b).  $n \geq k+1$ .

Since

$$\begin{aligned}
 (3.25) \quad \lim_{u_k \rightarrow \infty} G'_{k,k+1}(u_k, \varphi_i) &= c_k + \lim_{u_k \rightarrow \infty} V'_{k-1}(u_k, \varphi_i) + \lim_{u_k \rightarrow \infty} f'_{1k}(u_k, \varphi_i) \\
 &= c_k + \alpha^k \lim_{u_k \rightarrow \infty} \sum_{i_1=1}^m \cdots \sum_{i_k=1}^m p_{i_1 i_1} \cdots p_{i_{k-1} i_k} \int_0^\infty \cdots \\
 &\quad \int_0^\infty f'_1(u_k - t - t_1 - \cdots - t_{k-1}, \varphi_{i_k}) \varphi_i(t) \cdots \varphi_{i_{k-1}}(t_{k-1}) dt \cdots dt_{k-1} > 0,
 \end{aligned}$$

and

$$\begin{aligned}
 (3.26) \quad G'_{k,k+1}(0, \varphi_i) &= c_k + V'_{k-1}(0, \varphi_i) + f'_{1k}(0, \varphi_i) \\
 &= c_k - \alpha^{k-1}v + \alpha^k \sum_{i_1=1}^m \cdots \sum_{i_k=1}^m p_{i_1 i_1} \cdots p_{i_{k-1} i_k} \int_0^\infty \cdots \\
 &\quad \int_0^\infty f'_1(-t - t_1 - \cdots - t_{k-1}, \varphi_{i_k}) \varphi_i(t) \cdots \varphi_{i_{k-1}}(t_{k-1}) dt \cdots dt_{k-1} \\
 &< c_k - \alpha^{k-1}v + \alpha^k f'_1(0, \varphi_{i_k}) < 0.
 \end{aligned}$$

By (3.14 c, d),  $G_{k,k+1}(u_k, \varphi_i)$  is strictly convex for  $u_k > 0$ . By lemma A, case (a), it is proved inductively that  $\lim_{u_j \rightarrow \infty} G'_{j,k+1}(u_j, \varphi_i) > 0$  for  $j=0, 1, \dots, k-1$ ,  $G'_{j,k+1}(0, \varphi_i) < 0$  for  $j=1, 2, \dots, k-1$  and  $\lim_{u_0 \rightarrow -\infty} G'_{0,k+1}(u_0, \varphi_i) < 0$ . There exists  $\bar{x}_{j,k+1}(\varphi_i) > 0$  such that  $G'_{j,k+1}(\bar{x}_{j,k+1}(\varphi_i), \varphi_i) = 0$ . Uniqueness follows from (3.14 c, d) (iii) is shown as in the case (a). Hence the theorem holds for  $k+1$ . It is easily seen, as in the case (a), that  $f'_{k+1}(0, \varphi_i) < 0$ . Assume that the theorem is true for  $n-1$ . Then

$$(3.27) \quad f'_{n-k}(x, \varphi_i) = \begin{cases} G'_{0,n-k}(x + m_{u_0}, \varphi_i) - c_0 & x + m_{u_0} < \bar{x}_{0,n-k}(\varphi_i), \\ -c_0 & x < \bar{x}_{0,n-k}(\varphi_i) < x + m_{u_0}, \\ G'_{0,n-k}(x, \varphi_i) - c_0 & \bar{x}_{0,n-k}(\varphi_i) < x \end{cases}$$

implies  $f'_{n-k}(x, \varphi_i) < 0$  for  $x \leq 0$ , hence

$$\begin{aligned}
 G'_{kn}(0, \varphi_i) &= c_k + V'_{k-1}(0, \varphi_i) + f'_{n-k,k}(0, \varphi_i) < c_k - \alpha^{k-1}v < 0 \\
 (3.28) \quad \lim_{u_k \rightarrow \infty} G'_{kn}(u_k, \varphi_i) &= c_k + \lim_{u_k \rightarrow \infty} V'_{k-1}(u_k, \varphi_i) + \lim_{u_k \rightarrow \infty} f'_{n-k,k}(u_k, \varphi_i) > 0.
 \end{aligned}$$

There exists  $\bar{x}_{kn}(\varphi_i) > 0$  such that  $G'_{kn}(u_k, \varphi_i) = 0$ . Uniqueness follows from (3.14 c, d). The assertion of (i) and (ii) for  $j=k-1, k-2, \dots, 0$  is shown as before. The last assertion is shown as in the case (a). Moreover we obtain from the discussion of Theorem 3.2 the following theorem.

**THEOREM 3.3.** *If conditions of Theorem 3.2. are satisfied, then the optimal ordering policy is of the following form.*

(i)  $n < k$

$$m_{0n}^i(x) = \begin{cases} \bar{x}_{0n}(\varphi_i) - x & \bar{x}_{0n}(\varphi_i) - m_{u_0} < x < \bar{x}_{0n}(\varphi_i), \\ 0 & \bar{x}_{0n}(\varphi_i) < x, \\ m_{u_0} & \bar{x}_{0n}(\varphi_i) - m_{u_0} > x. \end{cases} \quad \begin{matrix} n=1, 2, \dots, k-1; i=1, \\ 2, \dots, m, \end{matrix}$$

(3.29)

$$m_{jn}^i(u_{j-1}) = \begin{cases} x_{jn}(\varphi_i) - u_{j-1} \bar{x}_{jn}(\varphi_i) - m_{u_j} < u_{j-1} < x_{jn}(\varphi_i), \\ 0 & \bar{x}_{jn}(\varphi_i) < u_{j-1}, \\ m_{u_j} & x_{jn}(\varphi_i) - m_{u_j} > u_{j-1}. \end{cases} \quad \begin{matrix} n=1, 2, \dots, k-1; i=1, \\ 2, \dots, m, j=1, \dots, n, \end{matrix}$$

