SOME ESTIMATE PROCEDURES WITH A NONPARAMETRIC PRELIMINARY TEST I

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SOME ESTIMATE PROCEDURES WITH A NONPARAMETRIC PRELIMINARY TEST I

By

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§ 1. Introduction. The author has developed in [2] a "sometimes pool" procedure about sample medians as an estimate of the population median. In this paper, each estimate of a shift parameter of the population and the population median may be discussed along the same line as [2]. First we consider an estimate of a shift parameter.

Let \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) be two samples from the populations with continuous c.d.f. \( F(x) \) and \( F(x - \delta) \). Hodges and Lehmann [1] have defined as an estimate of \( \delta \) the following statistic (1) which is the median of the set of \( m \times n \) differences \( (Y_j - X_i) \)

\[
\hat{\delta}_{12} = \text{med}(Y_j - X_i).
\]

And they have shown some excellent properties about (1), for example,

(i) \( \hat{\delta}_{12} \) is a median unbiased estimate

(ii) the asymptotic efficiency of \( \hat{\delta}_{12} \) relative to the classical estimate \( \bar{Y} - \bar{X} \) (\( \bar{\cdot} \) denote sample mean) is \( 3/\pi \) in the case of normal \( F(x) \).

Now consider the case where there exists another sample \( Z_1, \ldots, Z_m \) with the c.d.f. \( F(x - \delta - \delta') \), \( \delta, \delta' \geq 0 \). In such case, it will be more effective to consider the estimation procedures after testing the hypothesis \( \delta = 0 \) against the alternative \( \delta > 0 \). Therefore we first perform a preliminary test by the Mann-Whitney statistic

\[
U_{23}(Y, Z) = \left( m_2 m_3 \right)^{-1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3} \phi(Y_j, Z_k)
\]

where

\[
\phi(y, z) = \begin{cases} 
1 & \text{for } y < z \\
0 & \text{otherwise} 
\end{cases}
\]

As a second problem, let \( X_1, \ldots, X_m \) be the sample from the population with continuous c.d.f. \( F(x - \theta) \) where \( F(x) \) is symmetric about zero. They have also derived an estimate \( \hat{\theta}_1 \) of \( \theta \) which is the median of \( \left( m + 1 \right) \) average \( \frac{1}{2} (X_i + X_j) \)
Suppose that there exists another sample $Y_1, \ldots, Y_n$ from the distribution $F(x - \theta - \delta)$, $\delta \geq 0$. Then we shall first consider testing the hypothesis $\delta = 0$ against the alternative $\delta > 0$ by the statistic

\[ U_{13}(X, Y) = (m_1, m_2)^{-1} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \phi(X_i, Y_j). \]

Section 2 is concerned with the "sometimes pool" procedure about an estimate of $\delta$ and the procedure for $\theta$ is dealt in section 3.

\section{An estimate about a shift parameter $\delta$}

First define the similar statistics as (1), (2) and (3)

\[ \hat{d}_{13} = \text{med} (Z_i - X_i) \]

\[ U_{13}(X, Z) = (m_1, m_3)^{-1} \sum_{i=1}^{m_1} \sum_{k=1}^{m_3} \phi(X_i, Z_k). \]

Then the estimate $\hat{\delta}$ of $\delta$ is formulated by the following

\[ \hat{\delta} = \hat{\delta}_{12} \quad \text{when} \quad U_{23} \geq u_a \]

\[ \hat{\delta} = \xi \hat{\delta}_{12} + (1 - \xi) \hat{\delta}_{13} \quad \text{when} \quad U_{23} < u_a \]

where $u_a = \frac{1}{2} z_{a/\sqrt{N_{23}}}/\sqrt{\lambda_{23}(1 - \lambda_{23})}$, $m_i + m_j = N_{ij}$, $\lambda_{ij} = m_i/N_{ij}$, and $1 - \Phi(z_a) = \alpha$, $\Phi(x)$ is the standard normal c.d.f. $N(0, 1)$. We shall intend to derive the asymptotic mean value and mean square error about $\delta$ of the estimate $\hat{\delta}$ (7) and moreover to evaluate the asymptotic efficiency relative to the "never pool" estimate $\hat{\delta}_{12}$. As the first step, consider the asymptotic joint distribution of $U_{23}$ and $\hat{\delta}_{12}$. The following are so easy to show that their proofs are omitted.

\[ \mu_{23} = E(U_{23}) = \int_{-\infty}^{\infty} F(x) \ dF(x - \delta) \]

\[ \sigma_{23}^2 = \text{Var}(1/\sqrt{N_{23}} U_{23}) \sim (r_{23} - \mu_{23}^2)/\lambda_{23} + (q_{23} - \mu_{23}^2)/(1 - \lambda_{23}) \]

\[ q_{23} = \int_{-\infty}^{\infty} F^2(x) \ dF(x - \delta), \quad r_{23} = \int \{1 - F(x - \delta)\}^2 dF(x) \]

where $\sim$ means asymptotic equality.

