RANDOM COMBINED FRACTIONAL FACTORIAL DESIGNS II : SAMPLING THEORY FROM THE FINITE POPULATION

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Random Combined Fractional Factorial Designs II: Sampling theory from the Finite Population

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0. Introduction

Let us consider the following randomization procedures in our designs (refer to Literature [12]).

(1) We have a design matrix

$$D=\left\| \begin{matrix} D_P \\ D_\Phi \\ D_R \end{matrix} \right\| \tag{0.1}$$

which is composed of three sub-matrices

$$D_P=\left\| \begin{matrix} x_{11} & x_{12} & \cdots\cdots & x_{1N} \\ x_{21} & x_{22} & \cdots\cdots & x_{2N} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ x_{P1} & x_{P2} & \cdots\cdots & x_{PN} \end{matrix} \right\| \tag{0.2}$$

$$D_\Phi=\left\| \begin{matrix} z_{11} & z_{12} & \cdots\cdots & z_{1N} \\ z_{21} & z_{22} & \cdots\cdots & z_{2N} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ z_{\Phi 1} & z_{\Phi 2} & \cdots\cdots & z_{\Phi N} \end{matrix} \right\| \tag{0.3}$$

and

$$D_R=\left\| \begin{matrix} \xi_{1\pi(1)} & \xi_{1\pi(2)} & \cdots\cdots & \xi_{1\pi(N)} \\ \xi_{2\pi(1)} & \xi_{2\pi(2)} & \cdots\cdots & \xi_{2\pi(N)} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \xi_{R\pi(1)} & \xi_{R\pi(2)} & \cdots\cdots & \xi_{R\pi(N)} \end{matrix} \right\| . \tag{0.4}$$

(2) In these sub-matrices, we have

$$\left\| \begin{matrix} D_P \\ D_\Phi \end{matrix} \right\| = \left\| \begin{matrix} x_{11} & x_{12} & \cdots\cdots & x_{1N} \\ x_{21} & x_{22} & \cdots\cdots & x_{2N} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ x_{P1} & x_{P2} & \cdots\cdots & x_{PN} \\ z_{11} & z_{12} & \cdots\cdots & z_{1N} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ z_{\Phi 1} & z_{\Phi 2} & \cdots\cdots & z_{\Phi N} \end{matrix} \right\| \tag{0.5}$$

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as an orthogonal array of size

$$N = P + \Phi \tag{0.6}$$

with two levels

$$\begin{cases} x_{pn} = +1 \quad \text{or} \quad -1 \\ z_{\phi n} = +1 \quad \text{or} \quad -1 \end{cases} \tag{0.7}$$

(3) With equal probability $1/N!$, we choose a permutation

$$\Pi = \begin{pmatrix} 1 & 2 & \cdots\cdots & N \\ \pi(1) & \pi(2) & \cdots\cdots & \pi(N) \end{pmatrix} \tag{0.8}$$

from the set of all possible permutations of N numbers $1, 2, \cdots, N-1$ and N.

In virtue of these randomization procedure, the variable

$$\dot{V}'_p = \sum_{n=1}^{N} x_{np} \quad v'_{\pi(n)}/N \qquad (p=1, 2, \cdots, P) \tag{0.9}$$

becomes a random variable, where

$$v'_{\pi(n)} = \sum_{r=1}^{R} \beta'_r \xi_{r\pi(n)} \qquad (n=1, 2, \cdots, N) \tag{0.10}$$

and $\beta'_1, \beta'_2, \cdots, \beta'_R$ are parameters. The purpose of this paper is to discuss the distribution of the random variable $\dot{V}'_p$.

1. A sampling distribution function from a finite population

1.1. Direct evaluation of probability distribution

First of all, let us consider the distribution function of statistics introduced from a classification of a finite population, $v_1, v_2, \cdots, v_N$, which can be classified into K $(<N)$ classes such that NP_1 elements of $\bar{v}_1$, NP_2 elements of $\bar{v}_2, \cdots$, NP_k elements of $\bar{v}_k, \cdots$, NP_K elements of $\bar{v}_K$ where $\bar{v}_k$ $(k=1, 2, \cdots, K)$ is the value of v_n in the k-th class.

These N points of v_n $(n=1, 2, \cdots, N)$ which randomly correspond to the N columns in the design matrix (0.1) by our randomization procedure are divided into 2 groups by an arbitrary p-th row in O.A. such that $+1$ of x_{pn} group contains $N_1 = \frac{N}{2}$ points and -1 of x_{pn} group contains $N_2 = \frac{N}{2}$ points, where suffix " 1 " stands for " + " sign group and suffix " 2 " stands for " − " sign group.

In these circumstances, N elements in the sequence $\{v_n\}$ are divided into two groups, such that the 1st group of these is the sequence $\{\dot{u}_{k1}\}$ $(k=1, 2, \cdots, K)$ and the 2nd group of these is the sequence $\{\dot{u}_{k2}\}$ $(k=1, 2, \cdots, K)$ in which $\dot{u}_{k1}$ and $\dot{u}_{k2}$ are the number of k-th class in the 1st and

2nd groups respectively. Then we have $\sum_{k=1}^{K}\dot{u}_{ki}=N_i$, $(i=1, 2)$ and $\sum_{i=1}^{2}\dot{u}_{ki}=NP_k$, $(k=1, 2, \cdots, K)$. Consequently, we get

$$\text{(1.1.1)} \qquad \dot{V}_p=\frac{1}{N}\sum_{n=1}^{N}\dot{v}_{\pi(n)}\, x_{pn} = \frac{2}{N}\sum_{k=1}^{K}v_k\ \dot{u}_{k1},$$

where we used the relation $\frac{1}{N}\sum_{n=1}^{N}\dot{v}_{\pi(n)}=\frac{1}{N}\sum_{r=1}^{R}\beta_r\sum_{n=1}^{N}\xi_{r\pi(n)}=0$. Then we get the probability, that for any assigned set $b_{11}, \cdots, b_{K1}$, we have $\dot{u}_{11}, \cdots, \dot{u}_{K1}=b_{11}, \cdots, b_{K1}$, is

$$\text{(1.1.2)} \qquad P_r(b_{11}=\dot{u}_{11}, \cdots, b_{K1}=\dot{u}_{K1})=\prod_{k=1}^{K}\binom{NP_k}{b_{k1}}\Big/\binom{N}{N_1}\,.$$

For any assigned set of $\{b_{k\mu}\}$ $k=1, 2, \cdots, K$, so as to simplify the following discussions, we shall denote the joint probability function of the random variables $\dot{u}_{1\mu}=b_{1\mu}$, $\dot{u}_{2\mu}=b_{2\mu}$, $\cdots$, $\dot{u}_{k\mu}=b_{k\mu}$ as $P\left\{\prod_{k=1}^{K}(\dot{u}_{k\mu}=b_{k\mu})\right\}$ (for any μ) and $P\left\{\prod_{k=1}^{K}\prod_{\mu=1}^{M}(\dot{u}_{k\mu}=b_{k\mu})\right\}$, where μ means the suffix representing the classification number such as $ij\cdots k$ $(i, j, \cdots, k=1, 2)$. Furthermore, we shall use the notation $b_{k\mu}$ for the assigned value corresponding random variable $\dot{u}_{k\mu}$.

Secondly, let us consider the case that N points of $v_{\pi(n)}$ are divided into $2^2=4$ groups by two arbitrary p and q-th rows in O. A. such that $\{+1, +1\}$ group contains $N_{11}=N/4$ points $\{+1, -1\}$ gronp contains $N_{12}=N/4$ points $\{-1, +1\}$ group contains $N_{21}=N/4$ points $\{-1, -1\}$ group contains $N_{22}=N/4$ points where suffices " 1 " and " 2 " stand for " + " and " − " sign group, respectively, then " 11 " stands for " ++ " sign group, and so on.

In these circumstances, N elements in the sequence $\{v_n\}$ $(n=1, 2, \cdots, N)$ are classified in the two-way classification as following page.

In Table 1.1.1 and in later discussions, we shall denote the sign " 0 " so as to show the suffix with which we have operate the summation as follows, $\sum_{i=1}^{2}\dot{u}_{kij}=\dot{u}_{k0j}$, $\sum_{i=1}^{2}b_{kij}=b_{k0j}$, and $\sum_{j=1}^{2}\dot{u}_{kij}=\dot{u}_{ki0}$, $\sum_{j=1}^{2}b_{kij}=b_{ki0}$.

The conditional probability, that for any assigned set of $\{b_{k11}\}$, $\{\dot{u}_{k11}\}$ $(k=1, 2, \cdots, K)$ elements are contained in N_{11} random samples from the population $N_{10}=\sum_{j=1}^{2}N_{ij}$ under the condition that for any assigned set of $\{b_{k10}\}$, $\{u_{k10}\}$ $(k=1, 2, \cdots, K)$ elements are contained in N_{10} random samples from the grand population N, is obtained from the expression

Table 1.1.1 Two-way classification table of $\{v_n\}$

$p \backslash q$	+	−	Sum
+	$\left.\begin{matrix}\dot{u}_{111}\\ \dot{u}_{211}\\ \cdots\\ \dot{u}_{K11}\end{matrix}\right\} N_{11}$	$\left.\begin{matrix}\dot{u}_{112}\\ \dot{u}_{212}\\ \vdots\\ \dot{u}_{K12}\end{matrix}\right\} N_{12}$	$\left.\begin{matrix}\dot{u}_{110}\\ \dot{u}_{210}\\ \vdots\\ \dot{u}_{K10}\end{matrix}\right\} N_{10}$
−	$\left.\begin{matrix}\dot{u}_{121}\\ \dot{u}_{221}\\ \vdots\\ \dot{u}_{K21}\end{matrix}\right\} N_{21}$	$\left.\begin{matrix}\dot{u}_{122}\\ \dot{u}_{222}\\ \vdots\\ \dot{u}_{K22}\end{matrix}\right\} N_{22}$	$\left.\begin{matrix}\dot{u}_{120}\\ \dot{u}_{220}\\ \vdots\\ \dot{u}_{K02}\end{matrix}\right\} N_{20}$
Sum	$\left.\begin{matrix}\dot{u}_{101}\\ \dot{u}_{201}\\ \vdots\\ \dot{u}_{K01}\end{matrix}\right\} N_{01}$	$\left.\begin{matrix}\dot{u}_{102}\\ \dot{u}_{202}\\ \vdots\\ \dot{u}_{K02}\end{matrix}\right\} N_{02}$	$\left.\begin{matrix}NP_1\\ NP_2\\ \vdots\\ NP_K\end{matrix}\right\} N$

$$P\left(\prod_{k=1}^{K}(\dot{u}_{k11}=b_{k11}) \;\Big|\; \prod_{k=1}^{K}(\dot{u}_{k10}=b_{k10})\right)=\prod\binom{b_{k10}}{b_{k11}}\Big/\binom{N_{10}}{N_{11}} .$$

Similarly, the conditional probability, that $\{b_{k21}\}$ $(k=1, 2, \cdots, K)$ elements are containing in N_{21} random samples from the population N_{20} under the condition that $\{b_{k20}\}$ $(k=1, 2, \cdots, K)$ elements are containing in N_{20} random samples from the grand population N, is obtained from the expression

$$P\left\{\prod_{k=1}^{K}(\dot{u}_{k21}=b_{k21}) \;\Big|\; \prod_{k=1}^{K}(\dot{u}_{k20}=b_{k20})\right\}=\prod_{k=1}^{K}\binom{b_{k20}}{b_{k21}}\Big/\binom{N_{20}}{N_{21}} .$$

Since these two random sequences $\{\dot{u}_{k11}\}$ and $\{\dot{u}_{k12}\}$ $(k=1, 2, \cdots, K)$ are mutually independently distributed, then we have the probability that the sequences $\{b_{k11}\}$ and $\{b_{k21}\}$ $(k=1, 2, \cdots, K)$ are jointly sampled, such that

$$(1.1.3)\qquad P\left\{\prod_{k=1}^{K}(\dot{u}_{k11}=b_{k11})\prod_{k=1}^{K}(\dot{u}_{k21}=b_{k21}) \;\Big|\; \prod_{k=1}^{K}(\dot{u}_{k10}=b_{k10})\right\}$$
$$=\prod_{k=1}^{K}\binom{b_{k10}}{b_{k11}}\binom{b_{k20}}{b_{k21}}\Big/\binom{N_{10}}{N_{11}}\binom{N_{20}}{N_{21}} .$$

Consequently, we get the joint probability distribution

$$(1.1.4)\qquad P\left\{\prod_{k=1}^{K}(\dot{u}_{k11}=b_{k11},\ \dot{u}_{k21}=b_{k21},\ \dot{u}_{k10}=b_{k10})\right\}$$
$$=\prod_{k=1}^{K}\frac{(NP_k)\,!}{(b_{k11})!(b_{k12})!(b_{k21})!\,(b_{k22})!}\Big/\frac{N!}{N_{11}!N_{12}!N_{21}!N_{22}!} .$$

Furthermore, in the case that N points of complete orthogonal array is equal to $N=2^J$, we can classify the N elements of the sequence $\{v_n\}$ $(n=1, 2, \cdots, N)$ into $N_{\underbrace{11\cdots1}_{J}}, N_{\underbrace{11\cdots2}_{J}}, \cdots, N_{\underbrace{22\cdots2}_{J}}$ elements, respectively, in a similar manner as following Table. In other words, we can represent these J-way classification in the following Figure.

Table 1.1.2 J-way $(2^J=N)$ classification table

	$p_1\ p_2\cdots\cdots p_J$		1	2	$\cdots\cdots$	K
1	$1\ 1\ \cdots\cdots\ 1$	$\underbrace{N_{11\cdots1}}_{J}$	$\underbrace{\dot{u}_{111\cdots1}}_{J}$	$\underbrace{\dot{u}_{211\cdots1}}_{J}$	$\cdots\cdots$	$\underbrace{\dot{u}_{K11\cdots1}}_{J}$
2	$1\ 1\ \cdots\cdots\ 2$	$\underbrace{N_{11\cdots2}}_{J}$	$\underbrace{\dot{u}_{111\cdots2}}_{J}$	$\underbrace{\dot{u}_{211\cdots2}}_{J}$	$\cdots\cdots$	$\underbrace{\dot{u}_{K11\cdots2}}_{J}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
N	$2\ 2\ \cdots\cdots\ 2$	$\underbrace{N_{22\cdots2}}_{J}$	$\underbrace{\dot{u}_{122\cdots2}}_{J}$	$\underbrace{\dot{u}_{222\cdots2}}_{J}$	$\cdots\cdots$	$\underbrace{\dot{u}_{K22\cdots2}}_{J}$

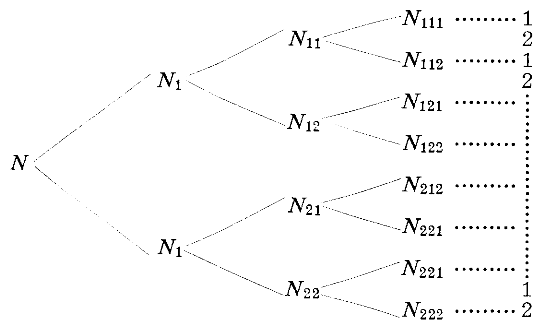


Fig. 1.1.1 J-way classification

Then we can evaluate the joint probability distribution that the sequences $\{\underbrace{\dot{u}_{k100\cdots0}}_{J}\}$ $(k=1,2,\cdots,K)$, $\{\underbrace{\dot{u}_{k110\cdots0}}_{J}\}$ $(k=1,2,\cdots,K)$, $\{\underbrace{\dot{u}_{k210\cdots0}}_{J}\}$ $(k=1,2,\cdots,K)$, $\cdots$, $\{\underbrace{\dot{u}_{k22\cdots1}}_{J}\}$ $(k=1,2,\cdots,K)$ are obtained for any assigned set of $\{\underbrace{b_{k10\cdots0}}_{J}\}$ $\{\underbrace{b_{k11\cdots0}}_{J}\}$ $\cdots$, $\{\underbrace{b_{k22\cdots1}}_{J}\}$ $(k=1,2,\cdots,K)$, such that

(1.1.5)
$$P\{\prod(\dot{u}_{k100\cdots0}=b_{k100\cdots0},\ \cdots,\ \dot{u}_{k22\cdots1}=b_{k22\cdots1})\}$$
$$=\frac{\prod_{k=1}^{K}\binom{NP_k}{b_{k10}}\binom{b_{k10}}{b_{k110}}\binom{b_{k20\cdots}}{b_{k21\cdots}}\cdots\cdots\binom{b_{k22\cdots0}}{b_{k22\cdots1}}}{\binom{N}{N_{10}}\binom{N_{10}}{N_{11}}\binom{N_{20}}{N_{21}}\cdots\cdots\binom{N_{22\cdots0}}{N_{22\cdots1}}}$$
$$=\frac{(NP_1)!(NP_2)!\cdots\cdots(NP_K)!}{N!}\quad.$$

1.2 Asymptotic distributions

1.2.1 Some transformations of variables

So as to study the asymptotic distribution of the probability, such as (1.1.2), (1.1.3) and (1.1.4), let us consider the several transformations of the variables.

(1) Let us introduce a random variable, which is obtained by the prescribed random J-way classifition of $N(=2^J)$ elements of $\{v_n\}$ $(n=1, 2, \cdots, N)$, such that

$$\dot{u}_{kn} \overset{d}{=} \dot{u}_{k\underbrace{12\cdots 1}_{J}}, \tag{1.2.1}$$

and which has the properties that we have $\dot{u}_{kn}=1$ for $\dot{v}_{\pi(n)}=\bar{v}_k$ and we have $\dot{u}_{kn}=0$ for $\dot{v}_{\pi(n)}\neq\bar{v}_k$. Then we get

$$\dot{v}_{\pi(n)} = \sum_{k=1}^{K} \bar{v}_k \dot{u}_{kn} . \tag{1.2.2}$$

(2) We shall define a random variable

$$\dot{V}_{kp} \overset{d}{=} \frac{1}{N} \sum_{n=1}^{N} x_{pn}\, \dot{u}_{kn} \tag{1.2.3}$$

where we put $\|x_{pn}\|$ $(p=1, 2, \cdots, N;\ n=1, 2, \cdots, N)$ as a complete O.A. with size $N(=P+\Phi)$ which is composed of two sub-matrices

$$D_P = \begin{Vmatrix} x_{11} & \cdots\cdots\cdots & x_{1N} \\ & \cdots\cdots\cdots & \\ & \cdots\cdots\cdots & \\ x_{P1} & \cdots\cdots\cdots & x_{PN} \end{Vmatrix} \tag{1.2.4}$$

and

$$D_\Phi = \begin{Vmatrix} x_{P+1,1} & \cdots & x_{P+1,N} \\ \cdots & \cdots & \\ \cdots & \cdots & \\ x_{N,1} & \cdots & x_{NN} \end{Vmatrix}, \tag{1.2.5}$$

where $\Phi=N-P$ then we get

$$\begin{aligned} \dot{V}_p &= \frac{1}{N} \sum_{n=1}^{N} x_{pn}\, \dot{v}_{\pi(n)} \\ &= \frac{1}{N} \sum_{k=1}^{K} \bar{v}_k \sum_{n=1}^{N} x_{pn}\, \dot{u}_{kn} \\ &= \sum_{k=1}^{K} \bar{v}_k\, V_{kp}, \qquad (p=1, 2, \cdots, N). \end{aligned} \tag{1.2.6}$$

(3) Let us introduce a complete orthogonal array

$$\left\|\eta_{\mu\psi}^{(M)}\right\| = \begin{Vmatrix} + & + & + & \cdots & + & + \\ + & + & + & \cdots & - & - \\ + & + & + & \cdots & + & + \\ + & + & + & \cdots & - & - \\ \cdot & \cdot & \cdot & \cdots & & \\ + & - & - & \cdots & - & + \\ + & - & - & \cdots & + & - \end{Vmatrix} \qquad \begin{pmatrix} \mu=1, 2, \cdots, M \\ \psi=1, 2, \cdots, M \end{pmatrix} \tag{1.2.7}$$

where $M=2^{J_1}(<N)$ is the size of the O.A. such as

$$(1.2.8)\qquad \|\eta_{\mu\psi}^{(2)}\| = \begin{Vmatrix} + & + \\ + & - \end{Vmatrix}, \qquad \begin{pmatrix}\mu=1,\ 2\\ \psi=1,\ 2\end{pmatrix}$$

$$(1.2.9)\qquad \|\eta_{\mu\psi}^{(4)}\| = \begin{Vmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{Vmatrix}, \qquad \begin{pmatrix}\mu=1,\ 2,\ 3,\ 4\\ \psi=1,\ 2,\ 3,\ 4\end{pmatrix}$$

$$(1.2.10)\qquad \|\eta_{\mu\psi}^{(8)}\| = \begin{Vmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & + & - & + & - \\ + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \end{Vmatrix}, \qquad \begin{pmatrix}\mu=1,\ 2,\ \cdots,\ 8\\ \psi=1,\ 2,\ \cdots,\ 8\end{pmatrix}$$

and so on.

(4) In virtue of the matrix $\|\eta_{\mu\psi}^{(M)}\|$, we can define a J_1-way classification matrix

$$(1.2.11)\qquad y_{\mu n}^{(M)} \overset{d}{=} \frac{1}{M}\sum_{\psi=1}^{M} \eta_{\mu\psi}^{(M)}\, w_{\psi n}\ ,$$

where $\|w_{\psi n}\|$ $(\psi=1,\ 2,\ \cdots,\ M;\ n=1,\ 2,\ \cdots,\ N)$ is the smallest complete sub-orthogonal array which contains

$$(1.2.12)\qquad \{x_{1n}\} = \|x_{11}\cdots x_{1N}\| \overset{d}{=} \|w_{11}\cdots w_{1N}\|$$

$$(1.2.13)\qquad \{x_{pn}\} = \|x_{p1}\cdots x_{pN}\| \overset{d}{=} \|w_{21}\cdots w_{2N}\|$$

and

$$(1.2.14)\qquad D_{\Phi} = \begin{Vmatrix} z_{11} & \cdots & z_{1N} \\ z_{21} & \cdots & z_{2N} \\ \cdots & \cdots & \cdots \\ z_{\Phi 1} & \cdots & z_{\Phi N} \end{Vmatrix} \overset{d}{=} \begin{Vmatrix} w_{31} & \cdots & w_{3N} \\ w_{41} & \cdots & w_{4N} \\ \cdots & \cdots & \cdots \\ w_{\Phi+2,1} & \cdots & w_{\Phi+2,N} \end{Vmatrix}$$

and where $\{x_{pn}\}$ $(n=1,\ 2,\cdots,\ N)$ for a particular $p(1\leqq p\leqq P)$, is a row vector of D_P corresponding to the statistic $\hat{\alpha}_p=\left(\alpha_p'+\dot{V}_p'+\frac{1}{N}\sum_{n=1}^{N} x_{pn}\dot{\varepsilon}_n\right)$ (refer to [12]). Then we get

(1.2.15)
$$w_{\psi n} = \begin{Vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{p1} & x_{p2} & \cdots & x_{pN} \\ z_{11} & z_{12} & \cdots & z_{1N} \\ \cdot & \cdot & \cdots & \cdot \\ z_{\Phi 1} & z_{\Phi 2} & \cdots & z_{\Phi N} \\ x_{i_1 1} & x_{i_1 2} & \cdots & x_{i_1 N} \\ \cdot & \cdot & \cdots & \cdot \\ x_{iQ1} & x_{iQ3} & \cdots & x_{iQN} \end{Vmatrix} = \begin{Vmatrix} w_{11} & w_{12} & \cdots & w_{1N} \\ w_{21} & w_{22} & \cdots & w_{2N} \\ \cdot & \cdot & & \cdot \\ w_{M1} & w_{M2} & \cdots & w_{MN} \end{Vmatrix},$$

where $i_q (q=1, 2, \cdots, Q)$ are additional row numbers in $1 \leqq i_q \leqq P$ and

(1.2.16) $$M=\Phi+2+Q.$$

Then we get

(1.2.17) $$\dot{V}_p = \frac{1}{N}\sum_{n=1}^{N} x_{pn}\dot{v}_{(\pi n)} = \frac{1}{N}\sum_{n=1}^{N} w_{2n}\dot{v}_{\pi(n)} = \dot{W}_2 \quad ,$$

and in general

(1.2.18) $$\dot{W}_\psi = \frac{1}{N}\sum_{n=1}^{N} w_{\psi n}\,\dot{v}_{\pi(n)} \qquad (\psi=1, 2, \cdots, M).$$

For example,

(4–1) for $J_1=1$, we have $M=2$ then

$$\eta^{(2)}_{\mu\psi} = \begin{Vmatrix} + & + \\ + & - \end{Vmatrix}, \qquad \begin{pmatrix} \mu=1, 2 \\ \psi=1, 2 \end{pmatrix}$$

$$y^{(2)}_{\mu n} = \begin{Vmatrix} 1 & 1 \cdots\cdots 1 & 0 & 0 \cdots\cdots 0 \\ 0 & \underbrace{0 \cdots\cdots 0}_{\frac{N}{2}} & 1 & \underbrace{1 \cdots\cdots 1}_{\frac{N}{2}} \end{Vmatrix},$$

(4–2) for $J_1=2$, we have $M=4$ then

$$\eta^{(4)}_{\mu\psi} = \begin{Vmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{Vmatrix}, \qquad \begin{pmatrix} \mu=1, 2, 3, 4 \\ \psi=1, 2, 3, 4 \end{pmatrix}$$

$$y^{(4)}_{\cdot n}= \underbrace{1 \cdots 1}_{\frac{N}{4}} \quad \underbrace{1 \cdots 1}_{\frac{N}{4}} \quad \underbrace{1 \cdots 1}_{\frac{N}{4}} \quad \underbrace{1 \cdots 1}_{\frac{N}{4}} \quad .$$

Consequently we get the relation between the random variables $\dot{u}_{k\mu}$ introduced in the random classification and the random variables $\dot{W}_{k\psi}$ introduced by the random pairing of $\dot{u}_{kn}$ ($k=1, 2, \cdots, K$; $n=1, 2, \cdots, N$) and $w_{\psi n}$ ($\psi=1, 2, \cdots, M$; $n=1, 2, \cdots, N$), such that

$$\text{(1.2.19)} \qquad \dot{u}_{k\mu}=\sum_{n=1}^{N} y^{(M)}_{\mu n}\dot{u}_{kn} = \frac{N}{M}\sum_{\psi=1}^{M} \eta^{(M)}_{\mu\psi}\dot{W}_{k\psi} ,$$

where

$$\text{(1.2.20)} \qquad \dot{W}_{k\psi}=\frac{1}{N}\sum_{n=1}^{N} w_{\psi n}\dot{u}_{kn} .$$

Inversely, we have

$$\text{(1.2.21)} \qquad \dot{W}_{k\psi}=\frac{M}{N}\sum_{\mu=1}^{M} \|\eta^{(M)}_{\mu\psi}\|^{-1}\dot{u}^{(M)}_{k\mu} ,$$

where $\|\eta^{(M)}_{\mu\psi}\|^{-1}$ is the inverse matrix of $\|\eta^{(M)}_{\mu\psi}\|$.