(ii)  $n \geq k$ 

$$m_{0n}^i(x) = \begin{cases} x_{0n}(\varphi_i) - x & x_{0n}(\varphi_i) - m_{u_0} < x < x_{0n}(\varphi_i), \\ 0 & \bar{x}_{0n}(\varphi_i) < x, \\ m_{u_0} & x_{0n}(\varphi_i) - m_{u_0} > x. \end{cases} \quad i=1, 2, \dots, m,$$

$$(3.30) \quad m_{jn}^i(u_{j-1}) = \begin{cases} x_{jn}(\varphi_i) - u_{j-1} \bar{x}_{jn}(\varphi_i) - m_{u_j} < u_{j-1} < x_{jn}(\varphi_i), \\ 0 & \bar{x}_{jn}(\varphi_i) < u_{j-1}, \\ m_{u_j} & x_{jn}(\varphi_i) - m_{u_j} > u_{j-1}. \end{cases} \quad \begin{matrix} i=1, 2, \dots, m; j=1, \\ 2, \dots, k-1, \end{matrix}$$

$$m_{kn}^i(u_{k-1}) = \begin{cases} \bar{x}_{kn}(\varphi_i) - u_{k-1} & u_{k-1} < \bar{x}_{kn}(\varphi_i), \\ 0 & u_{k-1} > \bar{x}_{kn}(\varphi_i). \end{cases} \quad i=1, 2, \dots, m,$$

where  $\bar{x}_{jn}(\varphi_i)$ ,  $\bar{x}_{0n}(\varphi_i)$ , are determined satisfying (3.15) and (3.16) respectively, and  $u_j$  is given by (3.2).

We next show that the analogue of a theorem, which is given in [3], holds true for comparing the critical numbers associated with two sequences of demand densities under our model. We say that  $\varphi$  is stochastically smaller than  $\psi$  and we write  $\varphi \subset \psi$  if  $\int_0^x \varphi(t) dt \geq \int_0^x \psi(t) dt$  for all  $x$ .

**THEOREM 3.4.** *Let conditions of Theorem 3.2 hold. If two sequences of demand densities  $\varphi_1, \varphi_2, \dots, \varphi_m$ , and  $\psi_1, \psi_2, \dots, \psi_m$ , are such that  $\varphi_i \subset \psi_i$  for  $i=1, 2, \dots, m$ , then*

$$(3.31) \quad \begin{aligned} (a) \quad & \bar{x}_{jn}(\varphi_i) \leq \bar{x}_{jn}(\psi_i) \quad j=0, 1, \dots, n; n=1, 2, \dots, k-1; i=1, 2, \dots, m, \\ & \bar{x}_{jn}(\varphi_i) \leq \bar{x}_{jn}(\psi_i) \quad j=0, 1, \dots, k; n \geq k; i=1, 2, \dots, m, \end{aligned}$$

and

$$(b) \quad f'_n(x, \varphi_i) \geq f'_n(x, \psi_i) \text{ for all } x, \quad i=1, 2, \dots, m.$$

**Proof.** We establish the proof by induction on  $n$ . There are two possibilities requiring separate treatment,  $n \leq k$  and  $n \geq k+1$ .

Case (a)  $n \leq k$ . When  $n=1$ , we consider demand densities  $\varphi_i$  and  $\psi_i$ , where  $\varphi_i \subset \psi_i$ . Integrating by parts, we obtain from (3.4)

$$G'_{11}(u, \varphi_i) = \begin{cases} (c_1 - v) + (v - \alpha w'(0)) \Phi_i(u) - \alpha \int_0^u w''(u-t) \Phi_i(t) dt & u < 0, \\ (c_1 - v) & u < 0, \end{cases}$$

$$(3.32) \quad \begin{cases} (c_0 - c_1 + v) + (h'(0) + p'(0) - v)\Phi_i(u) \\ \quad - p'(0) + \int_0^u h''(u-t)\Phi_i(t)dt \\ G'_{01}(u, \varphi_i) - H'_{01}(u, \varphi_i) = \quad - \int_u^\infty p''(t-u)(1-\Phi(t_i))dt & u > 0 \\ (c_0 - c_1 + v) - p'(u) - \int_0^\infty p''(t-u)(1-\Phi_i(t))dt & u < 0, \end{cases}$$

where

$$\Phi_i(t) = \int_0^t \varphi_i(\eta) d\eta \quad i=1, 2, \dots, m.$$

A similar expression is obtain for  $G'_{11}(u, \psi_i)$  and  $G'_{01}(u, \psi_i)$ , we simply replace  $\Phi_i(t)$  in (3.32) by  $\Psi_i(t)$ , where

$$\Psi_i(t) = \int_0^t \psi_i(\eta) d\eta \quad i=1, 2, \dots, m.$$

Since  $\varphi_i \subset \psi_i$ , it follows by definition that  $\Phi_i(t) \geq \Psi_i(t)$  for all  $t \geq 0$ ; hence, we get from (1.7 b), (1.3), (1.5), and (3.14)

$$(3.33) \quad G'_{11}(u_1, \varphi_i) \geq G'_{11}(u_1, \psi_i) \text{ for all } u_1 \quad i=1, 2, \dots, m,$$

and

$$G'_{01}(u_0, \varphi_i) - H'_{01}(u_0, \varphi_i) \geq G'_{01}(u_0, \psi_i) - H'_{01}(u_0, \psi_i) \text{ for all } u_0 \quad i=1, 2, \dots, m.$$

By (3.33) and (3.4),  $H_{01}(u_0, \varphi_i) \geq H_{01}(u_0, \psi_i)$  for all  $u_0$ . Hence

$$(3.34) \quad G'_{01}(u_0, \varphi_i) \geq G'_{01}(u_0, \psi_i) \text{ for all } u_0 \quad i=1, 2, \dots, m.$$