By the results of Hodges and Lehmann [1]
(8) \[ \text{Pr}(\hat{J}_{12} < a) = \text{Pr}(U_{12}(X, Y - a) < \mu_a) \]

, where \( \mu_a \) is the expected value of \( U_{12}(X, Y) \) under \( \mu = 0 \), that is \( \frac{1}{2} \), then we may get the identity

\[ G(u, a) = \text{Pr}\left[ \frac{U_{23} - \mu_{23}}{\sigma_{23}} < u, \frac{U_{12}}{\sqrt{N_{12}}} \left( \hat{J}_{12} - \mu_{12} \right) < a \right] = \text{Pr}\left[ \frac{U_{23} - \mu_{23}}{\sigma_{23}} < u, \frac{U_{12}}{\sqrt{N_{12}}} \left( X, Y - \frac{a}{\sqrt{N_{12}}} \right) - \mu_{12}, \sigma_{12} \right] \]

, where

\[ a_{12} = E U_{12}(X, Y, a) \frac{\int f^2(x) dx}{\sqrt{N_{12}}} \]

\[ \sigma_{12}^2 = \text{Var} U_{12}(X, Y - \frac{a}{\sqrt{N_{12}}}) \sim 1/12 \lambda_{12}(1 - \lambda_{12}) \]

and then

\[ \frac{1}{\sqrt{N_{12}}} \left( \frac{1}{2} - \mu_{12} \right) / \sigma_{12} \sim a_1 / 12 \lambda_{12} (1 - \lambda_{12}) \int f^2(x) dx \]

Since it is well-known that the joint c.d.f. of \( U \)-statistics is asymptotically normal, we may find the asymptotic expression

(9) \[ G(u, a) = \int \int g(x, y; 0, \mu_1, \mu_2; 1, 1; \rho_{12}) \ dx \ dy \]

, where \( g(x, y; \mu_1, \mu_2; \sigma_1, \sigma_2; \rho_{xy}) \) denote the bivariate normal density function with means \( \mu_1, \mu_2 \), variances \( \sigma_1, \sigma_2 \) and correlation \( \rho_{xy} \) and

(10) \[ \rho_{12} = \left[ \int \left[ F(x) - 1 - F(x - \delta) \right] dF(x) - \mu_{12} \rho_{23} \right] / \sqrt{\lambda_{23}(1 - \lambda_{12})} \times \sigma_{12}, \sigma_{23}. \]

The asymptotic joint density of \( \sqrt{N_{23}}(U_{23} - \mu_{23}) \) and \( \hat{J}_{12} \) may be obtained from (9) as follows,

(11) \[ g(x, y; 0, \delta; \sigma_2^2, 1/N_{12} k_{22}^2; \rho_{12}). \]

By the similar techniques, though slightly complicated, we may find the asymptotic joint density of \( \sqrt{N_{23}}(U_{23} - \mu_{23}) \) and \( \hat{J}_{13} \),

(12) \[ g(x, y; 0, \xi \delta + (1 - \xi)(\delta - \delta) ; \sigma_2^2, \sigma_1^2, \rho_{12}). \]

, where
\[\sigma_{1,23}^2 = \left[ -\frac{\xi^2}{N_{12}k_{12}} + \frac{(1-\xi)^2}{N_{12}k_{12}(1-\xi)} + \frac{2\xi(1-\xi)}{m_1} \right]/12 \left( \int f^2 dx \right)^2 \]

From the definition (7) of \(\hat{a}\) in (11), (12), the asymptotic probability of \(\hat{a}\) may be expressed as follows,

\[
Pr(\hat{a} < a) = Pr[\sqrt{N_{12}} (U_{12} - \mu_{12})/\sigma_{12} \geq v_a, \hat{a} < a] + Pr[\sqrt{N_{12}} (U_{12} - \mu_{12})/\sigma_{12} < v_a, \hat{a} < a] = \int \int g(x, y; 0, \delta; 1, 1/N_{12}k_{12}; \rho_{12}) \, dx \, dy + \int \int g(x, y; 0, \delta + (1-\xi) \delta; 1, \sigma_{12}; \rho_{12}) \, dx \, dy,
\]

where \(v_a = \sqrt{N_{12}} (u_a - \mu_{12})/\sigma_{12}\).