($\dot{5}$) Let us define a standardized variate

$$\text{(1.2.22)} \qquad \dot{t}_{k\psi}=\dot{W}_{k\psi}/\sqrt{\sigma^2(\dot{W}_{k\psi})} ,$$

where $\sigma^2(\dot{W}_{k\psi})=aver\{\dot{W}^2_{k\psi}\}-aver^2\{\dot{W}_{k\psi}\}$. For the variate introduced by the prescribed random pairing $\dot{W}_{k\psi}=\frac{1}{N}\sum_{n=1}^{N} w_{\psi n}\,\dot{u}_{kn}$, we have $\sigma^2(\dot{W}_{k\psi})=\frac{1}{N^2}$

$\left\{N\langle\!\langle 2_{k\psi}\rangle\!\rangle+N(N-1)\langle\!\langle 1^2_{k\psi}\rangle\!\rangle-N^2\langle\!\langle 1_{k\psi}\rangle\!\rangle^2\right\}$. Using Tukey's theorem [10] of random pairing which will be discussed in the next chapter of this paper, then we get

$$\text{(1.2.23)} \qquad \sigma^2(\dot{W}_{k\psi})=\begin{cases}0 & , \text{ for } \psi=1\\ \dfrac{P_k(1-P_k)}{N-1} & , \text{ for } \psi\geqq 2 .\end{cases}$$

Hence, we have

$$\text{(1.2.24)} \qquad t_{k\psi}=\dot{W}_{k\psi}\Big/\sqrt{\frac{P_k(1-P_k)}{N-1}} , \text{ for } \psi\geqq 2 .$$

Using the result

$$\dot{W}_{k1}=\frac{1}{N}\sum_{n=1}^{N} w_{1n}\dot{u}_{kn}=P_k\ , \tag{1.2.25}$$

we get

$$\dot{u}_{k\mu}=\frac{NP_k}{M}(1+\dot{U}_{k\mu}^{(M)})\ , \tag{1.2.26}$$

where

$$\dot{U}_{k\mu}\overset{d}{=}\sum_{\psi=2}^{M}\eta_{\mu\psi}^{(M)}\dot{t}_{k\psi}\sqrt{\frac{1-P_k}{P_k(N-1)}}\ . \tag{1.2.27}$$

1.2.2. Asymptotic joint distribution function of statistics

In virtue of the results in section 1.1.2, for any assigned set of

$$b_{k\mu}=\begin{matrix} b_{11} & b_{12} & \cdots & b_{1M} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{K1} & b_{K2} & \cdots & b_{KM} \end{matrix}\ ,$$

we can evaluate the joint probability distribution function of the set of the statistics

$$\dot{u}_{k\mu}=\begin{matrix} \dot{u}_{11} & \dot{u}_{12} & \cdots & \dot{u}_{1M} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dot{u}_{K1} & \dot{u}_{K2} & \cdots & \dot{u}_{KM} \end{matrix}$$

is given by

$$P\left\{\prod_{k=1}^{K}\prod_{\mu=1}^{M}(\dot{u}_{k\mu}=b_{k\mu})\right\}=\prod_{k=1}^{K}\frac{(NP_k)\,!}{\prod_{\mu=1}^{M}(b_{k\mu})!}\Bigg/\frac{N\,!}{\prod_{\mu=1}^{M}(N_\mu)\,!}\ . \tag{1.2.28}$$

We can evaluate the limiting value of

$$\log P\left\{\prod_{k=1}^{K}\prod_{\mu=1}^{M}(\dot{u}_{k\mu}=b_{k\mu})\right\}=\sum_{k=1}^{K}\log(NP_k)!-\log N! + \sum_{\mu=1}^{M}\log(N_\mu)\,!-\sum_{k=1}^{K}\sum_{\mu=1}^{M}\log b_{k\mu}! \tag{1.2.29}$$

by virtue of the Stiring's formula. After some calculations, we get

$$\log P\left\{\prod_{k=1}^{K}\prod_{\mu=1}^{M}(\dot{u}_{k\mu}=b_{k\mu})\right\}=-\frac{1}{2}(K-1)(M-1)\log(2\pi N) -\frac{1}{2}(M-1)\log\Big(\prod_{k=1}^{K}P_k\Big)+\frac{M}{2}(K-1)\log M -\sum_{k=1}^{K}\sum_{\mu=1}^{M}\Big(b_{k\mu}+\frac{1}{2}\Big)\log(1+B_{k\mu})+\varepsilon_1'\ , \tag{1.2.30}$$

where

$$(1.2.31)\qquad \varepsilon_1'=\frac{1}{12N}\left\{\sum_{k=1}^{K}\frac{1}{P_k}-1+M^2-\sum_{\mu=1}^{M}\sum_{k=1}^{K}\frac{M}{P_k(1+B_{k\mu})}+o\left(N^{-\frac{3}{2}}\right)\right\}$$

and $B_{k\mu}\overset{d}{=}\frac{M}{NP_k}b_{k\mu}-1$. Consequently, we get

$$(1.2.32)\qquad P\left\{\prod_{k=1}^{K}\prod_{\mu=1}^{M}(\dot{u}_{k\mu}=b_{k\mu})\right\}=\frac{M^{\frac{M(K-1)}{2}}}{(2\pi N)^{\frac{(K-1)(M-1)}{2}}\left(\prod_{k=1}^{K}P_k\right)^{\frac{M-1}{2}}}e^{-\sum_{\mu=1}^{M}\sum_{k=1}^{K}h_{k\mu}+\varepsilon_1'}$$

where

$$(1.2.33)\qquad h_{k\mu}=\left(b_{k\mu}+\frac{1}{2}\right)\log(1+B_{k\mu})\ .$$

Substituting the definition

$$B_{k\mu}=\frac{M}{NP_k}b_{k\mu}-1\ ,$$

we get

$$(1.2.34)\qquad h_{k\mu}=\left(\frac{NP_k}{M}B_{k\mu}+\frac{NP_k}{M}+\frac{1}{2}\right)\left\{B_{k\mu}-\frac{B_{k\mu}^2}{2}+\frac{B_{k\mu}^3}{3}-\cdots\right\}\ .$$

Since

$$\dot{U}_{k\mu}=\sqrt{\frac{1-P_k}{P_k(N-1)}}\sum_{\psi=2}^{M}\dot{t}_{k\psi}\,\eta_{\mu\psi}^{(M)}\ ,$$

and

$$\sum_{\mu=1}^{M}\dot{U}_{k\mu}=\sqrt{\frac{1-P_k}{P_k(N-1)}}\sum_{\psi=2}^{M}\dot{t}_{k\psi}\sum_{\mu=1}^{M}\eta_{\mu\psi}^{(M)}=0\ ,$$

then we get

$$(1.2.35)\qquad \sum_{k=1}^{K}\sum_{\mu=1}^{M}h_{k\mu}=\sum_{k=1}^{K}\sum_{\mu=1}^{M}\frac{NP_k}{2}MB_{k\mu}^2+\varepsilon_2'+o(N^{-2/3})\ ,$$

where

$$(1.2.36)\qquad \varepsilon_2'=-\sum\sum\frac{NP_k}{6M}B_{k\mu}^3+\sum\sum\left\{\frac{NP_k}{12M}B_{k\mu}^4-\frac{B_{k\mu}^2}{4}\right\}\ .$$

Conseqnently, we get

THEOREM A

For any assigned set of $b_{k\mu}$ $(k=1,2,\cdots,K;\ \mu=1,2,\cdots,M)$ *the joint distribution function of the statistics* $\dot{u}_{k\mu}$ $(k=1,2,\cdots,K;\ \mu=1,2,\cdots,M)$ *is given by*

$$(1.2.37)\qquad P\left\{\prod_{k=1}^{K}\prod_{\mu=1}^{M}(\dot{u}_{k\mu}=b_{k\mu})\right\}=ce^{-\sum_k\sum_\mu\frac{NP_k}{2M}B_{k\mu}^2+\varepsilon_1'+\varepsilon_2'}\ ,$$

where

$$(1.2.38\text{-}1)\quad c=\frac{M^{\frac{M(K-1)}{2}}}{(2\pi N)^{\frac{(K-1)(M-1)}{2}}\left(\prod_{k=1}^{K}P_k\right)^{\frac{M-1}{2}}}$$

$$(1.2.38\text{-}2)\quad \varepsilon_1'=-\sum_k\sum_\mu\frac{M}{12NP_k(1+B_{k\mu})}+\frac{1}{12N}\left\{\sum_{k=1}^{K}\frac{1}{P_k}-1+M^2\right\}+o(N^{-3/2})$$

$$(1.2.38\text{-}3)\quad \varepsilon_2'=-\frac{1}{6M}\sum_k\sum_\mu NP_kB_{k\mu}^3+\sum_k\sum_\mu\left\{\frac{NP_k}{12M}B_{k\mu}^4-\frac{1}{4}B_{k\mu}^2\right\}+o(N^{-3/2})$$

In the expression (1.2.38), we can put

$$(1.2.39)\quad B'_{k\mu}\overset{d}{=}\frac{NP_k}{M}B_{k\mu}=b_{k\mu}-\frac{NP_k}{M}$$

then

$$(1.2.40)\quad \sum_{\mu=1}^{M}B'_{k\mu}=\sum_{\mu=1}^{M}b_{\mu k}-NP_k=0$$

and

$$(1.2.41)\quad \sum_{k=1}^{K}B'_{k\mu}=\sum_{k=1}^{K}b_{k\mu}-\frac{N}{M}\sum_{k=1}^{K}P_k=0\ .$$

Consequently, we have

$$(1.2.42)\quad \sum_{k=1}^{K}\sum_{\mu=1}^{M}\frac{NP_k}{M}B_{k\mu}^2=\sum_{k=1}^{K}\sum_{\mu=1}^{M}\frac{M}{NP_k}(B'_{k\mu})^2$$

then

$$(1.2.43)\quad P\left\{\prod_{k=1}^{K}\prod_{\mu=1}^{M}(\dot{u}_{k\mu}=b_{k\mu})\right\}=ce^{-\frac{1}{2}\sum_{k=1}^{K}\sum_{\mu=1}^{M}\frac{M}{NP_k}(B_{k\mu})^2+\varepsilon''}\ ,$$

where

$$(1.2.44)\quad c=\frac{M^{\frac{M}{2}(K-1)}}{(2\pi N)^{\frac{1}{2}(K-1)(M-1)}\left(\prod_{k=1}^{K}P_k\right)^{\frac{1}{2}(M-1)}}$$

and

$$(1.2.45)\quad \varepsilon''=\varepsilon_1'+\varepsilon_2'\ .$$

Since we get already in (1.2.16)

$$\dot{u}_{k\mu}=\sum_{\psi=1}^{M}\left(\frac{1}{M}\eta_{\mu\psi}^{(M)}\right)(N\dot{W}_{k\psi})\ ,$$

then we obtain

$$(1.2.46)\quad \sum_{k=1}^{K}\sum_{\mu=1}^{M}\frac{M}{NP_k}B'^2_{k\mu}=\sum_{k=1}^{K}\sum_{\psi=2}^{M}\frac{1}{NP_k}(NA_{k\psi})^2\ ,$$

where $A_{k\psi}$ stands for the assigned set corresponding to the set of statistics $\dot{W}_{k\psi}$. Consequently, we get

(1.2.47) $$P\left\{\prod_{k=1}^{K}\prod_{\psi=2}^{M}(N\dot{W}_{k\psi}=NA_{k\psi})\right\}=c'e^{-\frac{1}{2}\sum_{k=1}^{K}\sum_{\psi=2}^{M}\frac{1}{NP_k}(NA_{k\psi})^2+\varepsilon''},$$

where

(1.2.48) $$c'=\frac{M^{\frac{M}{2}(K-1)}}{(2\pi N)^{\frac{(K-1)(M-1)}{2}}\left(\prod_{k=1}^{K}P_k\right)^{\frac{M-1}{2}}}\left|\frac{\partial(b_{11},\cdots,b_{(K-1)M})}{\partial(NA_{11},\cdots,NA_{(K-1)M})}\right|.$$

In the coefficient c', we put

(1.2.49) $$\left|\frac{\partial(b_{11},\cdots,b_{K-1,M})}{\partial(NA_{11},\cdots,NA_{(K-1)M})}\right| = \begin{vmatrix} |\partial_{1,\psi_1,\psi_2}| & & & \\ & |\partial_{2,\psi_1,\psi_2}| & & \\ & & \ddots & \\ & & & |\partial_{K-1,\psi_1,\psi_2}| \end{vmatrix},$$

where

$$\partial_{k,\psi_1,\psi_2}\overset{d}{=}\frac{\partial b_{k\psi_1}}{\partial(NA_{k\psi_2})},$$

and

$$|\partial_{k,\psi_1,\psi_2}|=\frac{\partial(b_{k1},\cdots,b_{kM})}{\partial(NA_{k1},\cdots,NA_{kM})}.$$

Since $\left\|\frac{\eta_{\mu\psi}^{(M)}}{\sqrt{M}}\right\|$ is an orthogonal matrix, then

$$|\partial_{k,\psi_1,\psi_2}|=\left|\frac{1}{M}\eta_{\mu\psi}^{(M)}\right|=\left(\frac{1}{\sqrt{M}}\right)^M$$

and then we get

(1.2.50) $$\frac{\partial(b_{11},\cdots,b_{K-1,M})}{\partial(NA_{11},\cdots,NA_{K-1,M})}=\left(\frac{1}{\sqrt{M}}\right)^{M(K-1)}.$$

The coefficient (1.2.48) becomes

(1.2.51) $$c'=\frac{1}{(2\pi N)^{\frac{1}{2}(K-1)(M-1)}\left(\prod_{k=1}^{K}P_k\right)^{\frac{M-1}{2}}}$$

then

$$P\left\{\prod_{k=1}^{K}\prod_{\psi=2}^{M}(N\dot{W}_{k\psi}=NA_{k\psi})\right\}=C'e^{-\frac{1}{2}\sum_{k=1}^{K}\sum_{\psi=2}^{M}\frac{1}{NP_k}(NA_{k\psi})^2+\varepsilon''},$$

Here we shall introduce the variable

(1.2.52) $$A'_{k\psi}\overset{d}{=}\sqrt{N}\,A_{k\psi}$$

then the sum of $P\left\{\prod_{k=1}^{K}\prod_{\psi=2}^{M}(N\dot{W}_{k\psi}=NA_{k\psi})\right\}$

(1.2.53) $$\sum\cdots\sum P\left\{\prod_{k=1}^{K}\prod_{\psi=2}^{M}(N\dot{W}_{k\psi}=NA_{k\psi})\right\}$$

$$\rightarrow\quad C''\int\cdots\int e^{-\frac{1}{2}\sum_{k=1}^{K}\sum_{\psi=2}^{M}\frac{1}{P_k}A_{k\psi}^{2\prime}}\,dA'_{12}\cdots dA'_{k-1,M}\ ,\qquad N\rightarrow\infty$$

where

(1.2.54) $$C''=\frac{1}{(2\pi)^{\frac{1}{2}(K-1)(M-1)}\left(\prod_{k=1}^{K}P_k\right)^{\frac{1}{2}(M-1)}}\ .$$

On the other hand, we have

(1.2.55) $$\sum_{k=1}^{K}P_k^{-1}(A'_{\psi k})^2=\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}$$

where $\boldsymbol{A}^T$ is the transposed matrix of $\boldsymbol{A}$ such that

$$\boldsymbol{A}^T=\|A'_{1\psi}\ ,\ A'_{2\psi}\ ,\ \cdots\ ,\ A'_{K-1,\psi}\|$$

and

(1.2.56) $$\boldsymbol{P}=\left\|\begin{matrix}P_1^{-1}+P_K^{-1} & P_K^{-1} & \cdots & P_K^{-1}\\ P_K^{-1} & P_2^{-1}+P_K^{-1} & \cdots & P_K^{-1}\\ P_K^{-1} & & & \vdots\\ \vdots & \vdots & & \vdots\\ P_K^{-1} & P_K^{-1} & \cdots & P_{K-1}^{-1}+P_K^{-1}\end{matrix}\right\|\ .$$

Since we have the determinant of $\boldsymbol{P}$ such as

(1.2.57) $$\det\,\boldsymbol{P}=\left(\prod_{k=1}^{K}P_k^{-1}\right)\Big/\sum_{j=1}^{K}\left(\frac{1}{P_j^{-1}}\right)$$

then the integral in the case that the order of $N^{-1/2}$ is negligible

(1.2.58) $$C''\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{-\frac{1}{2}\sum_{\psi=2}^{M}\sum_{k=1}^{K}\frac{1}{P_k}A_{k\psi}^{\prime 2}}dA'_{12}\cdots dA'_{K-1,M}=1\quad .$$

Consequently in the same cases, we get the moment generating functon of $A_\psi=\sum_{k=1}^{K}\bar{v}_kA_{k\psi}$ corresponding the random variable in the expression (1.2.6), such that

(1.2.59) $$q(A_\psi)=E\left(e^{\theta\sum_{k=1}^{K}\bar{v}_kA_{k\psi}}\right)$$

$$=e^{\frac{\theta^2}{2}\left(\frac{\mu_2}{N}\right)}\ ,$$

where $\mu_2=\sum_{k=1}^{K}\bar{v}_k^2P_k$. That is to say, the limiting distribution of $A_\psi=\sum_{k=1}^{K}\bar{v}_kA_{k\psi}$ is the normal distribution with mean 0, variance μ_2/N. Consequently, the joint limiting probability density function of A_2, A_3, $\cdots$, $A_{\phi+2}$ is given by

(1.2.60)
$$f\left(\sqrt{\frac{N}{\mu_2}}A_2, \sqrt{\frac{N}{\mu_2}}A_3, \cdots, \sqrt{\frac{N}{\mu_2}}A_{\Phi+2}\right) = \frac{1}{(2\pi)^{\frac{(\Phi+2)}{2}}} e^{-\frac{1}{2}\sum_{\psi=2}^{\Phi+2}\left(\sqrt{\frac{N}{\mu_2}}A_\psi\right)^2}$$

2 Generalized symmetric means

21 Symmetric means

First of all, let us consider the following results and notions by Tukey (refer to [9] and [10]) about the random pairings $\{x_{p1}, v_{\pi(1)}\}$, $\{x_{p2}, v_{\pi(2)}\}$, $\cdots$, $\{x_{pN}, v_{\pi(N)}\}$ of two sets of N numbers $x_{p1}, x_{p2}, \cdots, x_{pN}$ and $v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)}, \cdots, v_{\pi(N)}$, where $\pi(1), \pi(2), \cdots, \pi(N)$ is a permutation of the integers 1, 2, $\cdots$, N.

DEFINITION 2.1 ; (1) Let an a-th degree moment of multiplicative pairings be defined by

(2.1.1)
$$<a_p>=\frac{1}{N}\sum X_{pn}^a$$

(2.1.2)
$$X_{pn}=x_{pn}v_{\pi(n)}$$

then we get the averaging over all pairings, which is denoted by the notation "*aver*"

(2.1.3)
$$aver\{<a_p>\}\overset{d}{=}\langle\langle a_p\rangle\rangle =<a_p>^*<a>^{**} ,$$

where

(2.1.4)
$$<a_p>^*\overset{d}{=}\frac{1}{N}\sum_{n=1}^{N}x_{pn}^a$$

and

(2.1.5)
$$<a>^{**}\overset{d}{=}\frac{1}{N}\sum_{n=1}^{N}v_{\pi(n)}^a .$$

(2) Let us define a$(a+b+\cdots+e)$-th degree symmetric mean

(2.1.6)
$$<a_pb_p\cdots e_p>=\frac{1}{N^{(\nu)}}\sum_{1}^{N}{}^{\neq}_{n} X_{pn_1}^a X_{pn_2}^b\cdots X_{pn_\nu}^e$$

of the multiplicative pairings $X_{pn}=x_{pn}v_{\pi(n)}$, where $\sum_1^N{}^{\neq}_n$ means the distinct sum which is a sum taken over all subsequent subscripts, $\{n\}$, $n=1, 2, \cdots, N$, but with subscripts kept different when they are indicated by different letters, such as

(2.1.7)
$$\sum_{1}^{2}{}^{\neq}_{n} x_{n_1}x_{n_2}=x_1x_2+x_2x_1$$

and where $N^{(\nu)}$ is the number of terms in the summation, such that if we have different letters, $n_1, n_2, \cdots, n_\nu$,

(2.1.8) $$N^{(\nu)}=N\cdot(N-1)\cdots(N-\nu+1)\ .$$

Then we have

(2.1.9) $$aver\{<a_p b_p \cdots e_p>\} \overset{d}{=} \langle\langle a_p b_p \cdots e_p \rangle\rangle = <a_p b_p \cdots e_p>^{*}<ab \cdots e>^{**}\ ,$$

where

(2.1.10) $$<a_p b_p \cdots e_p>^{*} \overset{d}{=} \frac{1}{N^{(\nu)}} \sum_{1}^{N}{}_{n}^{\neq} x_{pn_1}^{a}\, x_{pn_2}^{b} \cdots x_{pn_\nu}^{e}$$

and

(2.1.11) $$<ab \cdots e>^{**} \overset{d}{=} \frac{1}{N^{(\nu)}} \sum_{1}^{N}{}_{n}^{\neq} v_{\pi(n_1)}^{a} \cdots v_{\pi(n_\nu)}^{e}\ .$$

2.2 Generalized Symmetric means: straightforward extensions of Tukey's symmetric means

2.2.1 Column Generalized Symmetric means

Let us define a generalized symmetric function of two way array

(2.2.1) $$\begin{vmatrix} Z_{11} \cdot \cdot \cdot Z_{1N} \\ Z_{21} \cdot \cdot \cdot Z_{2N} \end{vmatrix}$$

such that

(2.2.2) $$\begin{matrix} ① \\ ② \end{matrix}\begin{pmatrix} ab \cdots e \\ fg \cdots i \end{pmatrix}_c = \sum{}_{n}^{\neq} Z_{1n_1}^{a} Z_{1n_2}^{b} Z_{2n_1}^{f} Z_{2n_2}^{g} \cdots Z_{1n_m}^{e} Z_{2n_m}^{i}$$

for $(a+b+\cdots+e+f+g+\cdots+i)$-th degree generalized symmetric function with respect to column, with which we shall use the abbreviated notation "c. g. s. f." here-after, where subscript "c" means the distinct sum over the column numbers.

For example, $(a+b+c+d)$-th degree c. g. s. f. is

$$\begin{matrix} ① \\ ② \end{matrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}_c = \sum{}_{n}^{\neq} Z_{1n_1}^{a} Z_{1n_2}^{b} Z_{2n_1}^{c} Z_{2n_2}^{d}\ ,$$

and $(a+b+c+d+e+f)$-th degree c. g. s. f. is

$$\begin{matrix} ① \\ ② \end{matrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_c = \sum{}_{n}^{\neq} Z_{1n_1}^{a} Z_{1n_2}^{b} Z_{1n_3}^{c} Z_{1n_1}^{d} Z_{2n_2}^{e} Z_{2n_3}^{f}$$

and so on. In these abbreviated notations of c. g. s. f.'s, we shall use the preassigned subscripts "①", "②", etc, which represent the row number corresponding one in the population two-way array.

DEFINITION 2.2.: (1) For a two-way array

$$Z_{\phi n} = \begin{vmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdots & \\ Z_{\Phi 1} & Z_{\Phi 2} & \cdots & Z_{\Phi N} \end{vmatrix}, \quad \begin{pmatrix} \phi=1, \cdots, \Phi \\ n=1, \cdots, N \end{pmatrix} \tag{2.2.3}$$

the column generalized symmetric function is defined by

$$\begin{matrix} ① \\ ② \\ ③ \\ \vdots \\ (\Phi) \end{matrix} \begin{pmatrix} a & b & \cdots & e \\ f & g & \cdots & k \\ e & m & \cdots & o \\ \cdots & \cdots & \cdots & \cdots \\ p & q & \cdots & t \end{pmatrix}_c = \sum\nolimits_n^{\neq} Z_{1n_1}^a Z_{1n_2}^b \cdots Z_{1n\nu}^e Z_{2n_1}^f Z_{2n_2}^g \cdots Z_{2n\nu}^k \cdots \cdots Z_{\Phi n_1}^p Z_{\Phi n_2}^q \cdots Z_{\Phi n\nu}^t \,. \tag{2.2.4}$$

(2) As a straightforward extension of Tukey's symmetric means

$$<ab\cdots e> = \frac{[ab\cdots e]}{N^{(\nu)}}\,, \tag{2.2.5}$$

we can define a column generalized symmetric mean

$$\begin{matrix} ① \\ \cdots \\ (\Phi) \end{matrix} \left\langle \begin{matrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ p & q & \cdots & t \end{matrix} \right\rangle_c = \begin{matrix} ① \\ \vdots \\ (\Phi) \end{matrix} \begin{pmatrix} a & \cdots & e \\ \cdot & \cdots & \cdot \\ p & \cdots & t \end{pmatrix}_c \Big/ N^{(\nu)}\,, \tag{2.2.6}$$

which will be abbreviated as c. g. s. m. here-after.