Hence, it follows immediately from (3.33), (3.34) and Theorem 3.2, that

$$(3.35) \quad \begin{aligned} \bar{x}_{11}(\varphi_i) &\leq \bar{x}_{11}(\psi_i) \\ \bar{x}_{01}(\varphi_i) &\leq \bar{x}_{01}(\psi_i) \end{aligned} \quad i=1, 2, \dots, m.$$

If we compare (3.33), (3.34), and (3.35) with (3.17) and the corresponding equation for  $\psi_i$

$$(3.36) \quad f'_1(x, \psi_i) = \begin{cases} G'_{01}(x + m_{u_0}, \psi_i) - c_0 & x + m_{u_0} < \bar{x}_{01}(\psi_i), \\ -c_0 & x < \bar{x}_{01}(\psi_i) < x + m_{u_0}, \\ G'_{01}(x, \psi_i) - c_0 & \bar{x}_{01}(\psi_i) < x, \end{cases}$$

it is clear that

$$(3.37) \quad f'_1(x, \varphi_i) \geq f'_1(x, \psi_i) \text{ for all } x.$$

Thus we have proved the theorem for the one-period model.

Assume that we proved

$$\begin{aligned} \bar{x}_{jn}(\varphi_i) &\leq \bar{x}_{jn}(\psi_i) & j=0, 1, \dots, n-1; i=1, 2, \dots, m, \\ f'_{n-1}(x, \varphi_i) &\geq f'_{n-1}(x, \psi_i) \text{ for all } x & i=1, 2, \dots, m, \end{aligned}$$

$$G'_{j,n-1}(u_j, \varphi_i) - H'_{j,n-1}(u_j, \varphi_i) \geq G'_{j,n-1}(u_j, \psi_i) - H'_{j,n-1}(u_j, \psi_i) \text{ for all } u_j, \\ j=0, 1, \dots, n-2; i=1, 2, \dots, m,$$

$$G'_{j,n-1}(u_j, \varphi_i) \geq G'_{j,n-1}(u_j, \psi_i) \text{ for all } u_j, j=0, 1, \dots, n-1; i=1, 2, \dots, m,$$

for any  $m$  pair of demand densities  $\varphi_i$  and  $\psi_i$  satisfying  $\varphi_i \subset \psi_i$ . Recalling (3.9) and differentiating (3.4) with respect to  $u_j$  yields

$$(3.38) \quad \begin{aligned} G'_{nn}(u_n, \varphi_i) &= c_n + V'_{n-1}(u_n, \varphi_i) - W'_n(u_n, \varphi_i) \\ &= c_n - \alpha c_{n-1} + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty [c_{n-1} + V'_{n-2}(u_n - t, \varphi_{i_1}) \\ &\quad - W'_{n-1}(u_n - t, \varphi_{i_1})] \varphi_i(t) dt \\ &= c_n - \alpha c_{n-1} + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty G'_{n-1,n-1}(u_n - t, \varphi_{i_1}) \varphi_i(t) dt. \end{aligned}$$

Using our inductive assumption, integrating by parts, we get

$$(3.39) \quad \begin{aligned} G'_{nn}(u_n, \varphi_i) &\geq c_n - \alpha c_{n-1} + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty G'_{n-1,n-1}(u_n - t, \psi_{i_1}) \varphi_i(t) dt \\ &= c_n - \alpha c_{n-1} + \alpha \sum_{i_1=1}^m p_{ii_1} G'_{n-1,n-1}(u_n, \psi_{i_1}) \\ &\quad - \alpha \sum_{i_1=1}^m \int_0^\infty G''_{n-1,n-1}(u_n - t, \psi_{i_1}) (1 - \Phi_i(t)) dt \\ &\geq c_n - \alpha c_{n-1} + \alpha \sum_{i_1=1}^m p_{ii_1} G'_{n-1,n-1}(u_n, \psi_{i_1}) \\ &\quad - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty G''_{n-1,n-1}(u_n - t, \psi_{i_1}) (1 - \psi_i(t)) dt \\ &= G'_{nn}(u_n, \psi_i) \text{ for all } u_n \quad i=1, 2, \dots, m. \end{aligned}$$

Hence

$$\bar{x}_{nn}(\varphi_i) \leq \bar{x}_{nn}(\psi_i) \quad i=1, 2, \dots, m.$$

Assuming that  $G'_{jn}(u_j, \varphi_i) \geq G'_{jn}(u_j, \psi_i)$  for all  $u_j$  and  $j=n, n-1, \dots, n-r+1$  ( $1 \leq r \leq n$ ). Using the inductive assumption and (3.4), we obtain from the simple calculation

$$(3.40) \quad H'_{jn}(u_j, \varphi_i) \geq H'_{jn}(u_j, \psi_i) \text{ for all } u_j \quad j=n-1, n-2, \dots, n-r \quad (1 \leq r \leq n).$$

Hence from inductive assumption and (3.4), we have

$$\begin{aligned} G'_{n-r,n}(u_{n-r}, \varphi_i) - H_{n-r,n}(u_{n-r}, \varphi_i) &= (c_{n-r} - c_{n-r+1}) + L'_{n-r}(u_{n-r}, \varphi_i) + V'_{n-r-1}(u_{n-r}, \varphi_i) - V'_{n-r}(u_{n-r}, \varphi_i) \\ &= (c_{n-r} - c_{n-r+1}) - \alpha(c_{n-r-1} - c_{n-r}) \\ &\quad + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty [G'_{n-r-1,n-1}(u_{n-r} - t, \varphi_{i_1}) - H'_{n-r,n-1}(u_{n-r} - t, \varphi_{i_1})] \varphi_i(t) dt \\ &\geq (c_{n-r} - c_{n-r+1}) - \alpha(c_{n-r-1} - c_{n-r}) \\ &\quad + \alpha \sum_{i_1=1}^m p_{ii_1} [G'_{n-r-1,n-1}(u_{n-r}, \psi_{i_1}) - H'_{n-r-1,n-1}(u_{n-r}, \psi_{i_1})] \end{aligned}$$