The probability above leads to the asymptotic density \(w(a)\) of \(\hat{a}\)

\[
w(a) = \int g(x, a; 0, \delta; 1, 1/N_{12}k_{12}; \rho_{12}) \, dx + \int g(x, a; 0, \delta + (1-\xi) \delta; 1, \sigma_{12}; \rho_{12}) \, dx.
\]

From (14),

\[
E(\hat{a}) = \int \int a \, da \left[ \int g(x, a; 0, \delta; 1, 1/N_{12}k_{12}; \rho_{12}) \, dx + \int g(x, a; 0, \delta + (1-\xi) \delta; 1, \sigma_{12}; \rho_{12}) \, dx \right].
\]

After some computations, we may get the mean value

\[E(\hat{a}) = \delta + (1-\xi) \delta \Phi(v_a) + \left( \frac{\rho_{12}}{\sqrt{N_{12}k_{12}} - \sigma_{12} \rho_{12}} \right) \varphi(v_a)\]

By the similar computations,
Some estimate procedures with a nonparametric preliminary test

\[ M.S.E. (\hat{\theta}) = E(\hat{\theta} - \theta)^2 \]

\[ = \frac{1}{N_{12} k_{12}^2} \{ 1 - \Phi(v_a) \} \left[ \sigma_{1,23}^2 \Phi(v_a) + \left( \frac{\rho_{13}^2}{N_{12} k_{12}^2} - \sigma_{1,23}^2 \rho_{1,23} \right) v_a \varphi(v_a) \right] \]

\[ + \delta^2 (1 - \xi)^2 \Phi(v_a) - 2 \delta (1 - \xi) \sigma_{1,23} \rho_{1,23} \varphi(v_a). \]

We suppose \( \delta = 0 \) and determine the value of \( \xi \) minimizing the mean square error (16). From the equation \( d[M.S.E.(\hat{\theta})]/d\xi = 0 \), we may easily get the identity

\[ \{ \Phi(v_a) - v_a \varphi(v_a) \} (\xi - \lambda_{23}) = 0 \]

and we get \( \xi = \lambda_{23} \).

Though \( \xi = \lambda_{23} \) does not give the exact minimum value of the mean square error of \( \hat{\theta} \) in the case of \( \delta = 0 \), we adopt \( \xi = \lambda_{23} \) for sufficiently small \( \delta > 0 \). Lastly in the case where \( \delta \) is sufficiently small, say \( \delta = o(1/\sqrt{N}) \), we shall compute the asymptotic efficiency of \( \hat{\theta} \). It follows from (10) and (13) that

\[ \rho_{12} = -v_{12}(1 - \lambda_{23}), \quad \rho_{1,23} = o(1/\sqrt{N}). \]

Then from (16)

\[ M.S.E. (\hat{\theta}) = \left[ 1 - \lambda_{12} (1 - \lambda_{23}) \right] / 12 \left( \int f^2 dx \right) N_{12}. \]

Let the asymptotic efficiency of \( \hat{\theta} \) with regard to the "never pool" estimate \( \hat{\theta}_{12} \), which is defined by the reciprocal ratio of asymptotic variances, be denoted by \( e_{s,a} \), then we get the following

\[ e_{s,a} = \left[ 1 - \lambda_{12} (1 - \lambda_{23}) \right] \left( \phi(v_a) - v_a \varphi(v_a) \right) \]^{-1}.

In the case of \( \lambda_{ij} = 1/2 \), we get the Table I

<table>
<thead>
<tr>
<th>( v )</th>
<th>1.65</th>
<th>1.30</th>
<th>1.00</th>
<th>0.5</th>
<th>0</th>
<th>-0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1.243</td>
<td>1.205</td>
<td>1.176</td>
<td>1.148</td>
<td>1.143</td>
<td>1.138</td>
</tr>
</tbody>
</table>

§ 3. An estimate about the median \( \theta \). As stated in section 1, we assume that

\[ Pr(X \leq x) = F(x - \theta), \quad Pr(Y \leq y) = F(y - \theta - \delta) \]

and \( F(x) \) is symmetric about zero. In order to formulate the estimate \( \hat{\theta} \) of \( \theta \), we first define
\[ U_1(X) = \left( \frac{m_1 + 1}{2} \right)^{-1} \sum_{1 \leq i,j \leq m_1} \psi(X_i, X_j) \]

\[ U_2(Y) = \left( \frac{m_2 + 1}{2} \right)^{-1} \sum_{1 \leq i,j \leq m_2} \psi(Y_i, Y_j) \]

where \( \psi(x, x') = 1 \) or 0 if \( x + x' > 0 \) or otherwise and

\[ \theta_2 = \text{med} \left[ \frac{1}{2} (Y_i + Y_j) \right], \quad 1 \leq i \leq j \leq m_2 \]

is the median of \( \left( \frac{m_2 + 1}{2} \right) \) average \( (Y_i + Y_j)/2 \).