It must be noticed that a c. g. s. f.

$$\begin{matrix} ① \\ \vdots \\ \vdots \\ (\Phi) \end{matrix} \begin{pmatrix} a & \cdots & e \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ p & \cdots & t \end{pmatrix}_c$$

generates many other c. g. s. f.'s by the permutations with respect to columns, but c. g. s. f.'s permuted with respect to rows are not always equal to each other. That is to say, for example, we have

$$\begin{matrix} ① \\ ② \end{matrix} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_c = \begin{matrix} ① \\ ② \end{matrix} \begin{pmatrix} b & a & c \\ e & d & f \end{pmatrix}_c = \cdots = \begin{matrix} ① \\ ② \end{matrix} \begin{pmatrix} c & a & b \\ f & d & e \end{pmatrix}_c \tag{2.2.7}$$

but it is not always held that

$$\begin{matrix} ① \\ ② \end{matrix} \begin{pmatrix} a & b & \cdots & e \\ f & g & \cdots & i \end{pmatrix}_c = \begin{matrix} ① \\ ② \end{matrix} \begin{pmatrix} f & g & \cdots & i \\ a & b & \cdots & e \end{pmatrix}_c \,.$$

2.2.2 Multiplications of c. g. s. f.'s

The multiplication of these c. g. s. f.'s and c. g. s. m.'s is easily obtained as a slight extension of the multiplication of the brackets,

$$(2.2.8)\qquad \begin{aligned}(a)_c(b)_c &= \sum_n X_n^a \sum_m X_m^b \\ &= \sum X_n^{a+b} + \sum{}^{\neq} X_m^a X_p^b \\ &= (a+b)_c + (ab)_c \;.\end{aligned}$$

Thus

$$(2.2.9)\qquad \begin{aligned}\begin{pmatrix} a \\ 0 \end{pmatrix}_c \begin{pmatrix} 0 \\ b \end{pmatrix}_c &= (\sum Z_{fn}^a)_c (\sum Z_{gm}^q)_c \\ &= \sum Z_{fm}^a Z_{gm}^b + \sum{}^{\neq} Z_{fn} Z_{gm} \\ &= \begin{pmatrix} a \\ b \end{pmatrix}_c + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_c \;.\end{aligned}$$

We can simplify these calculations as

$$\begin{pmatrix} a \\ - \end{pmatrix}_c \begin{pmatrix} - \\ b \end{pmatrix}_c = \begin{pmatrix} a \\ b \end{pmatrix}_c + \begin{pmatrix} a & - \\ - & b \end{pmatrix}_c \;,$$

where dashes " − " denote the zero entry

$$(2.2.10)\qquad \begin{pmatrix} a & - \\ - & b \end{pmatrix} \overset{d}{=} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;.$$

For a matrix with more than two rows, in a similar way, we have, for example

$$\begin{aligned}
&\begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - \\ - & c & d \\ e & f & - \\ - & - & - \end{pmatrix}_c
\begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ g & h & - & - \end{pmatrix}_c \\
&= \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - \\ - & c & d \\ e & f & - \\ g & h & - \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - \\ - & c & d \\ e & f & - \\ g & - & h \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - \\ - & c & d \\ e & f & - \\ - & g & h \end{pmatrix}_c \\
&+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - \\ - & c & d \\ e & f & - \\ h & g & - \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - \\ - & c & d \\ e & f & - \\ h & - & g \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - \\ - & c & d \\ e & f & - \\ - & h & g \end{pmatrix}_c \\
&+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - & - \\ - & c & d & - \\ e & f & - & - \\ g & - & - & h \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - & - \\ - & c & d & - \\ e & f & - & - \\ - & g & - & h \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - & - \\ - & c & d & - \\ e & f & - & - \\ - & - & g & h \end{pmatrix}_c \\
&+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - & - \\ - & c & d & - \\ e & f & - & - \\ h & - & - & g \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - & - \\ - & c & d & - \\ e & f & - & - \\ - & h & - & g \end{pmatrix}_c
+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - & - \\ - & c & d & - \\ e & f & - & - \\ - & - & h & g \end{pmatrix}_c \\
&+ \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix}\begin{pmatrix} a & b & - & - & - \\ - & c & d & - & - \\ e & f & - & - & - \\ - & - & - & g & h \end{pmatrix}_c \;.
\end{aligned}$$

LEMMA 2.1 : *The product of two c. g. s. f.'s* $(\varphi_1 \times \nu_1)_c$ *and* $(\varphi_2 \times \nu_2)_c$ *by the* *following rule, such that, for $\nu_1 \geqq \nu_2$, is obtained*

$$(2.2.11)\qquad \begin{array}{c} ① \\ ② \\ \vdots \\ (\varphi_1) \\ (\varphi_1+1) \\ \vdots \\ (\varphi_1+\varphi_2) \end{array} \left(\begin{array}{cc} \boxed{\varphi_1 \times \nu_1} & \\ & \boxed{\varphi_2 \times \nu_2} \end{array} \right)_c$$

$$= \sum{}' \begin{array}{c} ① \\ \vdots \\ (\varphi_1+\varphi_2) \end{array} \Big((\varphi_1+\varphi_2) \times \nu_1 \Big)_c + \cdots + \sum{}'' \begin{array}{c} ① \\ \vdots \\ (\varphi_1+\varphi_2) \end{array} \Big((\varphi_1+\varphi_2) \times (\nu_1+1) \Big)_c$$

$$+ \sum{}''' \begin{array}{c} ① \\ \vdots \\ (\varphi_1+\varphi_2) \end{array} \Big((\varphi_1+\varphi_2) \times (\nu_1+2) \Big)_c + \cdots + \begin{array}{c} ① \\ \vdots \\ (\varphi_1+\varphi_2) \end{array} \left(\begin{array}{cc} \boxed{\varphi_1 \times \nu_1} & \\ & \boxed{\varphi_2 \times \nu_2} \end{array} \right)_c ,$$

where the following remarks should be considered.

(i) $\sum{}'$ *stands for summation over all possible* $(\varphi_1+\varphi_2)$ *rows and* ν_1 *columns g. s. f.'s ( )'s, which are generated by the combination of* ν_2 *in the* ν_1 *positions and permutations of* ν_2 *numbers.*

(ii) *Second summation* "$\sum{}''$" *stands for the summation over all possible* $\varphi_1+\varphi_2$ *rows and* ν_1+1 *columns g. s. f.'s which are generated by the rule that* ν_1+1*th column is chosen from the possible* ν_2 *columns and* ν_2-1 *possible columns are allocated in the possible* ν_1 *column* $\binom{\nu_1}{\nu_2-1}$ *and* ν_2-1 *columns permute each other all possible* $(\nu_2-1)!$ *permutations.*

(iii) *The third summation* "$\sum{}'''$" *stands for the summation over all possible* $\varphi_1+\varphi_2$ *rows and* ν_1+2 *columns g. s. f.'s which are generated by the rule that the possible* ν_1+1*th and* ν_2+2*th columns are chosen from the possible* ν_2 *columns in* $\binom{\nu_2}{2}$ *times and* ν_2-2 *possible columns are allocated in the possible* ν_1 *columns in* $\binom{\nu_1}{\nu_2-2}$ *times and* ν_2-2 *columns permute each other all possible* $(\nu_2-2)!$ *permutations, and so on.*

(iv) *The last term is obviously*

$$\begin{array}{c} ① \\ \vdots \\ (\varphi_1+\varphi_2) \end{array} \left(\begin{array}{cc} \boxed{\varphi_1 \times \nu_1} & \\ & \boxed{\varphi_2 \times \nu_2} \end{array} \right)_c ,$$

in which all ν_2 *columns are pushed out.*

The proof of this lemma being virtually identical with that of (2.2.8) and (2.2.9), will be omitted.

2.2.3 Hooke's g. s. m.

In virtue of R. Hooke's paper [6], we can define a generalized symmetric function of a two-way array $|Z_{\phi n}|$ $(\phi=1, 2, \cdots, \Phi; n=1, 2, \cdots, N)$ such that

(2.2.13)
$$\begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix} = \sum_{f}^{\mp} \begin{matrix} (f_1) \\ (f_2) \\ \\ \\ (f_\varphi) \end{matrix} \begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix}_c .$$

In this expression

$$\begin{matrix} (f_1) \\ \vdots \\ (f_\varphi) \end{matrix} \Big(\quad \Big)_c$$

is the c. g. s. f. of a bisample

$$\begin{Vmatrix} Z_{f_1 n_1} & \cdots & Z_{f_1 n_\nu} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ Z_{f_\varphi n_1} & \cdots & Z_{f_\varphi n_\nu} \end{Vmatrix}, \quad (\varphi \leqq \Phi;\ \nu \leqq N),$$

which is sampled from the population two way array

$$\begin{Vmatrix} Z_{11} & \cdots & Z_{1N} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ Z_{\Phi 1} & \cdots & Z_{\Phi N} \end{Vmatrix} .$$

Furthermore, we have

(2.2.13)
$$\begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix} = \begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix} \Big/ \Phi^{(\varphi)} N^{(\nu)}$$

as g. s. f. and its g. s. m. by R. Hooke.

2.2.4 Averaging over all pairings with respect to g. s. m. and c. g. s. m.'s

Let us extend slightly from the prescribed Tukey's Theorem, i. e.

DEFINITION 2.3: (1) Let us define the random pairings of a set of N vectors $Z_1=(z_{11}, \cdots, z_{\Phi 1}), \cdots, Z_N=(z_{1N}, \cdots, z_{\Phi N})$ and a set of N numbers $v_1, v_2, \cdots, v_N$ such that

$$\begin{pmatrix} Z_1 & Z_2 & \cdots & Z_N \\ v_{\pi(1)} & v_{\pi(2)} & \cdots & v_{\pi(N)} \end{pmatrix},$$

where $\{\pi(n)\}$ $(n=1, 2, \cdots, N)$ is a permutation Π, which is chosen from the possible $N!$ permutations of N numbers with equal probability $1/N!$.

(2) Let us define the multiplicative random pairing

$$Z_{\phi n}=z_{\phi n}v_{\pi(n)} \qquad (\phi=1, 2, \cdots, \Phi;\ n=1, 2, \cdots, N)$$

and generalized symmetric mean (g. s. m.) in sense by Hooke

$$\begin{pmatrix} a & \cdots\cdots & e \\ & \cdots\cdots\cdots\cdots & \\ f & \cdots\cdots & i \end{pmatrix} = \sum{}^{\neq}\sum{}^{\neq} Z^a_{\phi_1 n_1}\cdots\cdots Z^i_{\phi_s n_\nu}/\Phi^{(s)}N^{(\nu)} .$$

LEMMA 2.2: *The averaging over all possible g. s. m. is given by*

(2.2.14)
$$aver \begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix} \overset{d}{=} \left(\begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix}\right)$$
$$= \begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix}^{*} \langle a+\cdots+f,\ b+\cdots+g,\ \cdots,\ e+\cdots+i\rangle^{**} .$$

(proof)

We have

$$aver \begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix} = aver\ \frac{\sum^{\neq}\sum^{\neq} Z^a_{\phi_1 n_1}\cdots Z^i_{\phi_s \nu_n}}{N^{(\nu)}\Phi^{(s)}}$$

from the definition of g. s. m.

$$= \frac{\sum^{\neq}\sum^{\neq} z^a_{\phi_1 n_1}\cdots z^i_{\phi_s n_\nu}}{N^{(\nu)}\Phi^{(s)}}\ aver\ \{\dot{v}_1^{(a+\cdots+f)}\cdots \dot{v}_{n_\nu}^{(e+\cdots+i)}\}$$

from the definition of the random pairing

$$= \begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix}^{*} \langle a+\cdots+f,\ \cdots,\ e+\cdots+i\rangle^{**} .$$

COROLLARY OF LEMMA 4.2: *The averaging over all c. g. s. m.*

$$\begin{matrix} (\phi_1) \\ \cdot \\ (\phi_s) \end{matrix} \begin{pmatrix} a & b & \cdots & e \\ \cdot & \cdot & \cdots & \cdot \\ f & g & \cdots & i \end{pmatrix}_c = \sum{}_n^{\neq} Z^a_{\phi_1 n_1} \cdots Z^i_{\phi_s n_\nu}/N^{(\nu)}$$

is given by

(2.2.20)
$$\operatorname{aver}\begin{matrix}(\phi_1)\\ \cdot\\ (\phi_s)\end{matrix}\left(\begin{matrix} a & b & \cdots & e\\ \cdot & \cdot & \cdots & \cdot\\ f & g & \cdots & i\end{matrix}\right)_c^{d} \stackrel{d}{=} \begin{matrix}(\phi_1)\\ \cdot\\ (\phi_s)\end{matrix}\left(\begin{matrix} a & b & \cdots & e\\ \cdot & \cdot & \cdots & \cdot\\ f & g & \cdots & i\end{matrix}\right)_c$$
$$= \begin{matrix}(\phi_1)\\ \cdot\\ (\phi_s)\end{matrix}\left(\begin{matrix} a & b & \cdots & e\\ \cdot & \cdot & \cdots & \cdot\\ f & g & \cdots & i\end{matrix}\right)_c^{*} <a+\cdots+f,\ \cdots,\ e+\cdots+i>^{**}\ .$$

(proof)

In the similar manner to the proof of Lemma 2.2, we get

$$\operatorname{aver}\begin{matrix}(\phi_1)\\ \cdot\\ (\phi_s)\end{matrix}\left(\begin{matrix} a & \cdots\cdots & e\\ & \cdots\cdots\cdots & \\ f & \cdots\cdots & i\end{matrix}\right)_c = \operatorname{aver}\frac{\sum_n^{\mp} Z^a_{\phi_1 n_1}\cdots Z^i_{\phi_s n_\nu}}{N^{(\nu)}}$$
$$= \frac{\sum_n^{\mp} z^a_{\phi_1 n_1}\cdots z^i_{\phi_s n_\nu}}{N^{(\nu)}}\ \operatorname{aver}\ (v^{a+\cdots+f}_{\pi(n_1)}\cdots v^{e+\cdots+i}_{\pi(n_\nu)})$$
$$= \begin{matrix}(\phi_1)\\ \cdot\\ \cdot\\ (\phi_s)\end{matrix}\left(\begin{matrix} a & \cdots\cdots & e\\ & \cdots\cdots\cdots & \\ & \cdots\cdots\cdots & \\ f & \cdots\cdots & i\end{matrix}\right)_c^{*} <a+\cdots+f,\ \cdots,\ e+\cdots+i>^{**}\ .$$

2.3 Symmetric means of v_n's

2.3.1 Moments of v_n's

Let us consider the symmetric means

(2.3.1) $$\langle\langle ab\cdots e\rangle\rangle^{**}$$

of a sequence

(2.3.2) $$\| v_1,\ v_2,\ \cdots,\ v_N \| = \| \beta_1,\ \cdots,\ \beta_R \| \left\|\begin{matrix}\xi_{11} & \cdots & \xi_{1N}\\ \cdot & \cdots & \cdot\\ \cdot & \cdots & \cdot\\ \xi_{R1} & \cdots & \xi_{RN}\end{matrix}\right\|\ .$$

In virtue of the tables of symmetric function by David and Kendall [4], which will be abbreviated as S. F. Tables hereafter, we can easily calculate these symmetric means, for example

(2.3.3–1) $$<1^2>^{**} = \frac{1}{N^{(2)}}\{-(2)^{**}+(1)^{**2}\}\ ,$$

(2.3.3–2) $$<21>^{**} = \frac{1}{N^{(2)}}\{-(3)^{**}+(2)^{**}(1)^{**}\}\ ,$$

(2.3.3–3) $$<2>^{4**}=\frac{1}{N^{(4)}}\{-6(8)^{**}+8(6)^{**}(2)^{**}+3(4)^{**2}-6(4)^{**}(2)^{**2}+(2)^{**4}\}\ .$$

In these formulas, we need the following moments of the v_n's, for example

(2.3.4) $$(1)^{**}=\sum_1^N(\sum_r\beta_r\xi_{rn})=\sum_r\beta_r\sum_1^N\xi_{rn}=0\ ,$$

$$(2)^{**}=\sum_1^N(\sum_r\beta_r\xi_{rn})^2=\sum\beta_{r1}^2\sum_1^N\xi_{r1n}^2+\sum\beta_{r2}\beta_{r3}\sum_1^N\xi_{r2n}\xi_{r3n}\ .$$

Hence, we can define the moments of v_n's as follows,

(2.3.5) $$\mu_1'(v)\overset{d}{=}\frac{(1)^{**}}{N}=0\ ,\quad \mu_2'(v)\overset{d}{=}\frac{(2)^{**}}{N}=\mu_2\ ,\quad \mu_3'(v)\overset{d}{=}\frac{(3)^{**}}{N}=\mu_3\ ,$$

and so on, where μ_i' means i-th moment of v_n's about the origin and μ_i means ith moment of v_n's about the mean.

In the situation such that the matrix

$$\begin{Vmatrix}\xi_{11} & \cdots & \xi_{1N}\\ \cdot & \cdots & \cdot\\ \cdot & \cdots & \cdot\\ \xi_{R1} & \cdots & \xi_{RN}\end{Vmatrix}$$

is a O.A. in which we have

$$\sum_{n=1}^N\xi_{rn}\xi_{sn}\begin{cases}=0 & r\neq s\\ =N & r=s\end{cases}$$

$$\sum_{n=1}^N\xi_{rn}\xi_{sn}\cdots\xi_{vn}\begin{cases}=0 & r,\ s,\ \cdots,\ v \text{ are orthogonal}\\ =N & r,\ s,\ \cdots,\ v \text{ are alias,}\end{cases}$$

these moments of v_n's become the function only β_r's, such that for example $\mu_4=\sum\beta_r^4+\sum^{+}\beta_r^2\beta_s^2+\sum^{+}_{\text{alias}}\beta_r\beta_s\beta_t\beta_u$ where $\sum^{+}_{\text{alias}}$ means the summuation over all combinations of rows having the alias relation.

2.3.2 Symmetric means of v_n's

We shall present the symmetric means of v_n's in table which shows the formulas such that

2nd column = sum of entries in the 3rd, 4th, $\cdots$, columns and each entry is equal to $N^{a-(b)}$(Sum of terms of μ_i's). In general, we have for the r column symmetric functions

(2.3.6) $$[a\cdots e]=A_1(a)\cdots(e)+A_2(a+b)(c)\cdots(e)+\cdots+A_r(a+b+\cdots+e)$$

where A_1, $\cdots$, A_r are constants obtained by the combinatorial calculations. Hence, in the case that the $s_1{}^*$ unit entries are contained in the symmetric function of v_n's, we have $(1)^{**}=0$ then

Table 2-1 Symmetric means of v_n's

Degree		$o(1)$	$o(N^{-1})$	$o(N^{-2})$	$o(N^{-3})$	$o(N^{-4})$	$o(N^{-5})$
2	$<2>$**	μ_2					
	$<1^2>$**		$N^{1-(2)}$ $-\mu_2$				
4	$<4>$**	μ_4					
	$<2^2>$**	$N^{2-(2)}$ μ_2^2	$N^{1-(2)}$ $-\mu_4$				
	$<31>$**		$N^{1-(2)}$ $-\mu_4$				
	$<211>$**		$N^{2-(3)}$ $-\mu_2^2$	$N^{1-(4)}$ $2\mu_4$			
	$<1^4>$**			$N^{2-(4)}$ $3\mu_2^2$	$N^{1-(4)}$ $-6\mu_4$		
6	$<6>$**	μ_6					
	$<51>$**		$N^{1-(2)}$ $-\mu_6$				
	$<42>$**	$N^{2-(2)}$ $\mu_4\mu_2$	$N^{1-(2)}$ $-\mu_6$				
	$<41^2>$**		$N^{2-(3)}$ $-\mu_4\mu_2$	$N^{1-(3)}$ $2\mu_6$			
	$<3^2>$**	$N^{2-(2)}$ μ_3^2	$N^{1-(2)}$ $-\mu_6$				
	$<321>$**		$N^{2-(3)}$ $-\mu_3^2+\mu_4\mu_2$	$N^{1-(3)}$ $-\mu_6$			
	$<31^3>$**			$N^{2-(4)}$ $2\mu_3^2+3\mu_4\mu_2$	$N^{1-(4)}$ $-6\mu_6$		
	$<2^3>$**	$N^{3-(3)}$ μ_2^3	$N^{2-(3)}$ $-3\mu_4\mu_2$	$N^{1-(3)}$ $2\mu_6$			
	$<2^21^2>$**		$N^{3-(4)}$ $-\mu_2^3$	$N^{2-(4)}$ $2\mu_3^2$ $+5\mu_4\mu_2$	$N^{1-(4)}$ $-6\mu_6$		
	$<21^4>$**			$N^{3-(5)}$ $3\mu_2^3$	$N^{2-(5)}$ $-8\mu_3^2$ $-18\mu_4\mu_2$	$N^{1-(5)}$ $28\mu_6$	
	$<1^6>$**				$N^{3-(6)}$ $-15\mu_2^3$	$N^{2-(6)}$ $40\mu_3^2+$ $90\mu_4\mu_2$	$N^{1-(6)}$ $-120\mu_6$
8	$<8>$**	μ_8					
	$<71>$**	$N^{1-(2)}$ $-\mu_8$					
	$<62>$**	$N^{2-(2)}$ $\mu_6\mu_2$	$N^{1-(2)}$ $-\mu_6$				
	$<61^2>$**		$N^{2-(3)}$ $-\mu_6\mu_2$	$N^{1-(3)}$ $2\mu_8$			

Table 2-1 Continue

Degree		$o(1)$	$o(N^{-1})$	$o(N^{-2})$	$o(N^{-3})$	$o(N^{-4})$	$o(N^{-5})$	$o(N^{-6})$	$o(N^{-7})$
	<53>**	$N^{2-(2)}$ $\mu_5\mu_3$	$N^{1-(2)}$ $-\mu_8$						
	<521>**		$N^{2-(3)}$ $-\mu_5\mu_3$ $-\mu_6\mu_2$	$N^{1-(3)}$ $2\mu_8$					
	<51³>**			$N^{2-(4)}$ $2\mu_5\mu_3$ $3\mu_6\mu_2$	$N^{1-(4)}$ $-6\mu_8$				
	<4²>**	$N^{2-(2)}$ μ_4^2	$N^{1-(2)}$ $-\mu_8$						
	<431>**		$N^{2-(3)}$ $-\mu_4^2$ $-\mu_5\mu_3$	$N^{1-(3)}$ $2\mu_8$					
	<42²>**	$N^{3-(3)}$ $\mu_4\mu_2^2$	$N^{2-(3)}$ $-\mu_4^2$ $-2\mu_6\mu_2$	$N^{1-(3)}$ $2\mu_8$					
	<421²>**		$N^{3-(4)}$ $-\mu_4\mu_2^2$	$N^{2-(4)}$ $2\mu_4^2$ $2\mu_5\mu_3$ $3\mu_6\mu_2$	$N^{1-(4)}$ $-6\mu_8$				
8	<41⁴>**			$N^{3-(5)}$ $3\mu_4\mu_2^2$	$N^{2-(5)}$ $-6\mu_4^2$ $-8\mu_5\mu_3$ $-12\mu_6\mu_2$	$N^{1-(5)}$ $24\mu_8$			
	<3²2>**	$N^{3-(3)}$ $\mu_3^2\mu_2$	$N^{2-(3)}$ $-2\mu_5\mu_3$ $-\mu_6\mu_2$	$N^{1-(3)}$ $2\mu_8$					
	<3²1²>**		$N^{3-(4)}$ $-\mu_3^2\mu_2$	$N^{2-(4)}$ $2\mu_4^2$ $4\mu_5\mu_3$ $\mu_6\mu_2$	$N^{1-(4)}$ $-6\mu_8$				
	<32²1>**		$N^{3-(4)}$ $-2\mu_3^2\mu_2$ $-\mu_4\mu_2^2$	$N^{2-(4)}$ μ_4^2 $4\mu_5\mu_3$ $4\mu_6\mu_2$	$N^{1-(4)}$ $-6\mu_8$				
	<321³>**			$N^{3-(5)}$ $5\mu_3^2\mu$ $3\mu_4\mu_2^2$	$N^{2-(5)}$ $-6\mu_4^2$ $-14\mu_5\mu_3$ $-12\mu_6\mu_2$	$N^{1-(5)}$ $24\mu_8$			
	<31⁵>**				$N^{3-(6)}$ $-20\mu_3^2\mu_2$ $-15\mu_4\mu_2^2$	$N^{2-(6)}$ $30\mu_4^2$ $64\mu_5\mu_3$ $60\mu_6\mu_2$	$N^{1-(6)}$ $-120\mu_8$		
	<2⁴>**	$N^{4-(4)}$ μ_2^4	$N^{3-(4)}$ $-6\mu_4\mu_2^2$	$N^{2-(4)}$ $3\mu_4^2$ $8\mu_6\mu_2$	$N^{1-(4)}$ $-6\mu_8$				
	<2³1²>**		$N^{4-(5)}$ $-\mu_2^4$	$N^{3-(5)}$ $6\mu_3^2\mu_2$ $9\mu_4\mu_2^2$	$N^{2-(5)}$ $-6\mu_4^2$ $-12\mu_5\mu_3$ $-20\mu_6\mu_2$	$N^{1-(5)}$ $24\mu_8$			

Table 2-1 Continue

Degree		$o(1)$	$o(N^{-1})$	$o(N^{-2})$	$o(N^{-3})$	$o(N^{-4})$	$o(N^{-5})$	$o(N^{-6})$	$o(N^{-7})$
	$<2^21^4>^{**}$			$N^{4-(6)}$ $3\mu_2^4$	$N^{3-(6)}$ $-28\mu_3^2\mu_2$ $-33\mu_4\mu_2^2$	$N^{2-(6)}$ $30\mu_4^2$ $64\mu_5\mu_3$ $84\mu_6\mu_2$	$N^{1-(6)}$ $-120\mu_8$		
8	$<21^6>^{**}$				$N^{4-(7)}$ $15\mu_2^4$	$N^{3-(7)}$ $160\mu_3^2\mu_2$ $180\mu_4\mu_2^2$	$N^{2-(7)}$ $-180\mu_4^2$ $-384\mu_5\mu_3$ $-480\mu_6\mu_2$	$N^{1-(7)}$ $720\mu_8$	
	$<1^8>^{**}$					$N^{4-(8)}$ $105\mu_2^4$	$N^{3-(8)}$ $-1120\mu_3^2\mu_2$ $-1260\mu_4\mu_2^2$	$N^{2-(8)}$ $1260\mu_4^2$ $2688\mu_5\mu_3$ $3360\mu_6\mu_2$	$N^{1-(8)}$ $-5040\mu_8$

(2.3.7) $$<a\cdots e\underbrace{1\cdots 1}_{s_1^*}>^{**}=o(N^{-\frac{s_1^*}{2}}) .$$

2.4 Two way array finite population

2.4.1 Symmetric means of a row in O.A.

In virtue of O.A., we have

(2.4.1) $$(a_p)^*=\sum_{n=1}^{N}x_{pn}^a=0 \qquad \text{if “}a\text{” is odd}$$
$$=N \qquad \text{if “}a\text{” is even,}$$

then following the other sort of symmetric functions can be easily obtained in refering to S.F. Tables.

Using these results, we get the symmetric means of one row in O.A.

(2.4.2) $<abc\cdots e>^*=1$ if all $a, b, c, \cdots, e$ are even

(2.4.3) $<abc\cdots e>^*=0$ if the order of the symmetric mean, $(a+b+\cdots+e)$, is odd

(2.4.4) $$<abc\cdots e>^*=\frac{-1}{N-1}, \qquad \text{2 characters are odd}$$

$$=\frac{(-1)(-3)}{(N-1)(N-3)}, \qquad 4 \;\;〃\;\;〃$$

$$=\frac{(-1)(-3)(-5)}{(N-1)(N-3)(N-5)}, \qquad 6 \;\;〃\;\;〃$$

$$=\frac{(-1)(-3)(-5)(-7)}{(N-1)(N-3)(N-5)(N-7)}, \qquad 8 \;\;〃\;\;〃 .$$

In general, we have for the s. f.'s

(2.4.5) $[ab\cdots e]=A_1(a)(b)\cdots(e)+A_2(a+b)(c)\cdots(e)+\cdots+A_r(a+b+\cdots+e)$.

Since odd number has no partition which is constructed by all even entries, then the odd degree s. f. and s. m. denoted as $[(\text{odd})]^*$ and $<(\text{odd})>^*$, are

(2.4.6) $[(\text{odd})]^*=0$

and

(2.4.7) $<(\text{odd})>^*=0$.