$$\begin{aligned}
& -\alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty [G''_{n-r-1, n-1}(u_{n-r}-t, \psi_{i_1}) \\
& - H''_{n-r-1, n-1}(u_{n-r}-t, \psi_{i_1})](1-\Phi_i(t))dt \\
& \geq (c_{n-r}-c_{n-r+1}) - \alpha(c_{n-r-1}-c_{n-r}) \\
& + \alpha \sum_{i_1=1}^m p_{ii_1} [G'_{n-r-1, n-1}(u_{n-r}, \psi_{i_1}) - H'_{n-r-1, n-1}(u_{n-r}, \psi_{i_1})] \\
& - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty [G''_{n-r-1, n-1}(u_{n-r}-t, \psi_{i_1}) \\
& - H''_{n-r-1, n-1}(u_{n-r}-t, \psi_{i_1})](1-\Psi_i(t))dt \\
& = G'_{n-r, n}(u_{n-r}, \psi_i) - H'_{n-r, n}(u_{n-r}, \psi_i) \text{ for all } u_{n-r} \quad i=1, 2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
& G'_{1n}(u_1, \varphi_i) - H'_{1n}(u_1, \varphi_i) \\
& = (c_1 - c_2) - \alpha(c_0 - c_1) \\
& + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty [G'_{0, n-1}(u_1-t, \varphi_{i_1}) - H'_{0, n-1}(u_1-t, \varphi_{i_1})]\varphi_i(t)dt + V'(u_1, \varphi_i) \\
& \geq G'_{1n}(u_1, \psi_i) - H'_{1n}(u_1, \psi_i) \text{ for all } u_1 \quad i=1, 2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
& G'_{0n}(u_0, \varphi_i) = (c_0 - c_1) + L'(u_0, \varphi_i) - V'(u_0, \varphi_i) + H'_{0n}(u_0, \varphi_i) \\
& \geq (c_0 - c_1) + L'(u_0, \psi_i) - V'(u_0, \psi_i) + H'_{0n}(u_0, \psi_i) \\
& = G'_{0n}(u_0, \psi_i) \text{ for all } u_0. \quad i=1, 2, \dots, m.
\end{aligned}$$

From (3.40) and (3.41), we get

$$G'_{n-r, n}(u_{n-r}, \varphi_i) \geq G'_{n-r, n}(u_{n-r}, \psi_i) \text{ for all } u_{n-r} \quad i=1, 2, \dots, m.$$

The same argument used for the case  $n=1$ , now readily yields

$$\bar{x}_{jn}(\varphi_i) \leq \bar{x}_{jn}(\psi_i) \quad j=0, 1, \dots, n; \quad i=1, 2, \dots, m,$$

and

$$f'_n(x, \varphi_i) \geq f'_n(x, \psi_i) \text{ for all } x \quad i=1, 2, \dots, m.$$

Case (b)  $n \geq k+1$ . We begin the proof by showing that

$$(3.42) \quad V'_{l-1}(u, \varphi_i) \geq V'_{l-1}(u, \psi_i) \text{ for all } u \quad l=1, 2, \dots, k; \quad i=1, 2, \dots, m.$$

$$(3.43) \quad f'_{jl}(u, \varphi_i) \geq f'_{jl}(u, \psi_i) \text{ for all } u \quad l=1, 2, \dots, k; \quad j=1, 2, \dots, n-k; \quad i=1, 2, \dots, m.$$

We first establish (3.42) for the case  $l=1$ . From (2.4), we have

$$\begin{aligned}
& V'(u, \varphi_i) = v(\Phi_i(u) - 1) \\
& \geq v(\Psi_i(u) - 1) \\
& = V'(u, \psi_i) \text{ for all } u \geq 0, \quad i=1, 2, \dots, m,
\end{aligned}$$

and

$$V'(u, \psi_i) = -v = V'(u, \varphi_i) \text{ for all } u < 0 \quad i=1, 2, \dots, m.$$

Assuming that (3.42) holds for  $l-1$ , ( $1 \leq l-1 < k$ ), we advance the induction to  $l$ . Invoking the inductive assumption and (3.2), integrating by parts, we have

$$\begin{aligned}
(3.44) \quad V'_{k-1}(u, \varphi_i) &= \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V'_{k-2}(u-t, \varphi_{i_1}) \varphi_i(t) dt \\
&\geq \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V'_{k-2}(u-t, \psi_{i_1}) \varphi_i(t) dt \\
&= \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V'_{k-2}(u, \psi_{i_1}) - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V''_{k-2}(u-t, \psi_{i_1}) (1-\Phi_i(t)) dt \\
&\geq \alpha \sum_{i_1=1}^m p_{ii_1} V'_{k-2}(u, \psi_{i_1}) - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V'_{k-2}(u-t, \psi_{i_1}) (1-\Psi_i(t)) dt \\
&= V'_{k-1}(u, \psi_i) \text{ for all } u. \quad i=1, 2, \dots, m,
\end{aligned}$$

which proves (3.42).

We establish (3.43) by induction on  $l$  and  $j$ . Suppose first  $l=1, j=1$ , then from (3.2) and case (a),

$$\begin{aligned}
f'_{11}(u, \varphi_i) &= \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty f'_1(u-t, \varphi_{i_1}) \varphi_i(t) dt \\
&\geq \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty f'_1(u, \psi_{i_1}) - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty f''_1(u-t, \psi_{i_1}) (1-\Psi_i(t)) dt \\
&= f'_{11}(u, \psi_i) \text{ for all } u \quad i=1, 2, \dots, m.
\end{aligned}$$

We verify (3.43) for the integer  $l=1$  and  $j-1 (1 \leq j-1 < n-k)$  in a manner analogous to that for  $l=1$  and  $j=1$ . Now supposing that (3.43) is true for the integer  $l-1 (1 \leq l-1 < k)$  and  $j=1, 2, \dots, n-k$ , we extend the induction to integer  $l$ . It follows from the inductive assumption, (3.2) and case (a) that for all  $u$ , we have

$$\begin{aligned}
(3.45) \quad f'_{jl}(u, \varphi_i) &= \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty f'_{j,l-1}(u-t, \varphi_{i_1}) \varphi_i(t) dt \\
&\geq \alpha \sum_{i_1=1}^m p_{ii_1} f'_{j,l-1}(u, \psi_{i_1}) - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty f''_{j,l-1}(u-t, \psi_{i_1}) (1-\Psi_i(t)) dt \\
&= f'_{jl}(u, \psi_i) \text{ for all } u \quad \begin{array}{l} l=1, 2, \dots, k; j=1, 2, \dots, \\ n-k; i=1, 2, \dots, m, \end{array}
\end{aligned}$$

which proves (3.43).