The preliminary test is performed by Mann-Whitney statistic \( U_{12} \). We define an estimate \( \hat{\theta} \) of \( \theta \) as follows,

\[ \hat{\theta} = \hat{\theta}_1 \quad \text{when} \quad U_{12} \geq u_* \]

\[ \hat{\theta} = \lambda_{12} \theta_1 + (1 - \lambda_{12}) \theta \quad \text{when} \quad U_{12} < u_* \]

where \( u_* = \frac{1}{2} z_1 \sqrt{N_{12} \lambda_{12} (1 - \lambda_{12})} \).

By the similar considerations and computations as section 2, we may find the following values

\[ \mu_{12} = EU_{12} = \int_{-\infty}^{\infty} F(x) \, dF(x - \delta) \]

\[ \sigma_{12}^2 = \text{Var}(U_{12}) \sim (r_{12} - \mu_{12}^2)/\lambda_{12} + (q_{12} - \mu_{12}^2)/(1 - \lambda_{12}) \]

\[ \mu_i = EU_i \left( X - \theta - \frac{a}{\sqrt{m_i}} \right) \sim \frac{1}{2} - 2a \int \frac{f(x)}{\sqrt{m_i}} \, dx \]

\[ \sigma_i^2 = \text{Var} U_i \left( X - \theta - \frac{a}{\sqrt{m_i}} \right) \sim 1/3m_i, \quad i = 1, 2 \]

\[ \rho = \text{corr.} \left( U_{12}, U_1(X - \theta - \frac{a}{\sqrt{m_1}}) \right) = \sqrt{3} \int_{-\infty}^{\infty} \left| F(x) - 1 - F(x - \delta) \right| dF(x - \mu_{12}) / \sqrt{\lambda_{12} \sigma_{12}} \]

\[ \rho' = \text{corr.} \left( U_{12}, U_2(Y - \theta - \delta - \frac{a}{\sqrt{m_2}}) \right) = \sqrt{3} \int_{-\infty}^{\infty} \left| F(x) F(x + \delta) \right| dF(x - \mu_{12}) / \sqrt{1 - \lambda_{12} \sigma_{12}} \]

In fact, denoting \( Z = X - \theta - a/\sqrt{m_1} \),

\[ \sigma_i^2 = \left( \frac{m_i + 1}{2} \right)^2 \left[ (2m_i - 4) \left( \frac{m_i}{2} \right) \right] E\psi(Z_1, Z_2) \psi(Z_1, Z_2) + E\psi(Z_1, Z_2) \cdot \psi(Z_2, Z_3) \]
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(23) 
\[ -\left( \frac{m - 1}{2} \right)^2 E\psi(Z_1, Z_2)\psi(Z_2, Z_3) = -\mu^2 + o(1/m). \]

And

\[ \psi(Z_1, Z_2)\psi(Z_1, Z_3) = \int dF(x_1)dF(x_2)dF(x_3) \]

\[ \times \frac{x_1 + x_2 > 2a/\sqrt{m_1}}{x_1 + x_2 > 2a/\sqrt{m_1}} \]

\[ = \frac{1}{3} - 4a \int fFdF/\sqrt{m_1} + o(1/\sqrt{m}). \]

Similarly, we get

\[ \psi(Z_1, Z_2)\psi(Z_3, Z_4) = \frac{1}{4} - 2a \int f^2 dx/\sqrt{m_1} + o(1/\sqrt{m_1}) \]

\[ \psi(Z_1, Z_2)\psi(Z_2, Z_3) = \frac{1}{4} - a \int f^2 dx + f(0)/2\sqrt{m_1} + o(1/\sqrt{m_1}). \]

Substituting these values into (23), then we find \( \sigma_i^2 \sim 1/3m_i \). The other results in (22) may be similarly computed. By using the works of Hodges and Lehmann

(24) 
\[ Pr(\hat{\theta}_1 < a) = Pr(U_1(X - a) < \mu_0) \]

where \( \mu_0 \) is the expected value of \( U_1(X) \) under \( \theta = 0 \), that is, \( \mu_0 = 1/2 \), we shall derive the asymptotic joint density of \( U_{12} \) and \( \hat{\theta}_1 \).

\[ Pr[\sqrt{N_{12}}(U_{12} - \mu_{12}) < u, \sqrt{m_1}(\hat{\theta}_1 - \theta) < a] \]

\[ = Pr[\sqrt{N_{12}}(U_{12} - \mu_{12}) < u, \sqrt{m_1}|U_1(X - \theta - \frac{a}{\sqrt{m_1}}) - \mu_1| < \sqrt{m_1}(\frac{1}{2} - \mu_1)] \]

and then \( \sqrt{m_1}(\frac{1}{2} - \mu_1) \sim 2a \int f^2 dx (= ak) \) from (22).