Since the even degree s. f. contains even odd entries, we have

(2.4.8) $[ab\cdots e\alpha\beta\cdots\varepsilon]=A_1(a)(b)\cdots(\alpha+\beta)\cdots(\delta+\varepsilon)+\cdots+A_i(a+\cdots+\varepsilon)$,

where a, b, $\cdots$, e stand for even entries and α, β, $\cdots$, ε stand for odd entries. In virtue of formula (2.4.1), we have

(2.4.9) $[a\cdots e\alpha\cdots\varepsilon]^*=o(N^{s_1^*+\frac{s_2^*}{2}})$,

where s_1^*, s_2^* stand for the sum of even and odd entries, respectively. Consequently we get

(2.4.10) $<ab\cdots e\alpha\cdots\varepsilon>^*=o(N^{-\frac{s_2^*}{2}})$.

Furthermore, since the $2K$-th degree s. f. has the highest s_1^* when the all entries are 2, $s_1^*=K$, in the case that the number of entries k is larger than K, we have s_2^* unit entries,

(2.4.12) $s_2^*=k-K$,

then

(2.4.13) $<\overbrace{22\cdots2}^{s_1^*}\overbrace{1\cdots1}^{s_2^*}>^*=o(N^{-\frac{k-K}{2}})$.

2.4.2 Relations between rows of O.A.

So as to calculate the g. s. f.'s of $z_{\phi n}$ ($\phi=1, 2, \cdots, \Phi$; $n=1, 2, \cdots, N$) we shall average over all c. g. s. f.'s of possible bisamples,

$$\begin{vmatrix} z_{f_1 1} & \cdots & z_{f_1 N} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ z_{f_\varphi 1} & \cdots & z_{f_\varphi N} \end{vmatrix} .$$

Let us consider the c. g. s. f.'s of orthogonal array with two levels. In these O.A., we have alias relations (refer to Box & Hunter [2] [3] and Shimada [8]) with respect to particular three or more rows, such that

row No		relation
1	=	①
2	=	②
3	=	③
4	=	⑫

5	=	⑬
6	=	㉓
7	=	(123) .

In this table, the relation column means that for example the individuals in the ⑫ or ⑬ or ㉓ or (123) column are equal to the product of the individuals in the ① and ②, or ① and ③ or ② and ③ columns which are components of the number of relations ⑫ or ⑬ or ㉓ or (123). That is to say, $z_{1n}z_{2n}=z_{4n}$, $z_{2n}z_{3n}=z_{6n}$, and $z_{1n}z_{2n}z_{3n}=z_{7n}$, and so on. These relations are the well known "alias" relations.

In our calculations of c. g. s. f.'s of O.A., we shall represent the c. g. s. f.'s with relation numbers in places of row number so as to distinct the alias relation in these rows to orthogonal relation shall denote "alias" or "orth" at the bottom of c. g. s. f.'s and c. g. s. m.'s. With respect to the relation numbers it must be noticed that the third relation between the several rows sampled from the O. A. exists in addition to the above alias and orthogonal relations. That is to say, the case that the alias relation exists in a part of all these sampled rows, such that for example

(①, ②, ⑫, ③)
(①, ②, ⑫, ③, (123)) } →partially alias

and so on, in addition to the relations such that, for example

①, ②, ③, ④, ——→orthogonal
①, ②, ③, (123), ——→alias

Furthermore, for the orthogonal relation, we have ①, ②, ⋯, (φ) equivalent relation in the permuted row number ①, (φ), ⋯, ⑤ and for the alias relation ①, ②, ⋯, (φ), and (12⋯φ) we have equivalent relation in the permuted relation number, ⑤, ③, ⋯, (12⋯φ), ⋯, ②. On the other hand, in the third relation, we have not equivalent relation in the permuted sequence of the row numbers.

2.4.3 Diagonal c. g. s. f.'s of O.A.

In virtue of orthogonality of O.A., we have

(2.4.14) $$\begin{pmatrix}1\\1\end{pmatrix}_C=0\ ,$$

and in general

(2.4.15) $$\underset{orth}{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_C}=0\ ,$$

(2.4.16) $$\underset{alias}{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_C}=N\ ,$$

where $\underset{orth}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_C}$ represents the c. g. s. f. having rows in a orthogonal relation such that for example ①, ②, and ③, and $\underset{alias}{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_C}$ represents the c. g. s. f. having the rows in an alias relations, such that for example ①, ② and ⑫.

Since the orthogonal or alias relations are held in the permutation of rows, the column g. s. f.'s, which are generated by the permutations of rows are equal to each other in the case the alias relation and orthogonal relation are held in the sampled rows.

On the other hand, for a rows of z_{pn} we can also denote

$$\textcircled{1}(1)_C\textcircled{1}(1)_C = \begin{matrix}\textcircled{1} \\ \textcircled{1}\end{matrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_C + \begin{matrix}\textcircled{1} \\ \textcircled{1}\end{matrix}\begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}_C$$

$$\begin{matrix}\textcircled{1} \\ \textcircled{1}\end{matrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_C\textcircled{1}(1) = \begin{matrix}\textcircled{1} \\ \textcircled{1} \\ \textcircled{1}\end{matrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_C + \begin{matrix}\textcircled{1} \\ \textcircled{1} \\ \textcircled{1}\end{matrix}\begin{pmatrix} 1 & - \\ 1 & - \\ - & 1 \end{pmatrix}_C$$

and so on. In general, all s. f.'s in the S. F. Tables by David and Kendall can be denoted as the c. g. s. f.'s partitioned into several rows so as to partition into unit entries.

Similarly, in the case the alias and orthogonal relations are in our O. A., we have

$$(1)(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}$$

and

$$\underset{alias}{(1)(1)(1)} = \underset{alias}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} + 3\underset{alias}{\begin{pmatrix} 1 & - \\ 1 & - \\ - & 1 \end{pmatrix}} + \underset{alias}{\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}}$$

and so on, where ( )'s are the abbreviated notation of c. g. s. f.'s with no subscript and no pre-assigned row number so as to represent the analogy of s. f.'s for a row.

We can tabulate these relations in the style of the S.F. Tables. such that for example

(2.4.17)

	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & - \\ 1 & - \\ - & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	1	−1	2
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1)$	1	1	−3
$(1)(1)(1)$	1	3	1

and so on.

These tables are, of course, equivalent to the S.F. Tables except the notation of s. f.'s in the 1st column and 1st row, which are partitioned to different rows.

In our third condition, in which the alias relation exists in a part of the sampled rows, we have also the formulas (2.4.17) etc.

Consequently, we can use these tables in the same manner to converge the product-sums, such as $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and so on, into terms of c. g. s. f.'s or vice versa.

Then we get easily

$$\begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}_C = 0 ,$$

$$\underset{orth}{\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}_C} = 0 , \qquad \underset{alias}{\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}_C} = 2N ,$$

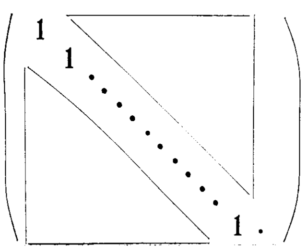
$$\underset{orth}{\begin{pmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{pmatrix}_C} = 0 , \qquad \underset{alias}{\begin{pmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{pmatrix}_C} = -6N ,$$

and

$$\underset{orth}{\begin{pmatrix} 1 & - & - & - & - \\ - & 1 & - & - & - \\ - & - & 1 & - & - \\ - & - & - & 1 & - \\ - & - & - & - & 1 \end{pmatrix}_C} = 0 , \qquad \underset{alias}{\begin{pmatrix} 1 & - & - & - & - \\ - & 1 & - & - & - \\ - & - & 1 & - & - \\ - & - & - & 1 & - \\ - & - & - & - & 1 \end{pmatrix}_C} = 24N .$$

2.4.4 Ordinary c. g. s. f.'s of O.A.

In virtue of our O.A. with two levels, any g. s. f.'s containing odd entries can convert the g. s. f.'s which have at most only one odd entry in any column. Furthermore, any c. g. s. f. with only one odd entry in any column can be obtained as the last term of the expansions of the product of the non-zero c. g. s. f.'s which consist of two sorts of c. g. s. f.'s, such that one of these is $[1^s]$ and the other one is the diagonal c. g. s. f.'s such as



First of all, let us calculate $\begin{pmatrix}1&1&-&-\\-&-&1&1\end{pmatrix}_C$,

then

(2.4.18) $$\begin{pmatrix}1&1&-&-\\-&-&1&1\end{pmatrix}_C=(\,1\ \ 1\,)(\,1\ \ 1\,)-2\begin{matrix}\textcircled{1}\\\textcircled{2}\end{matrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}-4\begin{matrix}\textcircled{1}\\\textcircled{2}\end{matrix}\begin{pmatrix}1&1&-\\-&1&1\end{pmatrix}.$$

Substituting the results

$$\begin{matrix}\textcircled{1}\\\textcircled{2}\end{matrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\textcircled{12}(\,1\ \ 1\,)\quad=-N$$

and

$$\begin{matrix}\textcircled{1}\\\textcircled{2}\end{matrix}\begin{pmatrix}1&1&-\\-&1&1\end{pmatrix}=\begin{matrix}\textcircled{1}\\\textcircled{2}\\\textcircled{12}\end{matrix}\begin{pmatrix}1&-&-\\-&1&-\\-&-&1\end{pmatrix}=2N,$$

then we get

$$\begin{matrix}\textcircled{1}\\\textcircled{2}\end{matrix}\begin{pmatrix}1&1&-&-\\-&-&1&1\end{pmatrix}=N^2-6N.$$

Using these symbolic recurrence procedures, we can calculate the g. s. f.'s in refering to the above c. g. s. f. Tables.

On the other hand, we have a theorem, such that, the order of N of the product, $F(N)$, of non-zero polynomials, $f_1(N), \cdots, f_l(N)$, is equal to the sum of these orders of the original polynomials. In virtue of the theorem, with respect to the all g. s. m.'s which are derived from the multiplication of several non-zero c. g. s. f.'s, we have

(2.4.19) $$[(s\times h)]^*_C=o(N^{s^*-h}),$$

where $[(s\times h)]^*$ is a c. g. s. m. of our O.A. with s rows and h columns and s^* stands for the sum of order of N in the original non-zero c. g. s. f.'s. Furthermore, we have

$$\begin{pmatrix}1&-&-&-&-&-\\-&1&-&-&-&-\\-&-&1&-&-&-\\-&-&-&1&-&-\\-&-&-&-&1&-\\-&-&-&-&-&1\end{pmatrix}=o\Bigg(\underset{alias}{\begin{pmatrix}1&-&-\\-&1&-\\-&-&1\end{pmatrix}}\underset{alias}{\begin{pmatrix}1&-&-\\-&1&-\\-&-&1\end{pmatrix}}\Bigg)=o(N^2),$$

$$\begin{pmatrix}1&-&-&-&-&-&-\\-&1&-&-&-&-&-\\-&-&1&-&-&-&-\\-&-&-&1&-&-&-\\-&-&-&-&1&-&-\\-&-&-&-&-&1&-\\-&-&-&-&-&-&1\end{pmatrix}=o\Bigg(\underset{alias}{\begin{pmatrix}1&-&-&-\\-&1&-&-\\-&-&1&-\\-&-&-&1\end{pmatrix}}\underset{alias}{\begin{pmatrix}1&-&-\\-&1&-\\-&-&1\end{pmatrix}}\Bigg)=o(N^2),$$

and

$$\begin{pmatrix}1&-&-&-&-&-&-&-\\-&1&-&-&-&-&-&-\\-&-&1&-&-&-&-&-\\-&-&-&1&-&-&-&-\\-&-&-&-&1&-&-&-\\-&-&-&-&-&1&-&-\\-&-&-&-&-&-&1&-\\-&-&-&-&-&-&-&1\end{pmatrix}=o\Bigg(\underset{alias}{\begin{pmatrix}1&-&-&-\\-&1&-&-\\-&-&1&-\\-&-&-&1\end{pmatrix}}\underset{alias}{\begin{pmatrix}1&-&-&-\\-&1&-&-\\-&-&1&-\\-&-&-&1\end{pmatrix}}\Bigg)=o(N^2),$$

and, for c. g. s. f. with h columns in diagonal, we have

(2.4.20)
$$\overbrace{\begin{matrix} \diagdown & & 0 \\ & 1 & \\ 0 & & \diagdown \end{matrix}}^{h}_{\textit{diagonal}} = o(N^{\frac{h}{3}}) \ .$$

Consequently, refering to the formulas (2.4.9) and (2.4.19), we get

(2.4.21)
$$s^* \leqq \sum_i \left(s_{i1}^* + \frac{s_{i2}^*}{2} \right) ,$$

where s_{i1}^* and s_{i2}^* are the sum of even and odd entries of i-th original c. g. s. f., for example we have $[21111][11] = N^{(5)}N^{(2)} <21111><11>$, and from this

$$s^* \leqq \left(5 - \frac{4}{2}\right) + \left(2 - \frac{2}{2}\right) = 3 + 1 \ ,$$

then we get easily

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ - & - & - & 1 & 1 \end{bmatrix} = o(N^{-1})$$

and

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & - \\ - & - & - & - & 1 & 1 \end{bmatrix} = o(N^{-2}) \ .$$

2.4.6 g. s. m.'s of O.A.

In this section, tables which make possible the calculation of the higher moments of λ_2, will be presented. In virtue of O.A. all g. s. m.'s with h columns can be classified into several g. s. m.'s with h entries which have only one entry in a column as follows.

1 columns:

all entries are even

$$[(s \times 1)]^* = 1$$

2 columns:

all are even

$$[(s \times 2)]^* = 1$$

$$\begin{matrix} \text{①} \\ \text{②} \\ \text{⑫} \end{matrix} \underset{\textit{alias}}{\begin{pmatrix} \cdot & 1 & 1 \\ & 1 & 1 \\ & 1 & 1 \end{pmatrix}^*} = 1$$

$$\begin{matrix} \text{①} \\ \text{②} \end{matrix} \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix}^* = \frac{-1}{N-1}$$

3 columns:

$[(s\times 3)]^* = 1$ all are even

$\begin{matrix}① \\ ②\end{matrix}\begin{bmatrix} 1 & 1 & - \\ - & - & - \end{bmatrix}^* = \frac{-1}{N-1}$

$\begin{matrix}① \\ ②\end{matrix}\begin{bmatrix} 1 & 1 & - \\ - & - & 1 \end{bmatrix}^* = 0$

$\begin{matrix}① \\ ② \\ ⑫\end{matrix}\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}^*_{alias} = \frac{2}{(N-1)(N-2)}$

$\begin{matrix}① \\ ② \\ ③\end{matrix}\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}^* = 0$

4 columns:

$[(s\times 4)]^* = 1$ all are even

$\begin{matrix}① \\ ②\end{matrix}\begin{bmatrix} 1 & 1 & - & - \\ - & - & - & - \end{bmatrix}^* = \frac{-1}{N-1}$

$\begin{matrix}① \\ ②\end{matrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ - & - & - & - \end{bmatrix}^* = \frac{3}{(N-1)(N-3)}$

$\begin{matrix}① \\ ②\end{matrix}\begin{bmatrix} 1 & 1 & - & - \\ - & - & 1 & 1 \end{bmatrix}^* = \frac{N-6}{(N-1)(N-2)(N-3)}$

$\begin{matrix}① \\ ② \\ ⑫\end{matrix}\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}^*_{alias} = \frac{2}{(N-1)(N-2)}$

$\begin{matrix}① \\ ② \\ ③\end{matrix}\begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}^*_{orth} = 0$

$\begin{matrix}① \\ ② \\ ③ \\ ④\end{matrix}\begin{pmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{pmatrix}^*_{alias} = \frac{-6}{(N-1)(N-2)(N-3)}$

$\begin{matrix}① \\ ② \\ ③ \\ ④\end{matrix}\begin{pmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{pmatrix}^*_{orth} = 0$

5 and more columns:

For our present purpose with respect to the calculation of the 4th and less degree moments of λ_2, we have

$$s^* \leqq 4 ,$$

because s^* is the sum of $(s_1^* + s_2^*/2)$'s of the original s. f.'s which are constructed by the partitions of the all even partitions of number 8. In general, we have for K-th moment of λ_2

(2.4.22) $$s^* \leqq K$$

because the original s. f.'s are constructed by the partitions of all even partitions of number "$2K$", and s^* is equal to the sum of the numbers of even and a half of odd entries in these all s. f.'s.

In virtue of formula (2.4.21) and (2.4.22), we get

(2.4.23) $$[(s \times h)]^* = o(N^{-(h-k)}) \qquad \text{for } h>k\,.$$

3. Moments of λ_{1p} and λ_2

3.1 Expansions of λ_{1p}^S and λ_2^K

3.1.1 Expansions of λ_{1p}^S

In virtue of the results in our previous chapter in this paper, we can obtain the S-th degree moment (refer to Fisher [5], Robson [7], Behnken [1], and Wishert [11]) of

(3.1.1) $$\dot{V}_p = \frac{1}{N}\sum x_{pn}\dot{v}_n = \frac{1}{N}\sum X_{pn} = \langle 1_p \rangle\ ,$$

such that

(3.1.2) $$aver\{\dot{V}_p^S\} = aver\{\langle 1_p\rangle^S\}$$

refering to the S.F. Table [4].

Henceforth expanding the formula

$$\lambda_{1p}^S = \left\{\frac{N}{2}(\alpha_p + \dot{V}_p)^2\right\}^S$$ and substituting (3.1.2), we get

(3.1.3–1) $$aver\{\lambda_{1p}\} = \left(\frac{N}{2}\right)\sum_{s=0}^{2}\binom{2}{s}\alpha_p^{2-s}\,aver\{\langle 1_p\rangle^s\}$$

(3.1.3–2) $$aver\{\lambda_{1p}^2\} = \left(\frac{N}{2}\right)^2\sum_{s=0}^{4}\binom{4}{s}\alpha_p^{4-s}\,aver\{\langle 1_p\rangle^s\}$$

(3.1.3–3) $$aver\{\lambda_{1p}^3\} = \left(\frac{N}{2}\right)^3\sum_{s=0}^{6}\binom{6}{s}\alpha_p^{6-s}\,aver\{\langle 1_p\rangle^s\}$$

(3.1.3–4) $$aver\{\lambda_{1p}^4\} = \left(\frac{N}{2}\right)^4\sum_{s=0}^{8}\binom{8}{s}\alpha_p^{8-s}\,aver\{\langle 1_p\rangle^s\}$$

and

(3.1.3–5) $$aver\{\lambda_{1p}^S\}=\left(\frac{N}{2}\right)^S\sum_{s=0}^{2S}\binom{2S}{s}\alpha_p^{2S-s}aver<1_p>^s .$$

Therefore, we have

(3.1.4–1) $$aver\lambda_{1p}=\frac{N}{2}\{\alpha_p^2+2\alpha_p aver<1_p>+aver<1_p>^2\} ,$$

$$=\frac{N}{2}\{\alpha_p^2+2\alpha_p\langle\langle 1_p\rangle\rangle+\frac{1}{N^2}(N\langle\langle 2_p\rangle\rangle+N(N-1)\langle\langle 1_p1_p\rangle\rangle)\}$$

and

(3.1.4–2) $$aver\{\lambda_{1p}^2\}=\left(\frac{N}{2}\right)^2\{\alpha_p^4+4\alpha_p^3 aver<1_p>+6\alpha_p^2 aver<1_p>^2$$

$$+4\alpha_p aver<1_p>^3+aver<1_p>^4\}$$

$$=\left(\frac{N}{2}\right)^2\alpha_p^4+\left(\frac{N}{2}\right)^2\sum_{i=1}^{4}\sum_{j=1}^{4}a_ib_jc_{ij} .$$

Table 3.1.1–1 Table of the 2nd term of righthand side of the formula (3.1.4–2)

	N	$N(N-1)$	$N(N-1)(N-2)$	$N(N-1)(N-2)(N-3)$
$4\alpha_p^3/N$	$\langle\langle 1_p\rangle\rangle$			
$6\alpha_p^2/N^2$	$\langle\langle 2_p\rangle\rangle$	$\langle\langle 1_p^2\rangle\rangle$		
$4\alpha_p/N^3$	$\langle\langle 3_p\rangle\rangle$	$3\langle\langle 2_p1_p\rangle\rangle$	$\langle\langle 1_p^3\rangle\rangle$	
$1/N^4$	$\langle\langle 4_p\rangle\rangle$	$4\langle\langle 3_p1_p\rangle\rangle+3\langle\langle 2_p^2\rangle\rangle$	$6\langle\langle 2_p1_p^2\rangle\rangle$	$\langle\langle 1_p^4\rangle\rangle$

In the above and following Tables, we shall show the double summation $\sum_{i=1}^{I}\sum_{j=1}^{J}a_ib_jc_{ij}$, as in the Table 3.1.1–2.

Table 3.1.1–2 Representation in table of the formula $\sum\sum a_ib_jc_{ij}$

	b_1	b_2	·	·	·	b_j	·	·	·	b_J
a_1	c_{11}	c_{12}	·	·	·	c_{1j}	·	·	·	c_{1J}
a_2	c_{21}	c_{22}	·	·	·	c_{2j}	·	·	·	c_{2J}
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
a_i	c_{i1}	c_{i2}	·	·	·	c_{ij}	·	·	·	c_{iJ}
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
a_I	c_{I1}	c_{I2}	·	·	·	c_{Ij}	·	·	·	c_{IJ}

Furthermore, we get

(3.1.4–3)
$$aver\{\lambda_{1p}^{3}\}=\left(\frac{N}{2}\right)^{3}\{\alpha_p^6+6\alpha_p^5 aver<1_p>+\binom{6}{2}\alpha_p^4 aver<1_p>^2$$
$$+\binom{6}{3}\alpha_p^3 aver<1_p>^3+\binom{6}{4}\alpha_p^2 aver<1_p>^4+6\alpha_p aver<1_p>^5$$
$$+aver<1_p>^6\}$$
$$=\left(\frac{N}{2}\right)^{3}\alpha_p^6+\left(\frac{N}{2}\right)^{3}\sum_{j=1}^{6}\sum_{i=1}^{6}a_ib_jc_{ij}\ ,$$

Table 3.1.1-3 Representation of the $\sum\sum a_ib_jc_{ij}$ term in the formula (3.1.4-3)

	$N^{(1)}$	$N^{(2)}$	$N^{(3)}$	$N^{(4)}$	$N^{(5)}$	$N^{(6)}$
$\frac{6}{N}\alpha_p{}^5$	$\langle\langle 1_p\rangle\rangle$					
$\frac{15}{N^2}\alpha_p{}^4$	$\langle\langle 2_p\rangle\rangle$	$\langle\langle 1_p{}^2\rangle\rangle$				
$\frac{20}{N^3}\alpha_p{}^3$	$\langle\langle 3_p\rangle\rangle$	$3\langle\langle 2_p1_p\rangle\rangle$	$\langle\langle 1_p{}^3\rangle\rangle$			
$\frac{15}{N^4}\alpha_p{}^2$	$\langle\langle 4_p\rangle\rangle$	$4\langle\langle 3_p1_p\rangle\rangle$ $3\langle\langle 2_p{}^2\rangle\rangle$	$6\langle\langle 2_p1_p{}^2\rangle\rangle$	$\langle\langle 1_p{}^4\rangle\rangle$		
$\frac{6}{N^5}\alpha_p$	$\langle\langle 5_p\rangle\rangle$	$5\langle\langle 4_p1_p\rangle\rangle$ $10\langle\langle 3_c2_p\rangle\rangle$	$10\langle\langle 3_p1_p{}^2\rangle\rangle$ $15\langle\langle 2_p{}^21_p\rangle\rangle$	$10\langle\langle 2_p1_p{}^3\rangle\rangle$	$\langle\langle 1_p{}^5\rangle\rangle$	
$\frac{1}{N^6}$	$\langle\langle 6_p\rangle\rangle$	$6\langle\langle 5_p1_p\rangle\rangle$ $15\langle\langle 4_p2_p\rangle\rangle$ $10\langle\langle 3_p{}^2\rangle\rangle$	$15\langle\langle 4_p1_p{}^2\rangle\rangle$ $66\langle\langle 3_p2_p1_p\rangle\rangle$ $15\langle\langle 2_p{}^3\rangle\rangle$	$20\langle\langle 3_p1_p{}^3\rangle\rangle$ $45\langle\langle 2_p{}^21_p{}^2\rangle\rangle$	$15\langle\langle 2_p1_p{}^4\rangle\rangle$	$\langle\langle 1_p{}^6\rangle\rangle$

(3.1.4–4)
$$aver\{\lambda_{1p}^{4}\}=\left(\frac{N}{2}\right)^{4}\{\alpha_p^8+8\alpha_p^7 aver<1_p>+\binom{8}{2}\alpha_p^6 aver<1_p>^2$$
$$+\binom{8}{3}\alpha_p^5 aver<1_p>^3+\binom{8}{4}\alpha_p^4 aver<1_p>^4+\binom{8}{5}\alpha_p^3 aver<1_p>^5$$
$$+\binom{8}{6}\alpha_p^2 aver<1_p>^6+8\alpha_p aver<1_p>^7+aver<1_p>^8\}$$
$$=\left(\frac{N}{2}\right)^{4}\alpha_p^8+\left(\frac{N}{2}\right)^{4}\sum_{i=1}^{8}\sum_{j=1}^{8}a_ib_jc_{ij}\ ,$$

and

(3.1.4–5)
$$aver\{\lambda_{1p}^{S}\}=\left(\frac{N}{2}\right)^{S}\{\alpha_p^{2S}+\binom{2S}{1}\alpha_p^{2S-1}aver<1_p>+$$
$$\cdots+aver<1_p>^{2S}\}$$
$$=\left(\frac{N}{2}\right)^{S}\alpha_p^{2S}+\left(\frac{N}{2}\right)^{S}\sum_{i=1}^{2S}\sum_{j=1}^{2S}a_ib_jc_{ij}\ .$$

Table 3.1.1-4 Representation of the $(\sum\sum a_i b_j c_{ij})$ term of the formula (3.1.4-4)