Since

$$\begin{aligned}
G'_{jn}(u_j, \varphi_i) - H'_{jn}(u_j, \varphi_i) &= G'_{j,n-1}(u_j, \varphi_i) - H'_{j,n-1}(u_j, \varphi_i) \text{ for all } u_j \\
&\quad j=0, 1, \dots, n-1; n \geq k+1; i=1, 2, \dots, m.
\end{aligned}$$

Using relations (3.42) and (3.43), we can prove the theorem for the  $n \geq k+1$  in the way as in the case (a).

**COROLLARY.** In Theorem 3.4, if  $\psi_i(x) = \varphi_i(x - a_i)$  for positive  $a_i > 0$ , then

$$\left( \bar{x}_{nn}(\varphi_i) + \sum_{l=0}^{n-1} a_l = \bar{x}_{nn}(\psi_i) \right) \quad \begin{array}{l} i=1, 2, \dots, m; l=0, 1, \dots, \\ n-1; n \leq k, \end{array}$$



$$\begin{aligned}
& \bar{x}_{jn}(\varphi_i) + \sum_{l=0}^{j-1} a_{il} < \bar{x}_{jn}(\psi_i) < \bar{x}_{jn}(\varphi_i) + \sum_{l=0}^{n-1} a_{il} & j=1, 2, \dots, n-1; i_l=1, 2, \dots, \\
& m; l=0, 1, \dots, n-1; & \\
& k \geq n \geq 2, & \\
& \bar{x}_{0n}(\varphi_i) + a_i \leq \bar{x}_{0n}(\psi_i) < \bar{x}_{0n}(\varphi_i) + \sum_{l=0}^{n-1} a_{il} & i_l=1, 2, \dots, m; l=0, 1, \dots, \\
& n-1; n \leq k, & \\
& \bar{x}_{01}(\varphi_i) + a_i = \bar{x}_{01}(\psi_i) & i=1, 2, \dots, m; n \leq k. \\
& \bar{x}_{kn}(\varphi_i) + \sum_{l=0}^{k-1} a_{il} \leq \bar{x}_{kn}(\psi_i) \leq \bar{x}_{kn}(\varphi_i) + \sum_{l=0}^{n-1} a_{il} & i_l=1, 2, \dots, m; l=0, 1, \dots, \\
& n-1; n \geq k+1, & \\
& \bar{x}_{jn}(\varphi_i) + \sum_{l=0}^{j-1} a_{il} < \bar{x}_{jn}(\psi_i) < \bar{x}_{jn}(\varphi_i) + \sum_{l=0}^{n-1} a_{il} & j=1, 2, \dots, k-1; i_l=1, 2, \dots, \\
& m; l=0, 1, \dots, n-1; & \\
& n \geq k+1, & \\
& \bar{x}_{0n}(\varphi_i) + a_i \leq \bar{x}_{0n}(\psi_i) < \bar{x}_{0n}(\varphi_i) + \sum_{l=0}^{n-1} a_{il} & i_l=1, 2, \dots; l=0, 1, \dots, n-1; \\
& n \geq k+1. & \\
& \left\{ \begin{aligned} & f'_1(x - a_i, \varphi_i) = f'_1(x, \psi_i) \text{ for all } x, \quad i=1, 2, \dots, m, \\ & f'_n(x - a_i, \varphi_i) \geq f'_n(x, \psi_i) \geq f'_n(x - a_i - a_{i_1} \dots - a_{i_{n-1}}, \varphi_i) \text{ for all } x, \\ & \quad i_l=1, 2, \dots, m; l=0, 1, \dots, n-1. \end{aligned} \right. &
\end{aligned}$$

Proof (by induction). Case (a).  $n \leq k$ . When  $n=1$ , we obtain from (3.4) upon making a simple change variable that

$$\begin{aligned}
(3.46) \quad G'_{11}(u_1, \psi_i) &= c_1 + V'(u_1, \psi_i) - W'_1(u_1, \psi_i) \\
&= c_1 + V'(u_1 - a_i, \varphi_i) - W'_1(u_1 - a_i, \varphi_i) \\
&= G'_{11}(u_1 - a_i, \varphi_i) \text{ for all } u_1, \quad i=1, 2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
(3.47) \quad H'_{01}(u_0, \psi_i) &= \begin{cases} G'_{11}(u_0 + m_{u_1}, \psi_i) & u_0 + m_{u_1} < \bar{x}_{11}(\psi_i), \\ 0 & u_0 < \bar{x}_{11}(\psi_i) < u_0 + m_{u_1}, \\ G'_{11}(u_0, \psi_i) & u_0 > \bar{x}_{11}(\psi_i), \end{cases} \\
&= \begin{cases} G'_{11}(u_0 - a_i + m_{u_1}, \varphi_i) & u_0 - a_i + m_{u_1} < \bar{x}_{11}(\varphi_i) \\ 0 & u_0 - a_i < \bar{x}_{11}(\varphi_i) < u_0 - a_i + m_{u_1}, \\ G'_{11}(u_0 - a_i, \varphi_i) & u_0 - a_i > \bar{x}_{11}(\varphi_i), \end{cases} \\
&= H'_{01}(u_0 - a_i, \varphi_i) \text{ for all } u_0, \quad i=1, 2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
(3.48) \quad G'_{01}(u_0, \psi_i) &= (c_0 - c_1) + L'(u_0, \psi_i) - V'(u_0, \psi_i) + H'_{01}(u_0, \psi_i) \\
&= (c_0 - c_1) + L'(u_0 - a_i, \varphi_i) - V'(u_0 - a_i, \varphi_i) + H'_{01}(u_0 - a_i, \varphi_i) \\
&= G'_{01}(u_0 - a_i, \varphi_i) \text{ for all } u_0, \quad i=1, 2, \dots, m.
\end{aligned}$$