Thus the asymptotic normality of \( \sqrt{N_{12}}(U_{12} - \mu_{12}) \) and \( \sqrt{m_1}(\hat{\theta}_1 - \theta) \) may be established from the fact of the asymptotic normality of U-statistics. The joint density may be written as

(25) 
\[ g(x, y; \theta, \phi; \sigma_i^2, 1/3k^2; \rho). \]

Similarly, we get the asymptotic joint density of \( \sqrt{N_{12}}(U_{12} - \mu_{12}) \) and \( \sqrt{N_{12}} \{ \lambda_{12}(\hat{\theta}_1 - \theta) + (1 - \lambda_{12})(\hat{\theta}_2 - \theta - \delta) \} \) as follows,
From (21) and (25), (26), we may derive the asymptotic distribution of \( \hat{\theta} \),
\[
Pr (\theta < a) = \int \int g(x, y; 0, 0; 1, 1/3k^2; \rho) \, dx \, dy
\]
where \( \nu_x = 1/\sqrt{N_{12}} (u_x - \mu_{12})^{a_{12}} \).

The asymptotic density \( w(a) \) of \( \hat{\theta} \) may be expressed as
\[
w(a) = \int_{-\infty}^{\nu_x} g(x, a; 0, \theta; 1, 1/3k^2m_1; \rho) \, dx
\]
\[
+ \int_{-\infty}^{\nu_x} g(x, a; 0, \theta + (1 - \lambda_{12})\delta; 1, 1/3k^2N_{12}; \rho') \, dx.
\]

By the similar computations as section 2, we get the following
\[
E(\hat{\theta}) = \theta + (1 - \lambda_{12})\delta \Phi(v_a) + \Phi(v_a) \left( \frac{\rho}{\sqrt{m_1}} - \frac{\rho'}{\sqrt{N_{12}}} \right) / \sqrt{3k}
\]
\[
M.S.E. (\hat{\theta}) = E(\hat{\theta} - \theta)^2
\]
\[
= \frac{1}{3} \Phi(v_a) \frac{1}{3} k^2 m_1 + \Phi(v_a) \frac{1}{3} k^2 N_{12} + v_a \varphi(v_a) \left( \frac{\rho}{\sqrt{m_1}} - \frac{\rho'}{\sqrt{N_{12}}} \right) / 3k^2
\]
\[
+ (1 - \lambda_{12})\delta^2 \Phi(v_a) - 2(1 - \lambda_{12})\delta \rho' \varphi(v_a) / \sqrt{3N_{12}} k.
\]
For sufficiently small \( \delta \), say \( \delta = o(1/\sqrt{N_{12}}) \), we get \( \rho = -\sqrt{1 - \lambda_{12}} \) and \( \rho' = o(1/\sqrt{N_{12}}) \) from (22) and then
\[
M.S.E. (\hat{\theta}) = \frac{1}{3} \Phi(v_a) \frac{1}{3} k^2 m_1 + \Phi(v_a) / 3k^2 N_{12}
\]
\[
+ (1 - \lambda_{12}) v_a \varphi(v_a) / 3k^2 m_1
\]

\( v_a = z_a / \sqrt{12\lambda_{12}(1 - \lambda_{12})} \).

Thus the asymptotic efficiency \( e_{s,a} \) of \( \hat{\theta} \) relative to the "never pool" estimate \( \theta_1 \) may be given by (30),
\[
e_{s,a} = \left[ 1 - (1 - \lambda_{12}) \Phi(v_a) - v_a \varphi(v_a) \right]^{-1}.
\]
When \( m_1 = m_2 \), we get the following Table II for \( \theta_{e,n} \).

<table>
<thead>
<tr>
<th>( v )</th>
<th>1.65</th>
<th>1.30</th>
<th>1.00</th>
<th>.5</th>
<th>0</th>
<th>-0.5</th>
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<td>1.428</td>
<td>1.347</td>
<td>1.333</td>
<td>1.320</td>
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References


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