	N	$N^{(2)}$	$N^{(3)}$	$N^{(4)}$	$N^{(5)}$	$N^{(6)}$	$N^{(7)}$	$N^{(8)}$
$\frac{8}{N}\alpha_p{}^7$	$\langle\langle 1_p \rangle\rangle$							
$\frac{28}{N^1}\alpha_p{}^6$	$\langle\langle 2_p \rangle\rangle$	$\langle\langle 1_p{}^2 \rangle\rangle$						
$\frac{56}{N^3}\alpha_p{}^5$	$\langle\langle 3_p \rangle\rangle$	$3\langle\langle 2_p 1_p \rangle\rangle$	$\langle\langle 1_p{}^3 \rangle\rangle$					
$\frac{70}{N^4}\alpha_p{}^4$	$\langle\langle 4_p \rangle\rangle$	$4\langle\langle 3_p 1_p \rangle\rangle$ $3\langle\langle 2_p{}^2 \rangle\rangle$	$6\langle\langle 2_p 1_p{}^2 \rangle\rangle$	$\langle\langle 1_p{}^4 \rangle\rangle$				
$\frac{56}{N^5}\alpha_p{}^3$	$\langle\langle 5_p \rangle\rangle$	$5\langle\langle 4_p 1_p \rangle\rangle$ $10\langle\langle 3_p 2_p \rangle\rangle$	$10\langle\langle 3_p 1_p{}^2 \rangle\rangle$ $15\langle\langle 2_p{}^2 1_p \rangle\rangle$	$10\langle\langle 2_p 1_p{}^3 \rangle\rangle$	$\langle\langle 1_p{}^5 \rangle\rangle$			
$\frac{28}{N^6}\alpha_p{}^2$	$\langle\langle 6_p \rangle\rangle$	$6\langle\langle 5_p 1_p \rangle\rangle$ $15\langle\langle 4_p 2_p \rangle\rangle$ $10\langle\langle 3_p{}^2 \rangle\rangle$	$15\langle\langle 4_p 1_p{}^2 \rangle\rangle$ $60\langle\langle 3_p 2_p 1_p \rangle\rangle$ $15\langle\langle 2_p{}^3 \rangle\rangle$	$20\langle\langle 3_p 1_p{}^3 \rangle\rangle$ $45\langle\langle 2_p{}^2 1_p{}^2 \rangle\rangle$	$15\langle\langle 2_p 1_p{}^4 \rangle\rangle$	$\langle\langle 1_p{}^6 \rangle\rangle$		
$\frac{8}{N^7}\alpha_p{}^1$	$\langle\langle 7_p \rangle\rangle$	$7\langle\langle 6_p 1_p \rangle\rangle$ $21\langle\langle 5_p 2_p \rangle\rangle$ $35\langle\langle 4_p 3_p \rangle\rangle$	$21\langle\langle 5_p 1_p{}^2 \rangle\rangle$ $105\langle\langle 4_p 2_p 1_p \rangle\rangle$ $70\langle\langle 3_p 1_p{}^2 \rangle\rangle$ $105\langle\langle 3_p 2_p{}^2 \rangle\rangle$	$35\langle\langle 4_p 1_p{}^3 \rangle\rangle$ $210\langle\langle 3_p 2_p 1_p{}^2 \rangle\rangle$ $105\langle\langle 2_p{}^3 1_p \rangle\rangle$	$35\langle\langle 3_p 1_p{}^4 \rangle\rangle$ $105\langle\langle 2_p{}^2 1_p{}^3 \rangle\rangle$	$21\langle\langle 2_p 1_p{}^5 \rangle\rangle$	$\langle\langle 1_p{}^7 \rangle\rangle$	
$\frac{1}{N^8}$	$\langle\langle 8_p \rangle\rangle$	$8\langle\langle 7_p 1_p \rangle\rangle$ $28\langle\langle 6_p 2_p \rangle\rangle$ $56\langle\langle 5_p 3_p \rangle\rangle$ $35\langle\langle 4_p{}^2 \rangle\rangle$	$28\langle\langle 6_p 1_p{}^2 \rangle\rangle$ $168\langle\langle 5_p 2_p 1_p \rangle\rangle$ $280\langle\langle 4_p 3_p 1_p \rangle\rangle$ $210\langle\langle 4_p 2_p{}^2 \rangle\rangle$ $280\langle\langle 3_p{}^2 2_p \rangle\rangle$	$56\langle\langle 5_p 1_p{}^3 \rangle\rangle$ $420\langle\langle 4_p 2_p 1_p{}^2 \rangle\rangle$ $280\langle\langle 3_p{}^2 1_p{}^2 \rangle\rangle$ $840\langle\langle 3_p 2_p{}^2 1_p \rangle\rangle$ $105\langle\langle 2_p{}^4 \rangle\rangle$	$70\langle\langle 4_p 1_p{}^4 \rangle\rangle$ $560\langle\langle 3_p 2_p 1_p{}^3 \rangle\rangle$ $420\langle\langle 2_p{}^3 1_p{}^2 \rangle\rangle$	$56\langle\langle 3_p 1_p{}^5 \rangle\rangle$ $210\langle\langle 2_p{}^2 1_p{}^4 \rangle\rangle$	$28\langle\langle 2_p 1_p{}^6 \rangle\rangle$	$\langle\langle 1_p{}^8 \rangle\rangle$

Table 3.1.1-5 Representation of the $(\sum\sum a_i b_j c_{ij})$ term of the formula (3.1.4-5)

	$N^{(1)}$	·	·	·	·	$N^{(2S)}$
$\frac{1}{N}\binom{2S}{1}\alpha_p{}^{2S-1}$	$\langle\langle 1_p \rangle\rangle$					
$\frac{1}{N^2}\binom{2S}{2}\alpha_p{}^{2S-2}$	$\langle\langle 2_p \rangle\rangle$	·	·	·	·	
$\frac{1}{N^3}\binom{2S}{3}\alpha_p{}^{2S-3}$	$\langle\langle 3_p \rangle\rangle$	·	·	·	·	
⋮	⋮	⋮	⋮	⋮	⋮	
$\frac{1}{N^2}\binom{2S}{2S}$	$\langle\langle 2S_p \rangle\rangle$	·	·	·	·	$\langle\langle 1_p{}^{2S} \rangle\rangle$

3.1.2 Expansions of λ_2^K

As we have seen in Lemma 2 of the previous paper [12], we have another statistic

$$\lambda_2 = \frac{N}{2}\sum_{\phi=1}^{\Phi} \dot{V}_\phi^2 \tag{3.1.5}$$

where

$$\Phi = N - P \ .$$

In virtue of the S. F. Table [4], we can expand the higher powers of λ_2, as follows,

(3.1.6-1) $$\lambda_2 = \frac{N}{2}[2]$$

(3.1.6-2) $$\lambda_2^2 = \left(\frac{N}{2}\right)^2 \{[4]+[22]\}$$

(3.1.6-3) $$\lambda_2^3 = \left(\frac{N}{2}\right)^3 \{[6]+3[42]+[222]\}$$

(3.1.6-4) $$\lambda_2^4 = \left(\frac{N}{2}\right)^4 \{[8]+4[62]+3[4^2]+6[42^2]+[2^4]\} \ ,$$

where square brackets [ ]'s are symmetric functions of $\dot{V}_1^2, \dot{V}_2^2, \cdots, \dot{V}_\Phi^2$, such that

(3.1.7) $$[ab\cdots e] = \sum_f^{\neq} \dot{V}_f^a \dot{V}_g^b \cdots \dot{V}_k^e \ .$$

In general, we have

(3.1.8) $$\lambda_2^K = \left(\frac{N}{2}\right)^K \sum_t \left(\frac{K!}{a!\cdots e!}\right)\left(\frac{1}{\alpha!\cdots \varepsilon!}\right)[(2a)^\alpha \cdots (2e)^\varepsilon] \ ,$$

where $\sum_t$ stands a summation over all even possible partitions of the number $2K$, $[(2a)^\alpha \cdots (2e)^\varepsilon]$. Substituting the relation $\dot{V}_f^2 = \langle 1_f \rangle^2$ to the above square brackets, we get

(3.1.9-1) $$[2] = \sum_f \langle 1_f \rangle^2$$

(3.1.9-2) $$[4] = \sum_f \langle 1_f \rangle^4$$

(3.1.9-3) $$[22] = \sum_f^{\neq} \langle 1_f \rangle^2 \langle 1_g \rangle^2$$

(3.1.9-4) $$[6] = \sum_f \langle 1_f \rangle^6$$

(3.1.9-5) $$[42] = \sum_f^{\neq} \langle 1_f \rangle^4 \langle 1_g \rangle^2$$

(3.1.9-6) $$[222] = \sum_f^{\neq} \langle 1_f \rangle^2 \langle 1_g \rangle^2 \langle 1_h \rangle^2$$

(3.1.9-7) $$[8] = \sum_f \langle 1_f \rangle^8$$

(3.1.9-8) $$[62] = \sum_f^{\neq} \langle 1_f \rangle^6 \langle 1_g \rangle^2$$

(3.1.9-9) $$[44] = \sum_f^{\neq} \langle 1_f \rangle^4 \langle 1_g \rangle^4$$

(3.1.9-10) $$[422] = \sum_f^{\neq} \langle 1_f \rangle^4 \langle 1_g \rangle^2 \langle 1_h \rangle^2$$

and

(3.1.9-11) $$[2222] = \sum_f^{\neq} \langle 1_f \rangle^2 \langle 1_g \rangle^2 \langle 1_h \rangle^2 \langle 1_i \rangle^2 \ .$$

These multiplication (3.1.9-3, 5, 6, 8, 9, and 10) of powers of angle brackets such as $\langle \ \ \rangle$'s are tabulated in Table 3.1.2. In this table, we shall mean that

Table 3.1.2 Multiplications of the angle brackets, $< >$'s

$N^4<1_f>^2<1_g>^2$	$[2_f]\,[2_g]$	$[1_f{}^2]\,[2_g]$	$[2_f]\,[1_g{}^2]$	$[1_f{}^2]\,[1_g{}^2]$	
$N^6<1_f>^2<1_g>^2$ $<1_h>^2$	$[2_f]\,[2_g]\,[2_h]$	$3[1_f{}^2]\,[2_g]\,[2_h]$		$3[1_f{}^2]\,[1_g{}^2]\,[2_h]$ $[1_f{}^2]\,[1_g{}^2]\,[1_h{}^2]$	
$N^8<1_f>^2<1_g>^2$ $<1_h>^2<1_i>^2$	$[2_f]\,[2_g]\,[2_h]\,[2_i]$	$4[1_f{}^2]\,[2_g]\,[2_h]\,[2_i]$ $6[1_f{}^2]\,[2_g]\,[1_h{}^2]\,[2_i]$		$4[1_f{}^2]\,[1_g{}^2]\,[1_h{}^2]\,[2_i]$ $[1_f{}^2]\,[_g{}^2]\,[1_h{}^2]\,[1_i{}^2]$	
$N^6<1_f>^4<1_g>^2$	$[4_f]\,[2_g]$ $[4_f]\,[1_g{}^2]$	$4[3_f1_f]\,[2_g]$ $4[3_f1_f]\,[1_g{}^2]$	$3[2_f{}^2]\,[2_g]$ $3[2_f{}^2]\,[1_g{}^2]$	$6[2_f1_f{}^2]\,[2_g]$ $6[2_f1_f{}^2]\,[1_g{}^2]$	$[1_f{}^4]\,[2_g]$ $[1_f{}^4]\,[1_g{}^2]$
$N^8<1_f>^4<1_g>^2$ $<1_h>^2$	$[4_f]\,[2_g]\,[2_h]$ $[4_f]\,[1_g{}^2]\,[2_h]$ $[4_f]\,[2_g]\,[1_h{}^2]$ $[4_f]\,[1_g{}^2]\,[1_h{}^2]$	$4[3_f1_f]\,[2_g]\,[2_h]$ $4[3_f1_f]\,[1_g{}^2]\,[2_h]$ $4[3_f1_f]\,[2_g]\,[1_h{}^2]$ $4[3_f1_f]\,[1^{g2}]\,[1_h{}^2]$	$3[2_f{}^2]\,[2_g]\,[2_h]$ $3[2_f{}^2]\,[1_g{}^2]\,[2_h]$ $3[2_f{}^2]\,[2_g]\,[1_h{}^2]$ $3[2_f{}^2]\,[1_g{}^2]\,[1_h{}^2]$	$6[2_f1_f{}^2]\,[2_g]\,[2_h]$ $6[2_f1_f{}^2]\,[1_g{}^2]\,[2_h]$ $6[2_f1_f{}^2[\,[2_g]\,[1_h{}^2]$ $6[2_f1_f{}^2]\,[1_g{}^2]\,[1_h{}^2]$	$[1_f{}^4]\,[2_g]\,[2_h]$ $[1_f{}^4]\,[1_g{}^2]\,[2_h]$ $[1_f{}^4]\,[2_g]\,[1_h{}^2]$ $[1_f{}^4]\,[1_g{}^2]\,[1_h{}^2]$
$N^8<1_f>^4<1_g>^4$	$[4_f]\,[4_g]$	$8[3_f1_f]\,[4_g]$ $16[3_f1_f]\,[3_g1_g]$	$6[2_f]\,4_g]$ $24[2_f]\,[3_g1_g]$ $9[2_f]\,[2_g{}^2]$	$12[2_f1_f{}^2]\,[4_g]$ $48[2_f1_f{}^2]\,[3_g1_g]$ $36[2_f1_f{}^2]\,[2_g{}^2]$ $36[2_f1_f{}^2]\,[2_g1_g{}^2]$	$2[1_f{}^4]\,[4_g]$ $8[1_f{}^4]\,[3_g1_g]$ $6[1_f{}^4]\,[2_g{}^2]$ $12[1_f{}^4]\,[2_g1_g{}^2]$ $[1_f{}^4]\,[1_g{}^4]$
$N^8<1_f>^6<1_g>^2$	$[6_f]\,[2_g]$ $60[3_f2_f1_f]\,[2_g]$ $[1_f{}^6]\,[2_g]$ $[6_f]\,[1_g{}^2]$ $60[3_f2_f1_f]\,[1_g{}^2]$ $[1_f{}^6]\,[1_g{}^2]$	$6[5_f1_f]\,[2_g]$ $15[2_f{}^3]\,[2_g]$ $6[5_f1_f]\,[1_g{}^2]$ $15[2_f{}^3]\,[1_g{}^2]$	$15[4_f2_f]\,[2_g]$ $20[3_f1_f{}^3]\,[2_g]$ $15[4_f2_f]\,[1_g{}^2]$ $20[3_f1_f{}^3]\,[1_g{}^2]$	$10[3_f{}^2]\,[2_g]$ $45[2_f{}^21_f{}^2]\,[2_g]$ $10[3_f{}^2]\,[1_g{}^2]$ $45[2_f{}^21_f{}^2]\,[1_g{}^2]$	$15[4_f1_f{}^2]\,[2_g]$ $15[2_f1_f{}^4]\,[2_g]$ $15[4_f1_f{}^2]\,[1_g{}^2]$ $15[2_f1_f{}^4]\,[1_g{}^2]$

(3.1.10) 1st column = sum of terms of 2nd, 3rd, $\cdots$, and last columns for example

(3.1.11) $$N^2\langle 1_f\rangle^2\langle 1_g\rangle^2=[2_f][2_g]+[1_f^2][2_g]+[2_f][1_g^2]+[1_f^2][1_g^2] \;.$$

Furthermore the multiplications of brackets are given by the method such that

$$[1_f^4][1_g^2][2_h]=\begin{matrix}\textcircled{f}\\ \textcircled{g}\\ \textcircled{h}\end{matrix}\begin{pmatrix}1&1&1&1&\\-&-&-&-&-\\-&-&-&-&-\end{pmatrix}_c\begin{matrix}\textcircled{f}\\ \textcircled{g}\\ \textcircled{h}\end{matrix}\begin{pmatrix}-&-\\1&1\\-&-\end{pmatrix}_c\begin{matrix}\textcircled{f}\\ \textcircled{g}\\ \textcircled{h}\end{matrix}\begin{pmatrix}-&-\\-&-\\2&-\end{pmatrix}_c N^{(4)}N^{(2)}N \;.$$

3.1.3 1st, 2nd, 3rd and 4th moments of λ_2

In virtue of these results, we can write down the moments of λ_2, as follows. We shall present the moments of λ_2 in the style of tables, as we have seen in the moments of λ_{1p}. In these tables, we shall mean $\sum_{i=1}^{I}\sum_{j=1}^{J}a_ib_jc_{ij}$, where a_i is i-th row entry in the 1st column b_j is j-th column entry in the 1st row and c_{ij} is the sum of i-th row and j-th column entry. In these expansions we shall use the relation

$$\sum\nolimits_f^{*} aver \begin{matrix}(\textcircled{f_1})\\ \\ (\textcircled{f_\nu})\end{matrix}\begin{pmatrix}a&\cdot&\cdot&\cdot&e\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ g&\cdot&\cdot&\cdot&i\end{pmatrix}_c=\Phi^{(\nu)}\left(\!\begin{pmatrix}a&\cdot&\cdot&\cdot&e\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ g&\cdot&\cdot&\cdot&i\end{pmatrix}\!\right) \;.$$

Furthermore, the coefficients of these g. s. m.'s are obtained by the combinatorial rules which can be seen in the section 2.1.2.

Then we get the higher moments of λ_2, as follows,

(3.1.12–1) $$aver\{\lambda_2\}=\frac{N}{2}\sum\nolimits_f aver\{\langle 1_f\rangle^2\}$$

$$=\frac{1}{2}\sum_{j=1}^{2}a_1b_jc_{1j} \;,$$

Table 3.1.3-1 Representation of the formula (3.1.12-1)

	$N^{(1)-1}$	$N^{(2)-1}$
Φ	$\left(\!\left(\begin{smallmatrix}2&-&-\\-&-&-\end{smallmatrix}\right)\!\right)$	$\left(\!\left(\begin{smallmatrix}1&1\\-&-\end{smallmatrix}\right)\!\right)$

(3.1.12–2) $$aver\{\lambda_2^2\}=\left(\frac{N}{2}\right)^2\{aver\sum\nolimits_f\langle 1_f\rangle^4+aver\sum\nolimits^{*}\langle 1_f\rangle^2\langle 1_g\rangle^2\}$$

$$=\left(\frac{1}{2}\right)^2\sum_{j=1}^{4}\sum_{i=1}^{2}a_ib_jc_{ij} \;,$$

Table 3.1.3-2 Representation of the formula (3.1.12-2)

	NN^{-2}	$\frac{N(N-1)}{N^2}$	$\frac{N(N-1)(N-2)}{N^2}$	$\frac{N(N-1)(N-2)(N-3)}{N^2}$
Φ	$\left[\begin{bmatrix} 4 & - \\ - & - \end{bmatrix}\right]$	$4\left[\begin{bmatrix} 3 & 1 \\ - & - \end{bmatrix}\right]$ $3\left[\begin{bmatrix} 2 & 2 \\ - & - \end{bmatrix}\right]$	$6\left[\begin{bmatrix} 2 & 1 & 1 \\ - & - & - \end{bmatrix}\right]$	$\left[\begin{bmatrix} 1 & 1 & 1 & 1 \\ - & - & - & - \end{bmatrix}\right]$
$\Phi^{(2)}$	$\left[\begin{bmatrix} 2 & - \\ 2 & - \end{bmatrix}\right]$	$\left[\begin{bmatrix} 2 & - \\ - & 2 \end{bmatrix}\right]$ $4\left[\begin{bmatrix} 1 & 1 \\ 2 & - \end{bmatrix}\right]$ $2\left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right]$	$2\left[\begin{bmatrix} 1 & 1 & - \\ - & - & 2 \end{bmatrix}\right]$ $4\left[\begin{bmatrix} 1 & 1 & - \\ - & 1 & 1 \end{bmatrix}\right]$	$\left[\begin{bmatrix} 1 & 1 & - & - \\ - & - & 1 & 1 \end{bmatrix}\right]$

(3.1.12-3)
$$aver\{\lambda_2^3\} = \left(\frac{1}{2}\right)^3 \sum_{j=1}^{6}\sum_{i=1}^{3} a_i b_j c_{ij} \ ,$$

and

(3.1.12-4)
$$aver\{\lambda_2^4\} = \left(\frac{1}{2}\right)^4 \sum_{j=1}^{8}\sum_{i=1}^{4} a_i b_j c_{ij} \ .$$

Table 3.1.3-3 Representation of the expansion of formula 3.1.12-3, aver$\{\lambda_2^3\}$

	$N^{(1)-3}$	$N^{(2)-3}$	$N^{(3)-3}$
Φ	$\left[\begin{bmatrix}6 & -\\ - & -\end{bmatrix}\right]$	$6\left[\begin{bmatrix}5 & 1\\ - & -\end{bmatrix}\right]15\left[\begin{bmatrix}4 & 2\\ - & -\end{bmatrix}\right]10\left[\begin{bmatrix}3 & 3\\ - & -\end{bmatrix}\right]$	$15\left[\begin{bmatrix}4 & 1 & 1\\ - & - & -\end{bmatrix}\right]60\left[\begin{bmatrix}3 & 2 & 1\\ - & - & -\end{bmatrix}\right]15\left[\begin{bmatrix}2 & 2 & 2\\ - & - & -\end{bmatrix}\right]$
$3\Phi^{(2)}$	$\left[\begin{bmatrix}4 & -\\ 2 & -\end{bmatrix}\right]$	$\left[\begin{bmatrix}4 & -\\ - & 2\end{bmatrix}\right]$	
		$2\left[\begin{bmatrix}1 & 1\\ 4 & -\end{bmatrix}\right]$	$\left[\begin{bmatrix}1 & 1 & -\\ - & - & 4\end{bmatrix}\right]$
		$4\left[\begin{bmatrix}3 & 1\\ 2 & -\end{bmatrix}\right]\ 4\left[\begin{bmatrix}3 & 1\\ - & 2\end{bmatrix}\right]$	$4\left[\begin{bmatrix}3 & 1 & -\\ - & - & 2\end{bmatrix}\right]$
		$8\left[\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix}\right]$	$8\left[\begin{bmatrix}1 & 1 & -\\ 1 & - & 3\end{bmatrix}\right]\ 8\left[\begin{bmatrix}1 & 1 & -\\ 3 & - & 1\end{bmatrix}\right]$
		$6\left[\begin{bmatrix}2 & 2\\ 2 & -\end{bmatrix}\right]$	$3\left[\begin{bmatrix}2 & 2 & -\\ - & - & 2\end{bmatrix}\right]$
		$6\left[\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\right]$	$12\left[\begin{bmatrix}1 & 1 & -\\ 2 & - & 2\end{bmatrix}\right]$
			$6\left[\begin{bmatrix}2 & 1 & 1\\ 2 & - & -\end{bmatrix}\right]\ 12\left[\begin{bmatrix}2 & 1 & 1\\ - & 2 & -\end{bmatrix}\right]$
			$12\left[\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & -\end{bmatrix}\right]\ 24\left[\begin{bmatrix}1 & 1 & 2\\ 1 & - & 1\end{bmatrix}\right]$
$\Phi^{(3)}$	$\left[\begin{bmatrix}2 & -\\ 2 & -\\ 2 & -\end{bmatrix}\right]$	$3\left[\begin{bmatrix}2 & -\\ 2 & -\\ - & 2\end{bmatrix}\right]$	$\left[\begin{bmatrix}2 & - & -\\ - & 2 & -\\ - & - & 2\end{bmatrix}\right]$
		$6\left[\begin{bmatrix}1 & 1\\ 2 & -\\ 2 & -\end{bmatrix}\right]\ 6\left[\begin{bmatrix}1 & 1\\ 2 & -\\ - & 2\end{bmatrix}\right]$	$12\left[\begin{bmatrix}1 & 1 & -\\ 2 & - & -\\ - & - & 2\end{bmatrix}\right]\ 3\left[\begin{bmatrix}1 & 1 & -\\ - & - & 2\\ - & - & 2\end{bmatrix}\right]$
		$12\left[\begin{bmatrix}1 & 1\\ 1 & 1\\ 2 & -\end{bmatrix}\right]$	$6\left(\begin{pmatrix}1 & 1 & -\\ 1 & 1 & -\\ - & - & 2\end{pmatrix}\right)\ 12\left(\begin{pmatrix}1 & 1 & -\\ 1 & - & 1\\ 2 & - & -\end{pmatrix}\right)\ 24\left(\begin{pmatrix}1 & 1 & -\\ 1 & - & 1\\ - & - & 2\end{pmatrix}\right)$
		$4\left[\begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & 1\end{bmatrix}\right]$	$24\left[\begin{bmatrix}1 & 1 & -\\ 1 & 1 & -\\ 1 & - & 1\end{bmatrix}\right]\ 8\left[\begin{bmatrix}1 & 1 & -\\ 1 & - & 1\\ - & 1 & 1\end{bmatrix}\right]$

$N^{(4)-3}$	$N^{(5)-3}$	$N^{(6)-3}$
$20\left[\left[\begin{matrix}3&1&1&1\\-&-&-&-\end{matrix}\right]\right]$ $45\left[\left[\begin{matrix}2&2&1&1\\-&-&-&-\end{matrix}\right]\right]$	$15\left[\left[\begin{matrix}2&1&1&1&1\\-&-&-&-&-\end{matrix}\right]\right]$	$\left[\left[\begin{matrix}1&1&1&1&1&1\\-&-&-&-&-&-\end{matrix}\right]\right]$
$4\left[\left[\begin{matrix}1&1&-&-\\-&-&1&3\end{matrix}\right]\right]$		
$3\left[\left[\begin{matrix}1&1&-&-\\-&-&2&2\end{matrix}\right]\right]$		
$6\left[\left[\begin{matrix}2&1&1&-\\-&-&-&2\end{matrix}\right]\right]$		
$24\left[\left[\begin{matrix}1&1&2&-\\1&-&-&1\end{matrix}\right]\right]$ $12\left[\left[\begin{matrix}1&1&2&-\\-&-&1&1\end{matrix}\right]\right]$	$6\left[\left[\begin{matrix}1&1&2&-&-\\-&-&-&1&1\end{matrix}\right]\right]$	
$4\left[\left[\begin{matrix}1&1&1&1\\2&-&-&-\end{matrix}\right]\right]$	$\left[\left[\begin{matrix}1&1&1&1&-\\-&-&-&-&2\end{matrix}\right]\right]$	
$12\left[\left[\begin{matrix}1&1&1&1\\1&1&-&-\end{matrix}\right]\right]$	$8\left[\left[\begin{matrix}1&1&1&1&-\\1&-&-&-&1\end{matrix}\right]\right]$	$\left[\left[\begin{matrix}1&1&1&1&-&-\\-&-&-&-&1&1\end{matrix}\right]\right]$
$3\left[\left[\begin{matrix}1&1&-&-\\-&-&2&-\\-&-&-&2\end{matrix}\right]\right]$		
$12\left[\left[\begin{matrix}1&1&-&-\\1&-&1&-\\-&-&-&2\end{matrix}\right]\right]$ $12\left[\left[\begin{matrix}1&1&-&-\\-&-&1&1\\2&-&-&-\end{matrix}\right]\right]$	$3\left[\left[\begin{matrix}1&1&-&-&-\\-&-&1&1&-\\-&-&-&-&2\end{matrix}\right]\right]$	
$6\left(\!\left(\begin{matrix}1&1&-&-\\1&1&-&-\\-&-&1&1\end{matrix}\right)\!\right)$ $8\left(\begin{matrix}1&1&-&-\\1&-&1&-\\1&-&-&1\end{matrix}\right)$ $24\left(\!\left(\begin{matrix}1&1&-&-\\1&-&1&-\\-&1&-&1\end{matrix}\right)\!\right)$	$12\left[\left[\begin{matrix}1&1&-&-&-\\-&1&1&-&-\\-&-&-&1&1\end{matrix}\right]\right]$	$\left[\left[\begin{matrix}1&1&-&-&-&-\\-&-&1&1&-&-\\-&-&-&-&1&1\end{matrix}\right]\right]$