Hence it follows immediately from (3.46), (3.47), (3.48), (3.17), (3.36) and Theorem 3.2. that

$$\begin{aligned}
& \bar{x}_{11}(\varphi_i) + a_i = \bar{x}_{11}(\psi_i) & i=1, 2, \dots, m, \\
& \bar{x}_{01}(\varphi_i) + a_i = \bar{x}_{01}(\psi_i) & i=1, 2, \dots, m, \\
& f'_1(x - a_i, \varphi_i) = f'_1(x, \psi_i) \text{ for all } x, & i=1, 2, \dots, m.
\end{aligned}$$

When  $n=2$ , we get from (3.4), (3.46), (3.47), (3.48), and strict convexity of  $L(u, \psi_i) - V(u, \psi_i)$  and  $V(u, \psi_i)$

$$\begin{aligned}
G'_{22}(u_2, \psi_i) &= c_2 + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V'(u_2 - t, \psi_{i_1}) \psi_i(t) dt \\
&\quad - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty W'_1(u_2 - t, \psi_{i_1}) \psi_i(t) dt \\
&= c_2 + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V'(u_2 - a_{i_1} - t, \varphi_{i_1}) \varphi_i(t - a_{i_1}) dt \\
&\quad - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty W'_1(u_2 - a_{i_1} - t, \varphi_{i_1}) \varphi_i(t - a_{i_1}) dt \\
&= c_2 + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty V'(u_2 - a_i - a_{i_1} - t, \varphi_{i_1}) \varphi_i(t) dt \\
&\quad - \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty W'_1(u_2 - a_i - a_{i_1} - t, \varphi_{i_1}) \varphi_i(t) dt \\
&= G'_{22}(u_2 - a_i - a_{i_1}, \varphi_i) \text{ for all } u_2, \quad i=1, 2, \dots, m. \\
H'_{12}(u_1, \psi_i) &= H'_{12}(u_1 - a_i - a_{i_1}, \varphi_i) \text{ for all } u_1, \quad i=1, 2, \dots, m. \\
G'_{12}(u_1, \psi_i) &= (c_1 - c_2) + L'_1(u_1, \psi_i) + V'(u_1, \psi_i) - V'_1(u_1, \psi_i) + H'_{12}(u_1, \psi_i) \\
&= (c_1 - c_2) + L'_1(u_1 - a_i - a_{i_1}, \varphi_i) + V'(u_1 - a_i, \varphi_i) \\
(3.49) \quad &\quad - V'_1(u_1 - a_i - a_{i_1}, \varphi_i) + H'_{12}(u_1 - a_i - a_{i_1}, \varphi_i) \\
&\geq G'_{12}(u_1 - a_i - a_{i_1}, \varphi_i) \text{ for all } u_1, \\
G'_{12}(u_1, \psi_i) &< G'_{12}(u_1 - a_i, \varphi_i) \text{ for all } u_1, \quad i=1, 2, \dots, m. \\
H'_{02}(u_0 - a_i, \varphi_i) &\geq H'_{02}(u_0, \psi_i) \geq H'_{02}(u_0 - a_i - a_{i_1}, \varphi_i) \text{ for all } u_0, \\
&\quad i=1, 2, \dots, m. \\
G'_{02}(u_0, \psi_i) &= (c_0 - c_1) + L'(u_0, \psi_i) - V'(u_0, \psi_i) + H'_{02}(u_0, \psi_i) \\
&= (c_0 - c_1) + L'(u_0 - a_i, \varphi_i) - V'(u_0 - a_i, \varphi_i) + H'_{02}(u_0, \psi_i) \\
&\leq G'_{02}(u_0 - a_i, \varphi_i) \text{ for all } u_0, \\
G'_{02}(u_0, \psi_i) &\geq (c_0 - c_1) + L'(u_0 - a_i, \varphi_i) - V'(u_0 - a_i, \varphi_i) + H'_{02}(u_0 - a_i, \varphi_i) \\
&> (c_0 - c_1) + L'(u_0 - a_i - a_{i_1}, \varphi_i) - V'(u_0 - a_i - a_{i_1}, \varphi_i) \\
&\quad + H'_{02}(u_0 - a_i - a_{i_1}, \varphi_i) \\
&= G'_{02}(u_0 - a_i - a_{i_1}, \varphi_i) \text{ for all } u_0, \quad i=1, 2, \dots, m.
\end{aligned}$$

Recalling  $V'(u, \varphi_0) = -v$  for all  $u \leq 0$ ,  $x_{12}(\varphi_i) > 0$  implies that

$$\begin{aligned}
G'_{12}(a_i, \psi_i) &= G'_{12}(-a_{i_1}, \varphi_i) < 0, \\
G'_{12}(u_1, \psi_i) &= G'_{12}(u_1 - a_i - a_{i_1}, \varphi_i) \text{ for all } u_1 \leq a_i, \\
G'_{12}(u_1, \psi_i) &> G'_{12}(u_1 - a_i - a_{i_1}, \varphi_i) \text{ for all } u_1 > a_i.
\end{aligned}$$

Hence  $\bar{x}_{12}(\psi_i) < \bar{x}_{12}(\varphi_i) + a_i + a_{i_1}$ . We get the theorem for  $n=2$  from above relations. The equality and inequality in part (c) depend on a relationship between  $\bar{x}_{02}(\psi_i)$ ,  $\bar{x}_{02}(\varphi_i) + a_i$ , and  $\bar{x}_{02}(\varphi_i) + a_i + a_{i_1} - m_{u_0}$ . For example, if  $\bar{x}_{02}(\varphi_i) + a_i > x > \bar{x}_{02}(\varphi_i) + a_i + a_{i_1} - m_{u_0}$ , then

$$f'_2(x-a_i, \varphi_i) = f'_2(x, \psi_i) = f'_2(x-a_i-a_{i1}, \varphi_i).$$

Assuming that Corollary is true for  $n-1$ , and that

$$(3.50) \quad \begin{aligned} G'_{n-1,n-1}(u_{n-1}, \psi_i) &= G'_{n-1,n-1}(u_{n-1}-a_i-a_{i1}-\cdots-a_{i_{n-2}}, \varphi_i) \text{ for all } u_{n-1}, \\ &\quad i_l=1, 2, \dots, m; l=0, 1, \dots, n-2, \\ L'_{n-1}(u, \psi_i) &= L'_{n-1}(u-a_i-a_{i1}-\cdots-a_{i_{n-1}}, \varphi_i) \text{ for all } u, \\ &\quad i_l=1, 2, \dots, m; l=0, 1, \dots, n-1, \\ V'_{n-1}(u, \psi_i) &= V'_{n-1}(u-a_i-a_{i1}-\cdots-a_{i_{n-1}}, \varphi_i) \text{ for all } u, \\ &\quad i_l=1, 2, \dots, m; l=0, 1, \dots, n-1, \end{aligned}$$

, then we have from the (3.4)

$$(3.51) \quad \begin{aligned} G'_{nn}(u_n, \psi_i) &= (c_n - \alpha c_{n-1}) + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty [c_{n-1} + V'_{n-2}(u_n-t, \psi_{i_1}) \\ &\quad - W'_{n-1}(u_n-t, \psi_{i_1})] \psi_i(t) dt \\ &= (c_n - \alpha c_{n-1}) + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty G'_{n-1,n-1}(u_n-t, \psi_{i_1}) \psi_i(t) dt \\ &= (c_n - \alpha c_{n-1}) + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty G'_{n-1,n-1}(u_n-a_{i_1}-a_{i_2}-\cdots \\ &\quad -a_{i_{n-1}}-t, \varphi_{i_1}) \varphi_i(t-a_i) dt \\ &= (c_n - \alpha c_{n-1}) + \alpha \sum_{i_1=1}^m p_{ii_1} \int_0^\infty G'_{n-1,n-1}(u_n-a_i-a_{i_1}-\cdots \\ &\quad -a_{i_{n-1}}-t, \varphi_{i_1}) \varphi_i(t) dt \\ &= G'_{nn}(u_n-a_i-a_{i1}-\cdots-a_{i_{n-1}}, \varphi_i) \text{ for all } u_n, \quad i=1, 2, \dots, m. \end{aligned}$$

From the definition of  $H'_{n-1,n}(u_{n-1}, \varphi_i)$ , and  $H'_{n-1,n}(u_{n-1}, \psi_i)$ ,  $H'_{n-1,n}(u_{n-1}, \psi_i) = H'_{n-1,n}(u_{n-1}-a_i-a_{i1}-\cdots-a_{i_{n-1}}, \varphi_i)$  for all  $u_{n-1}$ , Hence get from (3.4)

$$(3.52) \quad \begin{aligned} G'_{n-1,n}(u_{n-1}, \psi_i) &= (c_{n-1} - c_n) + L'_{n-1}(u_{n-1}, \psi_i) \\ &\quad + V'_{n-2}(u_{n-1}, \psi_i) - V'_{n-1}(u_{n-1}, \psi_i) + H'_{n-1,n}(u_{n-1}, \psi_i) \\ &= (c_{n-1} - c_n) + L'_{n-1}(u_{n-1}-a_i-a_{i1}-\cdots-a_{i_{n-1}}, \varphi_i) \\ &\quad + V'_{n-2}(u_{n-1}-a_i-a_{i1}-\cdots-a_{i_{n-2}}, \varphi_i) \\ &\quad - V'_{n-1}(u_{n-1}-a_i-a_{i1}-\cdots-a_{i_{n-1}}, \varphi_i) + H'_{n-1,n}(u_{n-1}, \psi_i) \\ &\geq G'_{n-1,n}(u_{n-1}-a_i-a_{i1}-\cdots-a_{i_{n-1}}, \varphi_i) \text{ for all } u_{n-1}, \\ G'_{n-1,n}(u_{n-1}, \psi_i) &< G'_{n-1,n}(u_{n-1}-a_i-a_{i1}-\cdots-a_{i_{n-2}}, \varphi_i) \text{ for all } u_{n-1}, \\ &\quad i_l=1, 2, \dots, m; l=0, 1, \dots, n-2. \end{aligned}$$

Hence we have, from the discussion as before,

$$\begin{aligned} \bar{x}_{nn}(\psi_i) &= \bar{x}_{nn}(\varphi_i) + \sum_{l=0}^{n-1} a_{il} & i_l=1, 2, \dots, m; l=0, 1, \dots, n-1, \\ \bar{x}_{n-1,n}(\varphi_i) + \sum_{l=0}^{n-2} a_{il} &< \bar{x}_{nn}(\psi_i) < \bar{x}_{nn}(\varphi_i) + \sum_{l=0}^{n-1} a_{il}, & i_l=1, 2, \dots, m; l=0, 1, \dots, n-1. \end{aligned}$$

We assume that Corollary holds for  $n$  and  $j=n, n-1, \dots, n-r+1$ , and that

$$\begin{aligned} G'_{jn}(u_j - a_i - a_{i_1} - \dots - a_{i_{j-1}}, \varphi_i) &> G_{jn}(u_j, \psi_i) \geq G'_{jn}(u_j - a_i - a_{i_1} - \dots - a_{i_j}, \varphi_i) \\ &\text{for all } u_j, \quad j=n-2, \dots, n-r+1 \quad i_l=1, 2, \\ &\quad \dots, n-1; \quad l=0, 1, \dots, n-1. \end{aligned}$$