Table 3.1. 3-4 Expansion of the formula 3.1.12-4, aver$\{\lambda_2^4\}$

	$N^{(1)-4}$	$N^{(2)-4}$	$N^{(3)-4}$	$N^{(4)-4}$
Φ	$\left[\!\left[\begin{smallmatrix}8&-\\-&-\end{smallmatrix}\right]\!\right]$	$8\left[\!\left[\begin{smallmatrix}7&1\\-&-\end{smallmatrix}\right]\!\right]$ $28\left[\!\left[\begin{smallmatrix}6&2\\-&-\end{smallmatrix}\right]\!\right]$ $56\left[\!\left[\begin{smallmatrix}5&3\\-&-\end{smallmatrix}\right]\!\right]$ $35\left[\!\left[\begin{smallmatrix}4&4\\-&-\end{smallmatrix}\right]\!\right]$	$28\left[\!\left[\begin{smallmatrix}6&1&1\\-&-&-\end{smallmatrix}\right]\!\right]$ $168\left[\!\left[\begin{smallmatrix}5&2&1\\-&-&-\end{smallmatrix}\right]\!\right]$ $280\left[\!\left[\begin{smallmatrix}4&3&1\\-&-&-\end{smallmatrix}\right]\!\right]$ $210\left[\!\left[\begin{smallmatrix}4&2&2\\-&-&-\end{smallmatrix}\right]\!\right]$ $280\left[\!\left[\begin{smallmatrix}3&3&2\\-&-&-\end{smallmatrix}\right]\!\right]$	$56\left[\!\left[\begin{smallmatrix}5&1&1&1\\-&-&-&-\end{smallmatrix}\right]\!\right]$ $420\left[\!\left[\begin{smallmatrix}4&2&1&1\\-&-&-&-\end{smallmatrix}\right]\!\right]$ $840\left[\!\left[\begin{smallmatrix}3&2&2&1\\-&-&-&-\end{smallmatrix}\right]\!\right]$ $280\left[\!\left[\begin{smallmatrix}3&3&1&1\\-&-&-&-\end{smallmatrix}\right]\!\right]$ $105\left[\!\left[\begin{smallmatrix}2&2&2&2\\-&-&-&-\end{smallmatrix}\right]\!\right]$
$4\Phi^{(2)}$	$\left[\!\left[\begin{smallmatrix}6&-\\2&-\end{smallmatrix}\right]\!\right]$	$\left[\!\left[\begin{smallmatrix}6&-\\-&2\end{smallmatrix}\right]\!\right]$		
		$6\left[\!\left[\begin{smallmatrix}5&1\\2&-\end{smallmatrix}\right]\!\right]$ $6\left[\!\left[\begin{smallmatrix}5&1\\-&2\end{smallmatrix}\right]\!\right]$	$6\left[\!\left[\begin{smallmatrix}5&1&-\\-&-&2\end{smallmatrix}\right]\!\right]$	
		$15\left[\!\left[\begin{smallmatrix}4&2\\2&-\end{smallmatrix}\right]\!\right]$ $15\left[\!\left[\begin{smallmatrix}4&2\\-&2\end{smallmatrix}\right]\!\right]$	$15\left[\!\left[\begin{smallmatrix}4&2&-\\-&-&2\end{smallmatrix}\right]\!\right]$	
		$20\left[\!\left[\begin{smallmatrix}3&3\\2&-\end{smallmatrix}\right]\!\right]$	$10\left[\!\left[\begin{smallmatrix}3&3&-\\-&-&2\end{smallmatrix}\right]\!\right]$	
			$15\left[\!\left[\begin{smallmatrix}4&1&1\\2&-&-\end{smallmatrix}\right]\!\right]$ $30\left[\!\left[\begin{smallmatrix}4&1&1\\-&2&-\end{smallmatrix}\right]\!\right]$	$15\left[\!\left[\begin{smallmatrix}4&1&1&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$
			$60\left[\!\left[\begin{smallmatrix}3&2&1\\2&-&-\end{smallmatrix}\right]\!\right]$ $60\left[\!\left[\begin{smallmatrix}3&2&1\\-&2&-\end{smallmatrix}\right]\!\right]$ $60\left[\!\left[\begin{smallmatrix}3&2&1\\-&-&2\end{smallmatrix}\right]\!\right]$	$60\left[\!\left[\begin{smallmatrix}3&2&1&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$
			$45\left[\!\left[\begin{smallmatrix}2&2&2\\2&-&-\end{smallmatrix}\right]\!\right]$	$15\left[\!\left[\begin{smallmatrix}2&2&2&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$
				$20\left[\!\left[\begin{smallmatrix}3&1&1&1\\2&-&-&-\end{smallmatrix}\right]\!\right]$ $60\left[\!\left[\begin{smallmatrix}3&1&1&1\\-&2&-&-\end{smallmatrix}\right]\!\right]$
				$90\left[\!\left[\begin{smallmatrix}2&2&1&1\\2&-&-&-\end{smallmatrix}\right]\!\right]$ $90\left[\!\left[\begin{smallmatrix}2&2&1&1\\-&-&2&-\end{smallmatrix}\right]\!\right]$
		$2\left[\!\left[\begin{smallmatrix}1&1\\6&-\end{smallmatrix}\right]\!\right]$	$\left[\!\left[\begin{smallmatrix}1&1&-\\-&-&6\end{smallmatrix}\right]\!\right]$	
		$12\left[\!\left[\begin{smallmatrix}5&1\\1&1\end{smallmatrix}\right]\!\right]$	$12\left[\!\left[\begin{smallmatrix}5&1&-\\1&-&1\end{smallmatrix}\right]\!\right]$ $12\left[\!\left[\begin{smallmatrix}5&1&-\\-&1&1\end{smallmatrix}\right]\!\right]$	$6\left[\!\left[\begin{smallmatrix}5&1&-&-\\-&-&1&1\end{smallmatrix}\right]\!\right]$
		$30\left[\!\left[\begin{smallmatrix}4&2\\1&1\end{smallmatrix}\right]\!\right]$	$30\left[\!\left[\begin{smallmatrix}4&2&-\\1&-&1\end{smallmatrix}\right]\!\right]$ $30\left[\!\left[\begin{smallmatrix}4&2&-\\-&1&1\end{smallmatrix}\right]\!\right]$	$15\left[\!\left[\begin{smallmatrix}4&2&-&-\\-&-&1&1\end{smallmatrix}\right]\!\right]$
		$20\left[\!\left[\begin{smallmatrix}3&3\\1&1\end{smallmatrix}\right]\!\right]$	$40\left[\!\left[\begin{smallmatrix}3&3&-\\-&1&1\end{smallmatrix}\right]\!\right]$	$10\left[\!\left[\begin{smallmatrix}3&3&-&-\\-&-&1&1\end{smallmatrix}\right]\!\right]$
			$60\left[\!\left[\begin{smallmatrix}4&1&1\\1&1&-\end{smallmatrix}\right]\!\right]$ $30\left[\!\left[\begin{smallmatrix}4&1&1\\-&1&1\end{smallmatrix}\right]\!\right]$	$30\left[\!\left[\begin{smallmatrix}4&1&1&-\\1&-&-&1\end{smallmatrix}\right]\!\right]$ $60\left[\!\left[\begin{smallmatrix}4&1&1&-\\-&1&-&1\end{smallmatrix}\right]\!\right]$
			$120\left[\!\left[\begin{smallmatrix}3&2&1\\1&1&-\end{smallmatrix}\right]\!\right]$ $120\left[\!\left[\begin{smallmatrix}3&2&1\\1&-&1\end{smallmatrix}\right]\!\right]$ $120\left[\!\left[\begin{smallmatrix}3&2&1\\-&1&1\end{smallmatrix}\right]\!\right]$	$120\left[\!\left[\begin{smallmatrix}3&2&1&-\\1&-&-&1\end{smallmatrix}\right]\!\right]$ $120\left[\!\left[\begin{smallmatrix}3&2&1&-\\-&1&-&1\end{smallmatrix}\right]\!\right]$ $120\left[\!\left[\begin{smallmatrix}3&2&1&-\\-&-&1&1\end{smallmatrix}\right]\!\right]$

$N^{(5)-4}$	$N^{(6)-4}$	$N^{(7)-4}$	$N^{(8)-4}$
$70\left[\left(\begin{smallmatrix}4&1&1&1&1\\-&-&-&-&-\end{smallmatrix}\right)\right]560\left[\left(\begin{smallmatrix}3&2&1&1&1\\-&-&-&-&-\end{smallmatrix}\right)\right]$ $420\left[\left(\begin{smallmatrix}2&2&2&1&1\\-&-&-&-&-\end{smallmatrix}\right)\right]$	$56\left[\left(\begin{smallmatrix}3&1&1&1&1&1\\-&-&-&-&-&-\end{smallmatrix}\right)\right]$ $210\left[\left(\begin{smallmatrix}2&2&1&1&1&1\\-&-&-&-&-&-\end{smallmatrix}\right)\right]$	$28\left[\left(\begin{smallmatrix}2&1&1&1&1&1&1\\-&-&-&-&-&-&-\end{smallmatrix}\right)\right]$	$\left[\left(\begin{smallmatrix}1&1&1&1&1&1&1&1\\-&-&-&-&-&-&-&-\end{smallmatrix}\right)\right]$
$20\left[\left(\begin{smallmatrix}3&1&1&1&-\\-&-&-&-&2\end{smallmatrix}\right)\right]$			
$45\left[\left(\begin{smallmatrix}2&2&1&1&-\\-&-&-&-&2\end{smallmatrix}\right)\right]$			
$15\left[\left(\begin{smallmatrix}2&1&1&1&1\\2&-&-&-&-\end{smallmatrix}\right)\right]60\left[\left(\begin{smallmatrix}2&1&1&1&1\\-&2&-&-&-\end{smallmatrix}\right)\right]$	$15\left[\left(\begin{smallmatrix}2&1&1&1&1&-\\-&-&-&-&-&2\end{smallmatrix}\right)\right]$		
	$6\left[\left(\begin{smallmatrix}1&1&1&1&1&1\\2&-&-&-&-&-\end{smallmatrix}\right)\right]$	$\left[\left(\begin{smallmatrix}1&1&1&1&1&1&-\\-&-&-&-&-&-&2\end{smallmatrix}\right)\right]$	
$15\left[\left(\begin{smallmatrix}4&1&1&-&-\\-&-&-&1&1\end{smallmatrix}\right)\right]$			
$60\left[\left(\begin{smallmatrix}3&2&1&-&-\\-&-&-&1&1\end{smallmatrix}\right)\right]$			

Table 3.1.3-5 Continue

	$N^{(1)}-4$	$N^{(2)}-4$	$N^{(3)}-4$	$N^{(4)}-4$
			90[[2 2 2 / 1 1 —]]	90[[2 2 2 — / 1 — — 1]]
				120[[3 1 1 1 / 1 1 — —]] 120[[3 1 1 1 / — 1 1 —]]
				90[[2 2 1 1 / 1 1 — —]] 90[[2 2 1 1 / — — 1 1]] 360[[2 2 1 1 / — 1 1 —]]
	[[4 — / 4 —]]	[[4 — / — 4]]		
		8[[3 1 / 4 —]] 8[[3 1 / — 4]]	8[[3 1 — / — — 4]]	
		12[[2 2 / 4 —]]	6[[2 2 — / — — 4]]	
			24[[1 1 2 / 4 — —]] 12[[1 1 2 / — — 4]]	12[[1 1 2 — / — — — 4]]
				8[[1 1 1 1 / 4 — — —]]
		16[[3 1 / 3 1]] 16[[3 1 / 1 3]]	16[[3 1 — / — 3 1]] 16[[3 1 — / — 1 3]] 16[[3 1 — / 3 — 1]] 16[[3 1 — / 1 — 3]]	16[[3 1 — — / — — 3 1]]
$3\Phi^{(2)}$		48[[2 2 / 3 1]]	48[[2 2 — / 3 — 1]] 48[[2 2 — / 1 — 3]]	24[[2 2 — — / — — 3 1]]
			96[[2 1 1 / 3 1 —]] 96[[2 1 1 / — 3 1]] 96[[2 1 1 / 1 3 —]]	48[[2 1 1 — / 3 — — 1]] 96[[2 1 1 — / — 3 — 1]] 48[[2 1 1 — / 1 — — 3]] 96[[2 1 1 — / — 1 — 3]]
				96[[1 1 1 1 / 3 1 — —]]
		18[[2 2 / 2 2]]	36[[2 2 — / 2 — 2]]	9[[2 2 — — / — — 2 2]]
			144[[2 1 1 / 2 2 —]] 72[[2 1 1 / — 2 2]]	72[[2 1 1 — / 2 — — 2]] 144[[2 1 1 — / — 2 — 2]]
				72[[1 1 1 1 / 2 2 — —]]
			72[[2 1 1 / 2 1 1]] 144[[2 1 1 / 1 2 1]]	144[[2 1 1 — / 2 1 — 1]] 144[[2 1 1 — / 1 2 — 1]] 144[[2 1 1 — / — 2 1 1]] 144[[2 1 1 — / 1 1 — 2]] 72[[2 1 1 — / — 1 1 2]]

$N^{(5)-4}$	$N^{(6)-4}$	$N^{(7)-4}$	$N^{(8)-4}$
$15\left[\begin{bmatrix} 2&2&2&-&- \\ -&-&-&1&1 \end{bmatrix}\right]$			
$40\left[\begin{bmatrix} 3&1&1&1&- \\ 1&-&-&-&1 \end{bmatrix}\right]$ $120\left[\begin{bmatrix} 3&1&1&1&- \\ -&1&-&-&1 \end{bmatrix}\right]$	$20\left[\begin{bmatrix} 3&1&1&1&-&- \\ -&-&-&-&1&1 \end{bmatrix}\right]$		
$180\left[\begin{bmatrix} 2&2&1&1&- \\ 1&-&-&-&1 \end{bmatrix}\right]$ $180\left[\begin{bmatrix} 2&2&1&1&- \\ -&-&1&-&1 \end{bmatrix}\right]$	$45\left[\begin{bmatrix} 2&2&1&1&-&- \\ -&-&-&-&1&1 \end{bmatrix}\right]$		
$120\left[\begin{bmatrix} 2&1&1&1&1 \\ 1&1&-&-&- \end{bmatrix}\right]$ $180\left[\begin{bmatrix} 2&1&1&1&1 \\ -&1&1&-&- \end{bmatrix}\right]$	$30\left[\begin{bmatrix} 2&1&1&1&1&- \\ 1&-&-&-&-&1 \end{bmatrix}\right]$ $120\left[\begin{bmatrix} 2&1&1&1&1&- \\ -&1&-&-&-&1 \end{bmatrix}\right]$	$15\left[\begin{bmatrix} 2&1&1&1&1&-&- \\ -&-&-&-&-&1&1 \end{bmatrix}\right]$	
	$30\left[\begin{bmatrix} 1&1&1&1&1&1 \\ 1&1&-&-&-&- \end{bmatrix}\right]$	$12\left[\begin{bmatrix} 1&1&1&1&1&1&- \\ 1&-&-&-&-&-&1 \end{bmatrix}\right]$	$\left[\begin{bmatrix} 1&1&1&1&1&1&-&- \\ -&-&-&-&-&-&1&1 \end{bmatrix}\right]$
$2\left[\begin{bmatrix} 1&1&1&1&- \\ -&-&-&-&4 \end{bmatrix}\right]$			
$48\left[\begin{bmatrix} 2&1&1&-&- \\ -&-&-&3&1 \end{bmatrix}\right]$			
$32\left[\begin{bmatrix} 1&1&1&1&- \\ 3&-&-&-&1 \end{bmatrix}\right]$ $32\left[\begin{bmatrix} 1&1&1&1&- \\ 1&-&-&-&3 \end{bmatrix}\right]$	$8\left[\begin{bmatrix} 1&1&1&1&-&- \\ -&-&-&-&1&3 \end{bmatrix}\right]$		
$36\left[\begin{bmatrix} 2&1&1&-&- \\ -&-&-&2&2 \end{bmatrix}\right]$			
$48\left[\begin{bmatrix} 1&1&1&1&- \\ 2&-&-&-&2 \end{bmatrix}\right]$	$6\left[\begin{bmatrix} 1&1&1&1&-&- \\ -&-&-&-&2&2 \end{bmatrix}\right]$		
$36\left[\begin{bmatrix} 2&1&1&-&- \\ 2&-&-&1&1 \end{bmatrix}\right]$ $72\left[\begin{bmatrix} 2&1&1&-&- \\ -&2&-&1&1 \end{bmatrix}\right]$ $72\left[\begin{bmatrix} 2&1&1&-&- \\ 1&-&-&2&1 \end{bmatrix}\right]$ $144\left[\begin{bmatrix} 2&1&1&-&- \\ -&1&-&2&1 \end{bmatrix}\right]$	$36\left[\begin{bmatrix} 2&1&1&-&-&- \\ -&-&-&2&1&1 \end{bmatrix}\right]$		

Table 3.1.3-6 Continue

	$N^{(1)-4}$	$N^{(2)-4}$	$N^{(3)-4}$	$N^{(4)-4}$
				$288\left[\!\left[\begin{smallmatrix}1&1&1&1\\2&1&1&-\end{smallmatrix}\right]\!\right]$
				$24\left[\!\left[\begin{smallmatrix}1&1&1&1\\1&1&1&1\end{smallmatrix}\right]\!\right]$
	$\left(\!\left(\begin{smallmatrix}4&-\\2&-\\2&-\end{smallmatrix}\right)\!\right)$	$2\left(\!\left(\begin{smallmatrix}4&-\\2&-\\-&2\end{smallmatrix}\right)\!\right)$ $\left(\!\left(\begin{smallmatrix}4&-\\-&2\\-&2\end{smallmatrix}\right)\!\right)$	$\left[\!\left[\begin{smallmatrix}4&-&-\\-&2&-\\-&-&2\end{smallmatrix}\right]\!\right]$	
		$4\left(\!\left(\begin{smallmatrix}3&1\\2&-\\2&-\end{smallmatrix}\right)\!\right)$ $8\left(\!\left(\begin{smallmatrix}3&1\\2&-\\-&2\end{smallmatrix}\right)\!\right)$ $4\left[\!\left[\begin{smallmatrix}3&1\\-&2\\-&2\end{smallmatrix}\right]\!\right]$	$8\left[\!\left[\begin{smallmatrix}3&1&-\\2&-&-\\-&-&2\end{smallmatrix}\right]\!\right]$ $8\left[\!\left[\begin{smallmatrix}3&1&-\\-&2&-\\-&-&2\end{smallmatrix}\right]\!\right]$ $4\left[\!\left[\begin{smallmatrix}3&1&-\\-&-&2\\-&-&2\end{smallmatrix}\right]\!\right]$	$4\left[\!\left[\begin{smallmatrix}3&1&-&-\\-&-&2&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$
		$6\left(\!\left(\begin{smallmatrix}2&2\\2&-\\2&-\end{smallmatrix}\right)\!\right)$ $6\left(\!\left(\begin{smallmatrix}2&2\\2&-\\-&2\end{smallmatrix}\right)\!\right)$	$3\left[\!\left[\begin{smallmatrix}2&2&-\\-&-&2\\-&-&2\end{smallmatrix}\right]\!\right]$ $12\left[\!\left[\begin{smallmatrix}2&2&-\\2&-&-\\-&-&2\end{smallmatrix}\right]\!\right]$	$3\left[\!\left[\begin{smallmatrix}2&2&-&-\\-&-&2&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$
			$6\left[\!\left[\begin{smallmatrix}2&1&1\\2&-&-\\2&-&-\end{smallmatrix}\right]\!\right]$ $24\left[\!\left[\begin{smallmatrix}2&1&1\\2&-&-\\-&2&-\end{smallmatrix}\right]\!\right]$ $12\left[\!\left[\begin{smallmatrix}2&1&1\\-&2&-\\-&2&-\end{smallmatrix}\right]\!\right]$ $12\left[\!\left[\begin{smallmatrix}2&1&1\\-&2&-\\-&-&2\end{smallmatrix}\right]\!\right]$	$12\left[\!\left[\begin{smallmatrix}2&1&1&-\\2&-&-&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$ $24\left[\!\left[\begin{smallmatrix}2&1&1&-\\-&2&-&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$ $6\left[\!\left[\begin{smallmatrix}2&1&1&-\\-&-&-&2\\-&-&-&2\end{smallmatrix}\right]\!\right]$
$6\Phi^{(3)}$				$4\left[\!\left[\begin{smallmatrix}1&1&1&1\\2&-&-&-\\2&-&-&-\end{smallmatrix}\right]\!\right]$ $12\left[\!\left[\begin{smallmatrix}1&1&1&1\\2&-&-&-\\-&2&-&-\end{smallmatrix}\right]\!\right]$
		$4\left(\!\left(\begin{smallmatrix}1&1\\4&-\\2&-\end{smallmatrix}\right)\!\right)$ $4\left(\!\left(\begin{smallmatrix}1&1\\4&-\\-&2\end{smallmatrix}\right)\!\right)$	$4\left[\!\left[\begin{smallmatrix}1&1&-\\4&-&-\\-&-&2\end{smallmatrix}\right]\!\right]$ $4\left[\!\left[\begin{smallmatrix}1&1&-\\-&-&4\\2&-&-\end{smallmatrix}\right]\!\right]$ $2\left[\!\left[\begin{smallmatrix}1&1&-\\-&-&4\\-&-&2\end{smallmatrix}\right]\!\right]$	$2\left[\!\left[\begin{smallmatrix}1&1&-&-\\-&-&4&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$
		$16\left(\!\left(\begin{smallmatrix}3&1\\1&1\\2&-\end{smallmatrix}\right)\!\right)$ $16\left(\!\left(\begin{smallmatrix}3&1\\1&1\\-&2\end{smallmatrix}\right)\!\right)$	$16\left[\!\left[\begin{smallmatrix}3&1&-\\1&1&-\\-&-&2\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-\\1&-&1\\2&-&-\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-\\1&-&1\\-&2&-\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-\\1&-&1\\-&-&2\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-\\-&1&1\\2&-&-\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-\\-&1&1\\-&2&-\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-\\-&1&1\\-&-&2\end{smallmatrix}\right]\!\right]$	$16\left[\!\left[\begin{smallmatrix}3&1&-&-\\1&-&1&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-&-\\-&1&1&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$ $8\left[\!\left[\begin{smallmatrix}3&1&-&-\\-&-&1&1\\2&-&-&-\end{smallmatrix}\right]\!\right]$ $8\left[\!\left[\begin{smallmatrix}3&1&-&-\\-&-&1&1\\-&2&-&-\end{smallmatrix}\right]\!\right]$ $16\left[\!\left[\begin{smallmatrix}3&1&-&-\\-&-&1&1\\-&-&2&-\end{smallmatrix}\right]\!\right]$
		$24\left[\!\left[\begin{smallmatrix}2&2\\1&1\\2&-\end{smallmatrix}\right]\!\right]$	$12\left[\!\left[\begin{smallmatrix}2&2&-\\1&1&-\\-&-&2\end{smallmatrix}\right]\!\right]$ $24\left[\!\left[\begin{smallmatrix}2&2&-\\1&-&1\\2&-&-\end{smallmatrix}\right]\!\right]$ $24\left[\!\left[\begin{smallmatrix}2&2&-\\1&-&1\\-&2&-\end{smallmatrix}\right]\!\right]$ $24\left[\!\left[\begin{smallmatrix}2&2&-\\1&-&1\\-&-&2\end{smallmatrix}\right]\!\right]$	$24\left[\!\left[\begin{smallmatrix}2&2&-&-\\1&-&1&-\\-&-&-&2\end{smallmatrix}\right]\!\right]$ $12\left[\!\left[\begin{smallmatrix}2&2&-&-\\-&-&1&1\\2&-&-&-\end{smallmatrix}\right]\!\right]$ $12\left[\!\left[\begin{smallmatrix}2&2&-&-\\-&-&1&1\\-&-&2&-\end{smallmatrix}\right]\!\right]$

$N^{(5)-4}$	$N^{(6)-4}$	$N^{(7)-4}$	$N^{(8)-4}$
$144\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - \\ 1 & 1 & - & - & 2 \end{pmatrix}\right]288\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - \\ 2 & 1 & - & - & 1 \end{pmatrix}\right]$	$48\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - & - \\ 2 & - & - & - & 1 & 1 \end{pmatrix}\right]$ $96\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - & - \\ 1 & - & - & - & 1 & 2 \end{pmatrix}\right]$	$12\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - & - & - \\ - & - & - & - & 2 & 1 & 1 \end{pmatrix}\right]$	
$96\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 \end{pmatrix}\right]$	$72\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - & - \\ 1 & 1 & - & - & 1 & 1 \end{pmatrix}\right]$	$16\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - & - & - \\ 1 & - & - & - & 1 & 1 & 1 \end{pmatrix}\right]$	$\begin{pmatrix} 1 & 1 & 1 & 1 & - & - & - & - \\ - & - & - & - & 1 & 1 & 1 & 1 \end{pmatrix}$
$6\left[\begin{pmatrix} 2 & 1 & 1 & - & - \\ - & - & - & 2 & - \\ - & - & - & - & 2 \end{pmatrix}\right]$			
$8\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - \\ 2 & - & - & - & - \\ - & - & - & - & 2 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - \\ - & - & - & - & 2 \\ - & - & - & - & 2 \end{pmatrix}\right]$	$\left[\begin{pmatrix} 1 & 1 & 1 & 1 & - & - \\ - & - & - & - & 2 & - \\ - & - & - & - & - & 2 \end{pmatrix}\right]$		
$8\left[\begin{pmatrix} 3 & 1 & - & - & - \\ - & - & 1 & 1 & - \\ - & - & - & - & 2 \end{pmatrix}\right]$			
$6\left[\begin{pmatrix} 2 & 2 & - & - & - \\ - & - & 1 & 1 & - \\ - & - & - & - & 2 \end{pmatrix}\right]$			

Table 3.1.3-7 Continue

	$N^{(1)-4}$	$N^{(2)-4}$	$N^{(3)-4}$	$N^{(4)-4}$
			$48\left[\left[\begin{matrix}2&1&1\\1&1&-\\2&-&-\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1\\1&1&-\\-&2&-\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1\\1&1&-\\-&-&2\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1\\-&1&1\\-&-&2\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&1&1\\-&1&1\\2&-&-\end{matrix}\right]\right]$	$48\left[\left[\begin{matrix}2&1&1&-\\1&1&-&-\\-&-&-&2\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&1&1&-\\-&1&1&-\\-&-&-&2\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&1&1&-\\1&-&-&1\\2&-&-&-\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1&-\\-&1&-&1\\2&-&-&-\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1&-\\1&-&-&1\\-&2&-&-\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1&-\\-&1&-&1\\-&2&-&-\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1&-\\-&-&1&1\\-&2&-&-\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&1&1&-\\1&-&-&1\\-&-&-&2\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&1&1&-\\-&1&-&1\\-&-&-&2\end{matrix}\right]\right]$
$6\Phi^{(3)}$				$48\left[\left[\begin{matrix}1&1&1&1\\1&1&-&-\\2&-&-&-\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}1&1&1&1\\1&1&-&-\\-&-&2&-\end{matrix}\right]\right]$
		$4\left[\left[\begin{matrix}1&1\\1&1\\4&-\end{matrix}\right]\right]$	$2\left[\left[\begin{matrix}1&1&-\\1&1&-\\-&-&4\end{matrix}\right]\right]$ $4\left[\left[\begin{matrix}1&1&-\\1&-&1\\4&-&-\end{matrix}\right]\right]$ $8\left[\left[\begin{matrix}1&1&-\\1&-&1\\-&-&4\end{matrix}\right]\right]$	$4\left[\left[\begin{matrix}1&1&-&-\\1&-&1&-\\-&-&-&4\end{matrix}\right]\right]$ $4\left[\left[\begin{matrix}1&1&-&-\\-&-&1&1\\4&-&-&-\end{matrix}\right]\right]$
		$12\left[\left[\begin{matrix}2&2\\1&1\\1&1\end{matrix}\right]\right]$	$48\left[\left[\begin{matrix}2&2&-\\1&1&-\\1&-&1\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&2&-\\1&-&1\\1&-&1\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&2&-\\1&-&1\\-&1&1\end{matrix}\right]\right]$	$12\left[\left[\begin{matrix}2&2&-&-\\1&1&-&-\\-&-&1&1\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&2&-&-\\1&-&1&-\\1&-&-&1\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&2&-&-\\1&-&1&-\\-&1&-&1\end{matrix}\right]\right]$ $48\left[\left[\begin{matrix}2&2&-&-\\1&-&1&-\\-&-&1&1\end{matrix}\right]\right]$ $6\left[\left[\begin{matrix}2&2&-&-\\-&-&1&1\\-&-&1&1\end{matrix}\right]\right]$

$N^{(5)-4}$	$N^{(6)-4}$	$N^{(7)-4}$	$N^{(8)-4}$
$24\left[\left[\begin{matrix}2&1&1&-&-\\1&-&-&1&-\\-&-&-&-&2\end{matrix}\right]\right]48\left[\left[\begin{matrix}2&1&1&-&-\\-&1&-&1&-\\-&-&-&-&2\end{matrix}\right]\right]$ $12\left[\left[\begin{matrix}2&1&1&-&-\\-&-&-&1&1\\2&-&-&-&-\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}2&1&1&-&-\\-&-&-&1&1\\-&2&-&-&-\end{matrix}\right]\right]24\left[\left[\begin{matrix}2&1&1&-&-\\-&-&-&1&1\\-&-&-&2&-\end{matrix}\right]\right]$	$12\left[\left[\begin{matrix}2&1&1&-&-&-\\-&-&-&1&1&-\\-&-&-&-&-&2\end{matrix}\right]\right]$		
$16\left[\left[\begin{matrix}1&1&1&1&-\\1&-&-&-&1\\2&-&-&-&-\end{matrix}\right]\right]48\left[\left[\begin{matrix}1&1&1&1&-\\1&-&-&-&1\\-&2&-&-&-\end{matrix}\right]\right]$ $16\left[\left[\begin{matrix}1&1&1&1&-\\1&-&-&-&1\\-&-&-&-&2\end{matrix}\right]\right]$ $24\left[\left[\begin{matrix}1&1&1&1&-\\1&1&-&-&-\\-&-&-&-&2\end{matrix}\right]\right]$	$16\left[\left[\begin{matrix}1&1&1&1&-&-\\1&-&-&-&1&-\\-&-&-&-&-&2\end{matrix}\right]\right]$ $8\left[\left[\begin{matrix}1&1&1&1&-&-\\-&-&-&-&1&1\\2&-&-&-&-&-\end{matrix}\right]\right]$ $4\left[\left[\begin{matrix}1&1&1&1&-&-\\-&-&-&-&1&1\\-&-&-&-&2&-\end{matrix}\right]\right]$	$2\left[\left[\begin{matrix}1&1&1&1&-&-&-\\-&-&-&-&1&1&-\\-&-&-&-&-&-&2\end{matrix}\right]\right]$	
$\left[\left[\begin{matrix}1&1&-&-&-\\-&-&1&1&-\\-&-&-&-&4\end{matrix}\right]\right]$			
$24\left[\left[\begin{matrix}2&2&-&-&-\\1&-&1&-&-\\-&-&-&1&1\end{matrix}\right]\right]12\left[\left[\begin{matrix}2&2&-&-&-\\-&-&1&1&-\\-&-&-&1&1\end{matrix}\right]\right]$	$3\left[\left[\begin{matrix}2&2&-&-&-&-\\-&-&1&1&-&-\\-&-&-&-&1&1\end{matrix}\right]\right]$		

Table 3.1.3-8 Continue

	$N^{(1)-4}$	$N^{(2)-4}$	$N^{(3)-4}$	$N^{(4)-4}$
		16[[3 1 / 1 1 / 1 1]]	32[[3 1 — / 1 1 — / 1 — 1]] 32[[3 1 — / 1 1 — / — 1 1]] 16[[3 1 — / 1 — 1 / 1 — 1]] 32[[3 1 — / — 1 1 / 1 — 1]] 16[[3 1 — / — 1 1 / — 1 1]]	16[[3 1 — — / 1 1 — — / — — 1 1]] 16[[3 1 — — / 1 — 1 — / 1 — — 1]] 32[[3 1 — — / 1 — 1 — / — 1 — 1]] 32[[3 1 — — / 1 — 1 — / — — 1 1]] 16[[3 1 — — / — 1 1 — / — 1 — 1]] 32[[3 1 — — / — 1 1 — / — — 1 1]] 8[[3 1 — — / — — 1 1 / — — 1 1]]
6Φ(3)			48[[2 1 1 / 1 1 — / 1 1 —]] 48[[2 1 1 / 1 1 — / 1 — 1]] 96[[2 1 1 / 1 1 — / — 1 1]] 24[[2 1 1 / — 1 1 / — 1 1]]	96[[2 1 1 — / 1 1 — — / 1 — — 1]] 96[[2 1 1 — / 1 1 — — / — 1 — 1]] 96[[2 1 1 — / 1 1 — — / — — 1 1]] 48[[2 1 1 — / — 1 1 — / 1 — — 1]] 96[[2 1 1 — / — 1 1 — / — 1 — 1]] 24[[2 1 1 — / 1 — — 1 / 1 — — 1]] 96[[2 1 1 — / 1 — — 1 / — 1 — 1]] 48[[2 1 1 — / — 1 — 1 / — 1 — 1]] 48[[2 1 1 — / — 1 — 1 / — — 1 1]]
				24[[1 1 1 1 / 1 1 — — / 1 1 — —]] 96[[1 1 1 1 / 1 — 1 — / 1 1 — —]] 24[[1 1 1 1 / 1 1 — — / — — 1 1]]
	((2 — / 2 — / 2 — / 2 —))	4((2 — / 2 — / 2 — / — 2)) 3((2 — / 2 — / — 2 / — 2))	6[[2 — — / 2 — — / — 2 — / — — 2]]	[[2 — — — / — 2 — — / — — 2 — / — — — 2]]
Φ(4)		8((1 1 / 2 — / 2 — / 2 —)) 24((1 1 / 2 — / 2 — / — 2))	24[[1 1 — / 2 — — / 2 — — / — — 2]] 24[[1 1 — / 2 — — / — 2 — / — — 2]] 24[[1 1 — / — — 2 / 2 — — / — — 2]] 4[[1 1 — / — — 2 / — — 2 / — — 2]]	24[[1 1 — — / 2 — — — / — — 2 — / — — — 2]] 12[[1 1 — — / — — 2 — / — — 2 — / — — — 2]]

$N^{(5)-4}$	$N^{(6)-4}$	$N^{(7)-4}$	$N^{(8)-4}$
$16\left[\left[\begin{matrix} 3 & 1 & — & — & — \\ 1 & — & 1 & — & — \\ — & — & — & 1 & 1 \end{matrix}\right]\right]$ $16\left[\left[\begin{matrix} 3 & 1 & — & — & — \\ — & 1 & 1 & — & — \\ — & — & — & 1 & 1 \end{matrix}\right]\right]$ $16\left[\left[\begin{matrix} 3 & 1 & — & — & — \\ — & — & 1 & 1 & — \\ — & — & 1 & — & 1 \end{matrix}\right]\right]$	$4\left[\left[\begin{matrix} 3 & 1 & — & — & — & — \\ — & — & 1 & 1 & — & — \\ — & — & — & — & 1 & 1 \end{matrix}\right]\right]$		
$48\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ 1 & 1 & — & — & — \\ — & — & — & 1 & 1 \end{matrix}\right]\right]$ $24\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ — & 1 & 1 & — & — \\ — & — & — & 1 & 1 \end{matrix}\right]\right]$ $24\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ 1 & — & — & 1 & — \\ 1 & — & — & — & 1 \end{matrix}\right]\right]$ $96\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ 1 & — & — & 1 & — \\ — & 1 & — & — & 1 \end{matrix}\right]\right]$ $48\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ 1 & — & — & 1 & — \\ — & — & — & 1 & 1 \end{matrix}\right]\right]$ $48\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ — & 1 & — & 1 & — \\ — & 1 & — & — & 1 \end{matrix}\right]\right]$ $48\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ — & 1 & — & 1 & — \\ — & — & 1 & — & 1 \end{matrix}\right]\right]$ $96\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ — & 1 & — & 1 & — \\ — & — & — & 1 & 1 \end{matrix}\right]\right]$ $12\left[\left[\begin{matrix} 2 & 1 & 1 & — & — \\ — & — & — & 1 & 1 \\ — & — & — & 1 & 1 \end{matrix}\right]\right]$	$24\left[\left[\begin{matrix} 2 & 1 & 1 & — & — & — \\ 1 & — & — & 1 & — & — \\ — & — & — & — & 1 & 1 \end{matrix}\right]\right]$ $48\left[\left[\begin{matrix} 2 & 1 & 1 & — & — & — \\ — & 1 & — & 1 & — & — \\ — & — & — & — & 1 & 1 \end{matrix}\right]\right]$ $24\left[\left[\begin{matrix} 2 & 1 & 1 & — & — & — \\ — & — & — & 1 & 1 & — \\ — & — & — & 1 & — & 1 \end{matrix}\right]\right]$	$6\left(\left(\begin{matrix} 2 & 1 & 1 & — & — & — & — \\ — & — & — & 1 & 1 & — & — \\ — & — & — & — & — & 1 & 1 \end{matrix}\right)\right)$	
$96\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — \\ 1 & 1 & — & — & — \\ 1 & — & — & — & 1 \end{matrix}\right]\right]$ $96\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — \\ 1 & 1 & — & — & — \\ — & — & 1 & — & 1 \end{matrix}\right]\right]$ $16\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — \\ 1 & — & — & — & 1 \\ 1 & — & — & — & 1 \end{matrix}\right]\right]$ $48\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — \\ 1 & — & — & — & 1 \\ — & 1 & — & — & 1 \end{matrix}\right]\right]$	$24\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — & — \\ 1 & 1 & — & — & — & — \\ — & — & — & — & 1 & 1 \end{matrix}\right]\right]$ $16\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — & — \\ 1 & — & — & — & 1 & — \\ 1 & — & — & — & — & 1 \end{matrix}\right]\right]$ $48\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — & — \\ 1 & — & — & — & 1 & — \\ — & 1 & — & — & — & 1 \end{matrix}\right]\right]$ $32\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — & — \\ 1 & — & — & — & 1 & — \\ — & — & — & — & 1 & 1 \end{matrix}\right]\right]$ $2\left[\left[\begin{matrix} 1 & 1 & 1 & 1 & — & — \\ — & — & — & — & 1 & 1 \\ — & — & — & — & 1 & 1 \end{matrix}\right]\right]$	$16\left(\left(\begin{matrix} 1 & 1 & 1 & 1 & — & — & — \\ 1 & — & — & — & 1 & — & — \\ — & — & — & — & — & 1 & 1 \end{matrix}\right)\right)$ $4\left(\left(\begin{matrix} 1 & 1 & 1 & 1 & — & — & — \\ — & — & — & — & 1 & 1 & — \\ — & — & — & — & — & 1 & 1 \end{matrix}\right)\right)$	$\left(\left(\begin{matrix} 1 & 1 & 1 & 1 & — & — & — & — \\ — & — & — & — & 1 & 1 & — & — \\ — & — & — & — & — & — & 1 & 1 \end{matrix}\right)\right)$
$4\left[\left[\begin{matrix} 1 & 1 & — & — & — \\ — & — & 2 & — & — \\ — & — & — & 2 & — \\ — & — & — & — & 2 \end{matrix}\right]\right]$			

Table 3.1.3-9 Continue

	$N^{(1)-4}$	$N^{(2)-4}$	$N^{(3)-4}$	$N^{(4)-4}$
		24 (1 1; 1 1; 2 —; 2 —) 24 (1 1; 1 1; 2 —; — 2)	48 [[1 1 —; 1 1 —; 2 — —; — — 2]] 12 [[1 1 —; 1 1 —; — — 2; — — 2]] 24 [[1 1 —; 1 — 1; 2 — —; 2 — —]] 96 [[1 1 —; 1 — 1; — 2 —; 2 — —]] 48 [[1 1 —; — 1 1; 2 — —; 2 — —]] 48 [[1 1 —; — 1 1; — — 2; 2 — —]]	12 [[1 1 — —; 1 1 — —; — — 2 —; — — — 2]] 48 [[1 1 — —; 1 — 1 —; 2 — — —; — — — 2]] 96 [[1 1 — —; 1 — 1 —; — 2 — —; — — — 2]] 24 [[1 1 — —; — — 1 1; 2 — — —; 2 — — —]] 72 [[1 1 — —; — — 1 1; — 2 — —; 2 — — —]] 24 [[1 1 — —; 1 — 1 —; — — — 2; — — — 2]]
$\Phi^{(4)}$		32 (1 1; 1 1; 1 1; 2 —)	16 [[1 1 —; 1 1 —; 1 1 —; — — 2]] 96 [[1 1 —; 1 1 —; 1 — 1; 2 — —]] 96 [[1 1 —; 1 1 —; 1 — 1; — 2 —]] 96 [[1 1 —; 1 1 —; 1 — 1; — — 2]] 96 [[1 1 —; 1 — 1; — 1 1; 2 — —]]	96 [[1 1 — —; 1 1 — —; 1 — 1 —; — — — 2]] 48 [[1 1 — —; 1 1 — —; — — 1 1; 2 — — —]] 192 [[1 1 — —; 1 — 1 —; — 1 — 1; 2 — — —]] 48 [[1 1 — —; 1 1 — —; — — 1 1; — — 2 —]] 32 [[1 1 — —; 1 — 1 —; — 1 1 —; — — — 2]] 96 [[1 1 — —; 1 — 1 —; 1 — — 1; — 2 — —]] 192 [[1 1 — —; 1 — 1 —; — — 1 1; — 2 — —]] 32 [[1 1 — —; 1 — 1 —; 1 — — 1; 2 — — —]]
		8 [[1 1; 1 1; 1 1; 1 1]]	48 [[1 1 —; 1 1 —; 1 — 1; 1 — 1]] 96 [[1 1 —; 1 1 —; 1 — 1; — 1 1]] 64 [[1 1 —; 1 1 —; 1 — 1; 1 1 —]]	16 [[1 1 — —; 1 1 — —; 1 1 — —; — — 1 1]] 96 [[1 1 — —; 1 1 — —; 1 — 1 —; 1 — — 1]] 96 [[1 1 — —; 1 1 — —; — 1 1 —; 1 — — 1]] 192 [[1 1 — —; 1 1 — —; 1 — 1 —; — — 1 1]] 12 [[1 1 — —; 1 1 — —; — — 1 1; — — 1 1]] 192 [[1 1 — —; — 1 1 —; 1 — 1 —; 1 — — 1]] 48 [[1 1 — —; 1 — 1 —; — — 1 1; — 1 — 1]]

$N^{(5)}-4$	$N^{(6)}-4$	$N^{(7)}-4$	$N^{(8)}-4$
24 [[1 1 — — —; 1 — 1 — —; — — — 2 —; — — — — 2]] 48 [[1 1 — — —; — — 1 1 —; 2 — — — —; — — — — 2]] 6 [[1 1 — — —; — — 1 1 —; — — — — 2; — — — — 2]]	6 [[1 1 — — — —; — — 1 1 — —; — — — — 2 —; — — — — — 2]]		
24 [[1 1 — — —; 1 1 — — —; — — 1 1 —; — — — — 2]] 32 [[1 1 — — —; 1 — 1 — —; 1 — — 1 —; — — — — 2]] 96 [[1 1 — — —; 1 — 1 — —; — 1 — 1 —; — — — — 2]] 48 [[1 1 — — —; 1 — 1 — —; — — — 1 1; 2 — — — —]] 192 [[1 1 — — —; 1 — 1 — —; — — — 1 1; — 2 — — —]]	48 [[1 1 — — — —; 1 — 1 — — —; — — — 1 1 —; — — — — — 2]] 24 [[1 1 — — — —; — — 1 1 — —; — — — — 1 1; 2 — — — — —]]	4 [1 1 — — — — —; — — 1 1 — — —; — — — — 1 1 —; — — — — — — 2]	
96 [[1 1 — — —; 1 1 — — —; 1 — 1 — —; — — — 1 1]] 48 [[1 1 — — —; 1 1 — — —; — — 1 1 —; — — 1 — 1]] 32 [[1 1 — — —; 1 — 1 — —; — 1 1 — —; — — — 1 1]] 16 [[1 1 — — —; 1 — 1 — —; 1 — — 1 —; 1 — — — 1]] 192 [[1 1 — — —; 1 — 1 — —; 1 — — 1 —; — 1 — — 1]] 192 [[1 1 — — —; 1 — 1 — —; — — 1 1 —; — 1 — — 1]]	12 [[1 1 — — — —; 1 1 — — — —; — — 1 1 — —; — — — — 1 1]] 32 [[1 1 — — — —; 1 — 1 — — —; 1 — — 1 — —; — — — — 1 1]] 96 [[1 1 — — — —; 1 — 1 — — —; — 1 — 1 — —; — — — — 1 1]] 98 [[1 1 — — — —; 1 — 1 — — —; — — — 1 1 —; — — — 1 — 1]]	24 [1 1 — — — — —; — — 1 1 — — —; 1 — — — 1 — —; — — — — — 1 1]	[1 1 — — — — — —; — — 1 1 — — — —; — — — — 1 1 — —; — — — — — — 1 1]

3.2 Approximation of the distribution of λ_{1p} and λ_2

3.2.1 Moments of λ_{1p} and non-central chi-square distribution

Let us calculate the higher moments of λ_{1p}. The S-th moments of λ_{1p} can be represented in the following formula, such that,

$$\text{(3.2.1)} \qquad aver\{\lambda_{1p}^S\} = \left(\frac{1}{2}\right)^S \sum_{r=0}^{2S} \binom{2S}{r} N^{S-\frac{r}{2}} \alpha_p^{2S-r} aver\{(1_p)^r\} \ ,$$

as we can see in the formula (3.1.4–1) (3.1.4–2) (3.1.4–3) and (3.1.4-4) for the 1st, 2nd, 3rd and 4th moments, respectively.

Since, for odd degree s. m., we have

$$\text{(3.2.2)} \qquad <a_p b_p \cdots e_p>^* = 0 \ ,$$

then we can transform the S-th moments of λ_{1p} ,

$$\text{(3.2.3)} \qquad aver\{\lambda_{1p}^S\} = \left(\frac{1}{2}\right)^S \sum_{s=0}^{S} \binom{2S}{2s} N^{S-s} \alpha_p^{2S-2s} S_s$$

and

$$\text{(3.2.4)} \qquad S_s = \sum_{h=1}^{2s} N^{(h)-s} S_{sh} \ ,$$

where S_{sh} means the sum of $2s$-th degree s. m.'s having h entries in s-th row in the following table.

Table 3.2.1 Representation of symmetric means in formula (3.2.3)

row No.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	N^{1-1} $\langle\langle 2_p\rangle\rangle$	$N^{(2)-1}$ $\langle\langle 1_p^2\rangle\rangle$						
2	N^{1-2} $\langle\langle 4_p\rangle\rangle$	$N^{(2)-2}$ $4\langle\langle 3_p1_p\rangle\rangle$ $3\langle\langle 2_p^2\rangle\rangle$	$N^{(3)-2}$ $6\langle\langle 2_p1_p^2\rangle\rangle$	$N^{(4)-2}$ $\langle\langle 1_p^4\rangle\rangle$				
3	N^{1-3} $\langle\langle 6_{\bar{p}}\rangle\rangle$	$N^{(2)-3}$ $6\langle\langle 5_p1_p\rangle\rangle$ $15\langle\langle 4_p2_p\rangle\rangle$ $10\langle\langle 3_p^2\rangle\rangle$	$N^{(3)-3}$ $15\langle\langle 4_p1_p^2\rangle\rangle$ $60\langle\langle 3_p2_p1_p\rangle\rangle$ $15\langle\langle 2_p^2\rangle\rangle$	$N^{(4)-3}$ $20\langle\langle 3_p1^3\rangle\rangle$ $45\langle\langle 2_p^21_p^2\rangle\rangle$	$N^{(5)-3}$ $15\langle\langle 2_p1_p^4\rangle\rangle$	$N^{(6)-3}$ $\langle\langle 1_p^6\rangle\rangle$		
4	N^{1-4} $\langle\langle 8_p\rangle\rangle$	$N^{(2)-4}$ $8\langle\langle 7_p1_p\rangle\rangle$ $28\langle\langle 6_p2_p\rangle\rangle$ $56\langle\langle 5_p3_p\rangle\rangle$ $35\langle\langle 4_p^2\rangle\rangle$	$N^{(3)-4}$ $28\langle\langle 6_p1_p^2\rangle\rangle$ $168\langle\langle 5_p2_p1_p\rangle\rangle$ $280\langle\langle 3_p4_p1_p\rangle\rangle$ $210\langle\langle 4_p2_p^2\rangle\rangle$ $280\langle\langle 3_p^22_p\rangle\rangle$	$N^{(4)-4}$ $56\langle\langle 5_p1_p\rangle\rangle$ $420\langle\langle 4_p2_p1_p^2\rangle\rangle$ $280\langle\langle 3_p^21_p^2\rangle\rangle$ $840\langle\langle 3_p2_p^21_p\rangle\rangle$ $105\langle\langle 2_p^4\rangle\rangle$	$N^{(5)-4}$ $70\langle\langle 4_p1_p^4\rangle\rangle$ $560\langle\langle 3_p2_p1_p^3\rangle\rangle$ $420\langle\langle 2_p^31_p^2\rangle\rangle$	$N^{(6)-4}$ $56\langle\langle 3_p1_p^5\rangle\rangle$ $210\langle\langle 2_p^21_p^4\rangle\rangle$	$N^{(7)-4}$ $28\langle\langle 2_p1_p^6\rangle\rangle$	$N^{(8)-4}$ $\langle\langle 1_p^8\rangle\rangle$

In the above s. m. table, the s. m.'s can be easily calculated as

$$\langle\!\langle a_p b_p \cdots e_p \rangle\!\rangle = \langle a_p b_p \cdots e_p \rangle^{*} \langle ab \cdots e \rangle^{**} ,$$

refering s. m.'s of v_n's and O.A. Then we have

(3.2.5)

$$\begin{array}{ll}
\langle\!\langle 1_p 1_p \rangle\!\rangle = o(N^{-2}) & \langle\!\langle 3_p 1_p \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 2_p 1_p 1_p \rangle\!\rangle = o(N^{-2}) & \langle\!\langle 1_p^4 \rangle\!\rangle = o(N^{-4}) \\
\langle\!\langle 5_p 1_p \rangle\!\rangle = o(N^{-2}) & \langle\!\langle 4_p 1_p 1_p \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 3_p 3_p \rangle\!\rangle = o(N^{-1}) & \langle\!\langle 3_p 2_p 1_p \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 3_p 1_p^3 \rangle\!\rangle = o(N^{-4}) & \langle\!\langle 2_p^2 1_p^2 \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 2_p 1_p^4 \rangle\!\rangle = o(N^{-4}) & \langle\!\langle 1_p^6 \rangle\!\rangle = o(N^{-6}) \\
\langle\!\langle 7_p 1_p \rangle\!\rangle = o(N^{-2}) & \langle\!\langle 6_p 1_p^2 \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 5_p 3_p \rangle\!\rangle = o(N^{-1}) & \langle\!\langle 5_p 2_p 1_p \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 5_p 1_p 1_p 1_p \rangle\!\rangle = o(N^{-4}) & \langle\!\langle 4_p 3_p 1_p \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 4_p 2_p 1_p^2 \rangle\!\rangle = o(N^{-2}) & \langle\!\langle 4_p 1_p^4 \rangle\!\rangle = o(N^{-4}) \\
\langle\!\langle 3_p^2 2_p \rangle\!\rangle = o(N^{-1}) & \langle\!\langle 3_p^2 1_p^2 \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 3_p 2_p^2 1_p \rangle\!\rangle = o(N^{-4}) & \langle\!\langle 3_p 2_p 1_p^3 \rangle\!\rangle = o(N^{-4}) \\
\langle\!\langle 3_p 1_p^5 \rangle\!\rangle = o(N^{-6}) & \langle\!\langle 2_p^3 1_p^2 \rangle\!\rangle = o(N^{-2}) \\
\langle\!\langle 2_p^2 1_p^4 \rangle\!\rangle = o(N^{-4}) & \langle\!\langle 2_p 1_p^6 \rangle\!\rangle = o(N^{-6}) \\
\langle\!\langle 1_p^8 \rangle\!\rangle = o(N^{-8}). &
\end{array}$$

If we denote the s. m.'s, containing odd entries, as $\langle \{\text{odd ent}\} \rangle$ and $\langle\!\langle \{\text{odd ent}\} \rangle\!\rangle$, we have from formulas (2.4.10),

$$\langle\!\langle \{\text{odd ent}\} \rangle\!\rangle = \langle \{\text{odd ent}\} \rangle^{*} \langle \{\text{odd ent}\} \rangle^{**} = o(N^{-1}) ,$$

and we have, for the all even entries s. m.'s

(3.2.7)

$$\langle\!\langle 2_p \rangle\!\rangle = \mu_2$$

$$\langle\!\langle 4_p \rangle\!\rangle = \mu_4$$

$$\langle\!\langle 2_p 2_p \rangle\!\rangle = \frac{N^2}{N^{(2)}} \mu_2^2 + o(N^{-1})$$

$$\langle\!\langle 6_p \rangle\!\rangle = \mu_6$$

$$\langle\!\langle 4_p 2_p \rangle\!\rangle = \frac{N^2}{N^{(2)}} \mu_4 \mu_2 + o(N^{-1})$$

$$\langle\!\langle 2_p 2_p 2_p \rangle\!\rangle = \frac{N^3}{N^{(3)}} \mu_2^3 + o(N^{-1})$$

$$\langle\!\langle 8_p \rangle\!\rangle = \mu_8$$

$$\langle\!\langle 6_p 2_p \rangle\!\rangle = \frac{N^2}{N^{(2)}} \mu_6 \mu_2 + o(N^{-1})$$

$$\langle\langle 4_p 4_p \rangle\rangle = \frac{N^2}{N^{(2)}}\mu_4^2 + o(N^{-1})$$

$$\langle\langle 4_p 2_p 2_p \rangle\rangle = \frac{N^3}{N^{(3)}}\mu_4\mu_2^2 + o(N^{-1})$$

$$\langle\langle 2_p 2_p 2_p 2_p \rangle\rangle = \frac{N^4}{N^{(4)}}\mu_2^4 + o(N^{-1}) \ .$$

Consequently, we get

(3.2.8) $$aver\{\lambda_{1p}\} = \left(\frac{1}{2}\right)\{N\alpha_p^2 + \mu_2\} + o(N^{-1}) \ .$$

If we put

(3.2.9) $$aver\{\lambda_{1p}\} \stackrel{d}{=} \frac{\mu_2}{2}\{2\tau_{1p} + 1\} + o(N^{-1}) \ ,$$

we get

(3.2.10) $$\tau_{1p} = N\frac{\alpha_p^2}{2\mu_2} + o(N^{-1}) \ .$$

In the case the $o(N^{-1})$ can be neglected, we have

(3.2.11) $$\alpha_p^2 = 2\tau_{1p}\mu_2 N^{-1} \ .$$

Substituting this result to the formula (3.2.3), we get

(3.2.12) $$aver\{\lambda_{1p}^S\} = \left(\frac{1}{2}\right)\sum_{s=0}^{S}\binom{2S}{2s}(2\tau_{1p}\mu_2)^{S-s}S_s \ .$$

So as to visualize the order of N of S_{sh}'s which are the elements of S_s, we shall tabulate the order of N of S_{sh} in the Table 3.2.1-1, and for the term of $N^{(h)-s}$ in the Table 3.2.1-2.