Then we obtain from (3.4)

$$\begin{aligned} (3.53) \quad H'_{jn}(u_j - a_i - a_{i_1} - \dots - a_{i_{j-1}}, \varphi_i) &\geq H'_{jn}(u_j, \psi_i) \geq H'_{jn}(u_j - a_i - a_{i_1} - \dots - a_{i_j}, \varphi_i) \\ &\text{for all } u_j, \quad j=n-2, \dots, n-r \\ &\quad i_l=1, 2, \dots, m; \quad l=0, 1, \dots, n-1. \end{aligned}$$

Hence, from the inductive assumption and (3.53) we have

$$\begin{aligned} G'_{n-r,n}(u_{n-r}, \psi_i) &= (c_{n-r} - c_{n-r+1}) + L'_{n-r}(u_{n-r}, \psi_i) + V'_{n-r-1}(u_{n-r}, \psi_i) \\ &\quad - V'_{n-r}(u_{n-r}, \psi_i) + H'_{n-r,n}(u_{n-r}, \psi_i) \\ &= (c_{n-r} - c_{n-r+1}) + L'_{n-r}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r}}, \varphi_i) \\ &\quad + V'_{n-r-1}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r-1}}, \varphi_i) \\ &\quad - V'_{n-r}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r}}, \varphi_i) + H'_{n-r,n}(u_{n-r}, \psi_i) \\ &\geq (c_{n-r} - c_{n-r+1}) + L'_{n-r}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r}}, \varphi_i) \\ &\quad + V'_{n-r-1}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r-1}}, \varphi_i) \\ &\quad - V'_{n-r}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r}}, \varphi_i) \\ &\quad + H'_{n-r,n}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r}}, \varphi_i) \\ &\geq G'_{n-r,n}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r}}, \varphi_i) \quad \text{for all } u_{n-r} \\ G'_{n-r,n}(u_{n-r}, \psi_i) &< G'_{n-r,n}(u_{n-r} - a_i - a_{i_1} - \dots - a_{i_{n-r-1}}, \varphi_i) \quad \text{for all } u_{n-r}, \\ &\quad i_l=1, 2, \dots, m; \quad l=0, 1, \dots, n-1. \end{aligned}$$

From the same argument used for the case  $n=1, n=2$ , we obtain Corollary for  $n \leq k$ .

Case (b)  $n \geq k+1$ . We have

$$\begin{aligned} f'_{il}(u - a_i - a_{i_1} - \dots - a_{i_l}, \varphi_i) &= f_{il}(u, \psi_i) \quad i_r=1, 2, \dots, m; \quad r=0, 1, \\ &\text{for all } u, \quad \dots, l; \quad l=1, 2, \dots, k. \end{aligned}$$

(3.54)

$$\begin{aligned} f'_{jl}(u - a_i - a_{i_1} - \dots - a_{i_l}, \varphi_i) &\geq f'_{jl}(u, \psi_i) \quad \text{for all } u, \\ &\geq f'_{jl}(u - a_i - a_{i_1} - \dots - a_{i_{j+l-1}}, \varphi_i) \quad j=2, \dots, n-k, \quad i_r=1, \\ &\quad 2, \dots, m; \quad r=0, 1, \dots, l; \quad l=1, 2, \dots, k, \end{aligned}$$

which can be proved inductively from the results of case (a) and (3.2). We can prove Corollary for the case under consideration by (3.54) in the same method as for the case (a). The detail of the proof are omitted. We established the proof of Corollary.

Specially  $\psi_i(x) = \varphi_i(x-a)$ , then we get

$$(a') \quad \begin{cases} \bar{x}_{nn}(\varphi_i) + na = \bar{x}_{nn}(\psi_i) & i=1, 2, \dots, m; \quad n \leq k, \\ \bar{x}_{jn}(\varphi_i) + ja < \bar{x}_{jn}(\psi_i) < \bar{x}_{jn}(\varphi_i) + na & j=1, 2, \dots, n-1; \quad k \geq n \geq 2; \quad i=1, \\ & 2, \dots, m, \end{cases}$$

$$\begin{aligned}
& \left\{ \begin{array}{ll} \bar{x}_{0n}(\varphi_i) + a \leq \bar{x}_{0n}(\psi_i) < \bar{x}_{jn}(\varphi_i) + na & i=1, 2, \dots, m; n \leq k, \\ \bar{x}_{01}(\varphi_i) + a = \bar{x}_{01}(\psi_i) & i=1, 2, \dots, m; n \leq k. \end{array} \right. \\
(b') & \left\{ \begin{array}{ll} \bar{x}_{kn}(\varphi_i) + ka \leq \bar{x}_{kn}(\psi_i) \leq \bar{x}_{kn}(\varphi_i) + na, & i=1, 2, \dots, m; n \geq k+1, \\ \bar{x}_{jn}(\varphi_i) + ja < \bar{x}_{jn}(\psi_i) < \bar{x}_{jn}(\varphi_i) + na, & j=1, 2, \dots, k-1; i=1, 2, \\ & \dots, m; n \geq k+1, \\ \bar{x}_{0n}(\varphi_i) + a \leq \bar{x}_{0n}(\psi_i) < \bar{x}_{0n}(\varphi_i) + na, & i=1, 2, \dots, m; n \geq k+1. \end{array} \right. \\
(c') & \left\{ \begin{array}{ll} f'_i(x-a, \varphi_i) = f'_i(x, \psi_i) & \text{for all } x, \quad i=1, 2, \dots, m. \\ f'_n(x-a, \varphi_i) \geq f'_n(x_1, \psi_i) \geq f'_n(x-na, \varphi_i) & \text{for all } x, \quad i=1, 2, \dots, m. \end{array} \right.
\end{aligned}$$

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Kumamoto University