Furthermore, for the terms which contain the higher order term than

Table 3.2.1-1 Order of N in the s. m.'s

s \ h	1	2	3	4	5	6	7	8
$s=1$	0	-2						
$s=2$	0	0, -1	-2	-4				
$s=3$	0	0, -1, -1	0, -2	-2, -4	-4	-6		
$s=4$	0	0, -1, -2	0, -1, -2	0, -2, -3, -4	-2, -4	-4, -6	-6	-8

Table 3.2.2-2 Order of N in $N^{(h)-s}$

s \ h	1	2	3	4	5	6	7	8
$s=1$	0	1						
$s=2$	-1	0	1	2				
$s=3$	-2	-1	0	1	2	3		
$s=4$	-3	-2	-1	0	1	2	3	4

N^{-1}, we have

$$N^{1-1}\langle\langle 2_p\rangle\rangle=\mu_2$$

$$N^{(2)-1}\langle\langle 1_p^2\rangle\rangle=\frac{1}{N-1}\mu_2$$

$$N^{1-2}\langle\langle 4_p\rangle\rangle=\frac{1}{N}\mu_4$$

$$N^{(2)-2}3\langle\langle 2_p^2\rangle\rangle=3\left(\mu_2^2-\frac{1}{N}\mu_4\right)$$

$$N^{(3)-2}6\langle\langle 2_p 1_p^2\rangle\rangle=6\mu_2^2\frac{1}{N-1}+o(N^{-2})$$

$$N^{(2)-3}15\langle\langle 4_p 2_p\rangle\rangle=\frac{15}{N}\mu_2\mu_4+o(N^{-2})$$

$$N^{(3)-3}15\langle\langle 2_p^3\rangle\rangle=15\left(\mu_2^3-3\frac{1}{N}\mu_4\mu_2\right)+o(N^{-2})$$

$$N^{(4)-3}45\langle\langle 2_p^2 1_p^2\rangle\rangle=45\frac{1}{N-1}\mu_2^3+o(N^{-2})$$

$$N^{(3)-4}210\langle\langle 4_p 2_p^2\rangle\rangle=210\frac{1}{N}\mu_4\mu_2^2+o(N^{-2})$$

$$N^{(4)-4}105\langle\langle 2_p^4\rangle\rangle=105\left(\mu_2^4-\frac{6}{N}\mu_2^2\mu_4\right)+o(N^{-2})$$

$$N^{(5)-4}420\langle\langle 2_p^3 1_p^2\rangle\rangle=420\mu_2^4\frac{1}{N-1}+o(N^{-2})\ .$$

In general, from the formulas (2.3.7), (2.4.12) and (2.4.13), we have for the $2s$-th degree s. m.

$$S_{sh}=o(N^{s}) \qquad \text{for } h<s \;,$$

and

$$=o(N^{2(s-h)}) \qquad \text{for } h>s \;,$$

then we get

$$S_s=\sum_{h=1}^{2s} N^{(h)-s}S_{sh}$$
$$=N^{(s)-s}\,S_{ss}+o(N^{-1}) \;.$$

Furthermore, we have

$$S_{ss}=C_s\langle\!\langle 2_p^s\rangle\!\rangle+o(N^{-1}) \;,$$

and

$$\langle\!\langle 2_p^s\rangle\!\rangle=N^{s-(s)}\mu_2^s+o(N^{-1}) \;.$$

Then we get

$$S_s=\frac{2s}{(2!)^s s!}\;\mu_2^s+o(N^{-1}).$$

Using these results, we easily get

(3.2.13–1) $$S_1=\mu_2+o(N^{-1})$$

(3.2.13–2) $$S_2=3\mu_2^2+o(N^{-1})$$

(3.2.13–3) $$S_3=15\mu_2^3+o(N^{-1})$$

(3.2.13–4) $$S_4=105\mu_2^4+o(N^{-1})$$

and

(3.2.13–5) $$S_s=1\cdot 3\cdot 5\,\cdots\,(2s-1)\,\mu_2^s+o(N^{-1}) \;.$$

Finally, we get

THEOREM B–1: *The moments of λ_{1p} are as follows*

(3.2.14–1) $$aver\;\lambda_{1p}=\frac{\mu_2}{2}\{2\tau_{1p}+1\}+\Delta_{1p}+o(N^{-2})$$

(3.2.14–2) $$aver\;\lambda_{1p}^2=\left(\frac{\mu_2}{2}\right)^2\{4\tau_{1p}^2+12\tau_{1p}+3\}+\Delta_{2p}+o(N^{-2})$$

(3.2.14–3) $$aver\;\lambda_{1p}^3=\left(\frac{\mu_2}{2}\right)^3\{8\tau_{1p}^3+60\tau_{1p}^2+90\tau_{1p}+15\}+\Delta_{3p}+o(N^{-2})$$

and

(3.2.14–4) $$aver\;\lambda_{1p}^4=\left(\frac{\mu_2}{2}\right)^4\{16\tau_{1p}^4+224\tau_{1p}^3+840\tau_{1p}^2+840\tau_{1p}+105\}$$
$$+\Delta_{4p}+o(N^{-2}) \;,$$

where

(3.2.15–1) $$\tau_{1p}=\frac{N\alpha_p^2}{2\mu_2}$$

(3.2.15–2)
$$\Delta_{1p}=\frac{1}{N-1}\binom{\mu_2}{2}$$

(3.2.15–3)
$$\Delta_{2p}=\left\{\frac{12\tau_{1p}}{N-1}-\frac{2}{N}\left(\frac{\mu_4}{\mu_2^2}\right)+\frac{6}{N-1}\right\}\binom{\mu_2}{2}^2$$

(3.2.15–4)
$$\Delta_{3p}=\left\{\frac{60\tau_{1p}^2}{N-1}-\frac{180\tau_{1p}}{3N}\left(\frac{\mu_4}{\mu_2^2}\right)+\frac{180\tau_{1p}}{N-1}\right.$$
$$\left.-\frac{30}{N}\left(\frac{\mu_4}{\mu_2^2}\right)+\frac{45}{N-1}\right\}\binom{\mu_2}{2}^3$$

(3.2.15–4)
$$\Delta_{4p}=\left[\frac{224\tau_{1p}^3}{N-1}-\left\{840\tau_{1p}^2\left(\frac{2}{3N}\left(\frac{\mu_4}{\mu_2^2}\right)-\frac{2}{N-1}\right)\right.\right.$$
$$-840\tau_{1p}\left(\frac{2}{N}\left(\frac{\mu_4}{\mu_2^2}\right)-\frac{3}{N-1}\right)$$
$$\left.\left.-\left(\frac{420}{N}\left(\frac{\mu_4}{\mu_2^2}\right)-\frac{420}{N-1}\right)\right\}\right]\binom{\mu_2}{2}^4 \quad .$$

In general, the S-th ($S<\infty$) *moment of* λ_{1p} *is*

(3.2.16)
$$\text{aver } \lambda_{1p}^S=\binom{\mu_2}{2}^S\left\{(2\tau_{1p})^S+\sum_{s=1}^{S}\prod_{r=1}^{s}(2r-1)\binom{2S}{2s}(2\tau_{1p})^{S-s}\right\}$$
$$+o(N^{-1}) \ .$$

On the other hand, we have the moment of
$$X^2=(\sqrt{2\tau}+x)^2$$
such that

(3.2.17)
$$E\{(\sqrt{2\tau}+x)^2\}^s=(2\tau)^s+\sum_{s=1}^{s}\binom{2S}{2s}(2\tau)^{S-s}E(x^{2s}) \ ,$$

where x is normally and independently distributed variate with mean 0 and variance 1. Then we get the moments of the non-central chi-square distribution with the degree of freedom 1 and the noncentral parameter τ, and we have

(3.2.18–1) $E\{(\sqrt{2\tau}+x)^2\}=2\tau+1$

(3.2.18–2) $E\{(\sqrt{2\tau}+x)^2\}^2=4\tau^2+12\tau+3$

(3.2.18–3) $E\{(\sqrt{2\tau}+x)^2\}^3=8\tau^3+60\tau^2+90\tau+15$

(3.2.18–4) $E\{(\sqrt{2\tau}+x)^2\}^4=16\tau^4+224\tau^3+840\tau^2+840\tau+105$

and in general

(3.2.18-5)
$$\mu_S'=(2\tau)^S+\sum_{s=1}^{S}\prod_{r=1}^{S}(2r-1)\binom{2S}{2s}(2\tau)^{S-s} \ .$$

From these results, it can be seen that the distribution of $\left\{\lambda_{1p}\cdot\binom{\mu_2}{2}^{-1}\right\}$

is approximately the non-central chi-square distribution with the degree of freedom 1 and non-central parameter

$$\tau_p = N\frac{\alpha_p^2}{2\mu_2} \quad .$$

3.2.2 Moments of λ_2 and central chi-squares

Now, we proceed into the calculation of the moments of λ_2. These moments of λ_2 can be represented in the following formula, as we have seen in section 3.1.2.

$$\text{(3.2.19)} \qquad aver\{\lambda_2^K\} = \left(\frac{1}{2}\right)^K \sum_{k=1}^{2K} aver\ G_k ,$$

$$averG_k = \sum{}' \left(\frac{N^{(k)}}{N^K}\right) \Phi^{(\varphi)} \ [(\varphi\times h)]_u^* <\{h\}>^{**} C_u ,$$

where $\sum'$ stands for the summation over the all g. s. m.'s with h columns, and where $[(\varphi\times h)]^*_u$ is u-th g. s. m. with φ rows and h columns of O. A. and C_u is the constant of the u-th g. s. m. as we can see the Table 3.1.1., 3.1.2, 3.1.3 and 3.1.4.

If we denote the g. s. m. consisted of even entries as $[\{\varphi\times h\}]^*_{\text{even ent}}$ and g. s. m. including odd entries as $[\{\varphi\times h\}]^*_{\text{odd ent}}$ we have

for $h<K$	for $K<h$
$\left(\frac{N^{(h)}}{N^K}\right)$	$\left(\frac{N^{(h)}}{N^K}\right)$
$[\{\varphi\times h\}]^*_{\text{even ent}}=1$	
$[\{\varphi\times h\}]^*_{\text{odd ent}}=o(N^\circ)$	$[\{\varphi\times h\}]^*_{\text{odd ent}}=o(N^{-(h-K)})$
$<\{h\}>^{**}=o(N^\circ)$	$<\{h\}>^{**}=o(N^{-(h-K)})$,

from the results of symmetric means of v_n's and g. s. m.'s of O. A., then we get

$$\text{(3.2.20)} \qquad S_h = o(N^{-1}) , \qquad \text{for } h \lesseqgtr K .$$

Consequently, it can be obtained that

$$\text{(3.2.21)} \qquad aver\{\lambda_2^K\} = \left(\frac{1}{2}\right) S_K + o(N^{-1}) ,$$

where

(3.2.22) $S_K = S(\text{even}) + S(\text{odd})$,

S(even) = Sum of symmetric means as $[\{\text{even}\}]^*$'s consisting of all even entries.

S(odd) = Sum of symmetric means as $[\{\text{odd}\}]^*$'s in odd entries.

As we have seen in the previous section for s. m.'s we also have at most the K all 2 entries in the $2K$-th degree g. s. m.'s which are containing the all even entries, such that

(3.2.23) $$<\{K\}>^{**}=<2\cdots 2>^{**} = \frac{(N\mu_2)^K}{N^{(K)}}+o(N^{-1}) \ .$$

On the contrary, since the all odd c. g. s. m.'s are containing at most $2K$ all odd entries in the $2K$-th degree g. s. m.'s, then we have no $\varphi(\geqq 3)\times K$ g. s. f. having all odd entries

$$\underbrace{\begin{pmatrix} 1 & 1 & 1\cdots\cdots\cdots 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1 & \cdots\cdots\cdots\cdots & 1 \end{pmatrix}}_{K}$$

and containing no zero entry in the $2K$-th degree g. s. m.'s.

Consequently refering the formula (2.2.20), we have $[\{\text{odd ent}\}]^{*}=o(N^{-1})$. Since we have in the previous section

(3.2.24) $$<\{K\}>^{**}=o(N^{0}) \ ,$$

we get

(3.2.25) $$S(\text{odd})=o(N^{-1}) \ .$$

Then we get

(3.2.26) $$S^{K}=\mu_2^{K}\sum_{\varphi=1}^{K}\Phi^{(\varphi)}C_{\varphi}(<2^{K}>)^{**}+o(N^{-1}) \ ,$$

where $C_{\varphi}(<2^{K}>^{**})$ is the sum of numerical coefficients of g. s. m.'s with φ rows and K columns of which all entries equal to 2. Furthermore, adding the calculations of the g. s. m.'s those order of N^{-1}, we finally get

THEOREM B–2: *The moments of λ_2 are*

(5.2.27–1) $$aver\ \{\lambda_2\}=\left(\frac{\mu_2}{2}\right)\Phi+\Delta_1+o(N^{-2})$$

(5.2.27–2) $$aver\ \{\lambda_2^2\}=\left(\frac{\mu_2}{2}\right)^2\{3\Phi+\Phi^{(2)}\}+\Delta_2+o(N^{-2})$$

(5.2.27–3) $$aver\ \{\lambda_2^3\}=\left(\frac{\mu_2}{2}\right)^3\{15\Phi+9\Phi^{(2)}+\Phi^{(3)}\}+\Delta_3+o(N^{-2})$$

and

(5.2.27–4) $$aver\ \{\lambda_2^4\}=\left(\frac{\mu_2}{2}\right)^4\{105\Phi+(15\times 4+9\times 3)\Phi^{(2)}+6\times 3\Phi^{(3)}+\Phi^{(4)}\} + \Delta_4+o(N^{-2}) \ ,$$

where

(5.2.28–1) $$\Delta_1 = \frac{\mu_2}{2}\Phi\frac{1}{N-1}$$

(5.2.28–2) $$\Delta_2 = \left(\frac{\mu_2}{2}\right)^2\left\{(6-2\delta_2)\Phi\right\}\frac{1}{N}$$

(5.2.28–3) $$\Delta_3 = \left(\frac{\mu_2}{2}\right)^3\left\{(-30\delta_2+45)\Phi-3(2\delta_2+3)\Phi^{(2)}+(-3+4\delta_1)\Phi^{(3)}\right\}\frac{1}{N}$$

(5.2.28–4) $$\Delta_4 = \left(\frac{\mu_2}{2}\right)^4\left\{(-420\delta_2+420)\Phi-(156\delta_2+228)\Phi^{(2)}+(-2\delta_2+24\delta_1-18)6\Phi^{(3)}+(16\delta_1-8)\Phi^{(4)}\right\}\frac{1}{N}\ ,$$

(3.2.29–1) $$\delta_1 \overset{d}{=} \frac{\mu_3^2}{\mu_2^3}r_3$$

(3.2.29–2) $$\delta_2 \overset{d}{=} \frac{\mu_4}{\mu_2^2}\ .$$

In general

(3.2.30) $$aver\ \{\lambda_2^K\} = \left(\frac{\mu_2}{2}\right)^K\sum_{\varphi}C_\varphi\{<2^K>^{**}\}\Phi^{(\varphi)}+o(N^{-1})\ .$$

Proof: In the formula (3.1.12–1), (3.1.12–2), (3.1.12–3) and (3.1.12–4), the all even g. s. m.'s in the K th column have the order of N^0. Furthermore in the $(K-1)$th, Kth and $(K+1)$th columns, we have the g. s. m.'s which have the order at most N^{-1}. These g. s. m.'s are as follows:

[$K=1$]

$$\left[\left[\begin{matrix}2 & -\\ - & -\end{matrix}\right]\right]N^{(1)-1}=\mu_2\ , \qquad \left[\left[\begin{matrix}1 & 1\\ - & -\end{matrix}\right]\right]N^{(2)-1}=\mu_2/N-1$$

[$K=2$]

$$\left[\left[\begin{matrix}4 & -\\ - & -\end{matrix}\right]\right]N^{(1)-2}=\mu_4/N\ , \qquad \left[\left[\begin{matrix}2 & 2\\ - & -\end{matrix}\right]\right]N^{(2)-2}=\mu_2^2-\mu_4\ N^{-1}$$

$$\left[\left[\begin{matrix}2 & -\\ 2 & -\end{matrix}\right]\right]N^{(1)-2}= \ //\ , \qquad \left[\left[\begin{matrix}2 & -\\ - & 2\end{matrix}\right]\right]N^{(2)-2}= \ //$$

$$\left[\left[\begin{matrix}2 & 1 & 1\\ - & - & -\end{matrix}\right]\right]N^{(3)-2}=\mu_2^2/N-1\ , \qquad \left[\left[\begin{matrix}1 & 1\\ 1 & 1\end{matrix}\right]\right]N^{(2)-2}=-\mu_2^2\frac{1}{N-1}$$

$$\left[\left[\begin{matrix}2 & - & -\\ - & 1 & 1\end{matrix}\right]\right]N^{(3)-2}= \ //\ ,$$

[$K=3$]

$$\left[\left[\begin{matrix}4&2\\-&-\end{matrix}\right]\right]N^{(2)-3}=\mu_4\mu_2N^{-1}+o(N^{-2})\ ,$$

$$\left[\left[\begin{matrix}4&-\\-&2\end{matrix}\right]\right]N^{(2)-3}=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&2\\2&-\end{matrix}\right]\right]N^{(2)-3}=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&-\\2&-\\-&2\end{matrix}\right]\right]N^{(2)-3}=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}1&1&2\\1&1&-\end{matrix}\right]\right]N^{(3)-3}=-\mu_2^3\frac{1}{N-1}+o(N^{-2}),$$

$$\left[\left[\begin{matrix}1&1&-\\1&1&-\\-&-&2\end{matrix}\right]\right]N^{(2)-3}=\quad〃\quad ,$$

$$\underset{alias}{\left[\left[\begin{matrix}1&1\\1&1\\1&1\end{matrix}\right]\right]}N^{(2)-3}=\mu_3^2\frac{1}{N}+o(N^{-2})\ ,$$

$$\left[\left[\begin{matrix}2&2&2\\-&-&-\end{matrix}\right]\right]N^{(3)-3}=\mu_2^3-3\mu_4\mu_2N^{-1}+o(N^{-2})$$

$$\left[\left[\begin{matrix}2&2&-\\-&-&2\end{matrix}\right]\right]N^{(3)-3}=\quad〃$$

$$\left[\left[\begin{matrix}2&-&-\\-&2&-\\-&-&2\end{matrix}\right]\right]N^{(3)-3}=\quad〃$$

$$\left[\left[\begin{matrix}2&2&1&1\\-&-&-&-\end{matrix}\right]\right]N^{(4)-3}=\mu_2^3\left(\frac{1}{N-1}\right)+o(N^{-2})$$

$$\left[\left[\begin{matrix}2&2&-&-\\-&-&1&1\end{matrix}\right]\right]N^{(4)-3}=\quad〃$$

$$\left[\left[\begin{matrix}2&1&1&-\\-&-&-&2\end{matrix}\right]\right]N^{(4)-3}=\quad〃$$

$$\left[\left[\begin{matrix}2&-&-&-\\-&2&-&-\\-&-&1&1\end{matrix}\right]\right]N^{(4)-3}=\quad〃$$

[$K=4$]

$$\left[\left[\begin{matrix}4&2&2\\-&-&-\end{matrix}\right]\right]N^{(3)-4}=\mu_4\mu_2^2N^{-1}+o(N^{-2}),$$

$$\left[\left[\begin{matrix}4&2&-\\-&-&2\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&2&2\\2&-&-\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&2&-\\-&-&4\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&2&-\\2&-&2\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}4&-&-\\-&2&-\\-&-&2\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&2&-\\-&-&2\\-&-&2\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&2&-\\2&-&-\\-&-&2\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\left[\left[\begin{matrix}2&-&-\\2&-&-\\-&2&-\\-&-&2\end{matrix}\right]\right]〃\quad=\quad〃\quad ,$$

$$\underset{alias}{\left[\left[\begin{matrix}2&1&1\\-&1&1\\-&1&1\end{matrix}\right]\right]}N^{(3)-4}=\mu_3^2\mu_2N^{-1}+o(N^{-2}),$$

$$\left[\left[\begin{matrix}2&2&2&2\\-&-&-&-\end{matrix}\right]\right]N^{(4)-4}=\mu_2^4-6\mu_4\mu_2^2N^{-1}+o(N^{-2})$$

$$\left[\left[\begin{matrix}2&2&2&-\\-&-&-&2\end{matrix}\right]\right]N^{(4)-4}=\quad〃$$

$$\left[\left[\begin{matrix}2&2&-&-\\-&-&2&2\end{matrix}\right]\right]〃\quad=\quad〃$$

$$\left[\left[\begin{matrix}2&2&-&-\\-&-&2&-\\-&-&&2\end{matrix}\right]\right]〃\quad=\quad〃$$

$$\left[\left[\begin{matrix}2&-&-&-\\-&2&-&-\\-&-&2&-\\-&-&-&2\end{matrix}\right]\right]〃\quad=\quad〃$$

$$\left[\left[\begin{matrix}2&2&1&1\\-&-&1&1\end{matrix}\right]\right]〃\quad=-\mu_2^4\frac{1}{N-1}+o(N^{-2})$$

$$\left[\left[\begin{matrix}2&1&1&-\\-&1&1&2\end{matrix}\right]\right]〃\quad=\quad〃$$

$$\left[\left[\begin{matrix}2&1&1&-\\-&1&1&-\\-&-&-&2\end{matrix}\right]\right]〃\quad=\quad〃$$

$$\left[\left[\begin{matrix}2&2&-&-\\-&-&1&1\\-&-&1&1\end{matrix}\right]\right]〃\quad=\quad〃$$

$$\left[\left[\begin{matrix}2&-&-&-\\-&2&-&-\\-&-&1&1\\-&-&1&1\end{matrix}\right]\right]〃\quad=\quad〃$$

$$\begin{array}{c} ① \\ ② \\ ⑫ \\ \end{array}\begin{pmatrix} 1 & 1 & - \\ 1 & 1 & - \\ 1 & 1 & - \\ - & - & 2 \end{pmatrix} N^{(3)-4}=\mu_3^2\mu_2 N^{-1}+o(N^{-1}), \quad \Big(\underset{alias}{①②⑫}\Big)$$

$$\left[\left[\begin{matrix} 2 & 2 & 2 & 1 & 1 \\ - & - & - & - & - \end{matrix}\right]\right] N^{(5)-4}=\mu_2^4\left(\frac{1}{N-1}\right)+o(N^{-2})$$

$$\left[\left[\begin{matrix} 2 & 2 & 1 & 1 & - \\ - & - & - & - & 2 \end{matrix}\right]\right] \text{//} \quad = \quad \text{//}$$

$$\left[\left[\begin{matrix} 2 & 2 & 2 & - & - \\ - & - & - & 1 & 1 \end{matrix}\right]\right] \text{//} \quad = \quad \text{//}$$

$$\left[\left[\begin{matrix} 2 & 1 & 1 & - & - \\ - & - & - & 2 & 2 \end{matrix}\right]\right] \text{//} \quad = \quad \text{//}$$

$$\left[\left[\begin{matrix} 2 & 1 & 1 & - & - \\ - & - & - & 2 & - \\ - & - & - & - & 2 \end{matrix}\right]\right] \text{//} \quad = \quad \text{//}$$

$$\left[\left[\begin{matrix} 2 & 2 & - & - & - \\ - & - & 2 & - & - \\ - & - & - & 1 & 1 \end{matrix}\right]\right] \text{//} \quad = \quad \text{//}$$

$$\left[\left[\begin{matrix} 1 & 1 & - & - & - \\ - & - & 2 & - & - \\ - & - & - & 2 & - \\ - & - & - & - & 2 \end{matrix}\right]\right] \text{//} \quad = \quad \text{//}$$

The other g. s. m.'s are the negligible terms those order are at most N^{-2} by virtue of the results of chapter 2 in this paper.

For the moments of the central chi-square distribution with the degrees of freedom Φ, we have

$$E\left(\sum_{\phi=1}^{\Phi} x_\phi^2\right)^K=\sum_{\phi=1}^{\Phi} C_\phi(<2^K>^{**})\Phi^{(\phi)}\ , \tag{3.2.30}$$

where x_1, x_2, $\cdots$, $x_{\Phi-1}$ and x_Φ are the mutual independent normal variates with same mean 0 and same variance 1, for example

$$\mu_1' \overset{d}{=} E\left(\sum_{\phi=1}^{\Phi} x_\phi^2\right)=\Phi$$

$$\mu_2' \overset{d}{=} E\left\{\left(\sum_{\phi=1}^{\Phi} x_\phi^2\right)^2\right\}$$

$$=\Phi E(x_\phi^4)+\Phi^{(2)}E(x_{\phi 1}^2)E(x_{\phi 2}^2)$$

and so on. Consequently, we finally obtain the conclusion that the asymptotic distribution of $\left\{\lambda_2\left(\mu_2/2\right)^{-1}\right\}$ is the central chi-square distribution with degrees of freedom Φ, in the condition that the terms $\{\Delta_K\}$ ($K=1$, 2, $\cdots$, 4) are negligible order.

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