SOME ESTIMATE PROCEDURES WITH A NONPARAMETRIC PRELIMINARY TEST I

Tamura, Ryoji
Shimane University

https://doi.org/10.5109/13017

出版情報：統計数理研究. 11 (3/4), pp.63-71, 1965-03. Research Association of Statistical Sciences
バージョン：
権利関係：

SOME ESTIMATE PROCEDURES WITH A NONPARAMETRIC PRELIMINARY TEST I

By

Ryoji TAMURA

(Received March 15, 1964)

§ 1. **Introduction.** The auther has developed in [2] a "*sometimes pool*" procedure about sample medians as an estimate of the population median. In this paper, each estimate of a shift parameter of the population and the population median may be discussed along the same line as [2]. First we consider an estimate of a shift parameter.

Let $X_1, \cdots, X_{m_1}$ and $Y_1, \cdots, Y_{m_2}$ be two samples from the populations with continuous *c.d.f.* $F(x)$ and $F(x-\Delta)$. Hodges and Lehmann [1] have defined as an estimate of Δ the following statistic (1) which is the median of the set of $m_1 m_2$ differences $(Y_j - X_i)$

$$(1) \qquad \hat{\Delta}_{12} = med(Y_j - X_i).$$

And they have shown some excellent properties about (1), for example,

(i) $\hat{\Delta}_{12}$ is a median unbiased estimate

(ii) the asymptotic efficiency of $\hat{\Delta}_{12}$ relative to the classical estimate $\bar{Y} - \bar{X}$ ($^-$ denote sample mean) is $3/\pi$ in the case of normal $F(x)$.

Now consider the case where there exists another sample $Z_1, \cdots, Z_{m_3}$ with the *c. d. f.* $F(x-\Delta-\delta)$, $\delta \geqq 0$. In such case, it will be more effective to consider the estimation procedures after testing the hypothesis $\delta = 0$ against the alternative $\delta > 0$. Therefore we first perform a preliminary test by the Mann-Whitney statistic

$$(2) \qquad U_{23}(Y,Z) = (m_2 m_3)^{-1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3} \phi(Y_j, Z_k)$$

, where
$$\phi(y, z) = \begin{cases} 1 & \text{for } y < z \\ 0 & \text{otherwise} \end{cases} .$$

As a second problem, let $X_1, \cdots, X_{m_1}$ be the sample from the population with continuous *c. d. f.* $F(x-\theta)$ where $F(x)$ is symmetric about zero. They have also derived an estimate $\hat{\theta}_1$ of θ which is the median of $\binom{m_1+1}{2}$ average $\frac{1}{2}(X_i + X_j)$

$$(3) \qquad \hat{\theta} = med\left[\frac{1}{2}(X_i + X_j)\right], \quad 1 \leq i \leq j \leq m_1 .$$

Suppose that there exists another sample $Y_1, \cdots, Y_{m_2}$ from the distribution $F(x-\theta-\delta)$, $\delta \geq 0$. Then we shall first consider testing the hypothesis $\delta = 0$ against the alternative $\delta > 0$ by the statistic

$$(4) \qquad U_{12}(X, Y) = (m_1 m_2)^{-1} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \phi(X_i, Y_j).$$

Section 2 is concerned with the "*sometimes pool*" procedure about an estimate of Δ and the procedure for θ is dealt in section 3.

§ 2. An estimate about a shift parameter Δ. First define the similar statistics as (1), (2) and (3)

$$(5) \qquad \hat{\Delta}_{13} = med\ (Z_k - X_i)$$

$$(6) \qquad U_{13}(X, Z) = (m_1 m_3)^{-1} \sum_{i=1}^{m_1} \sum_{k=1}^{m_3} \phi(X_i, Z_k).$$

Then the estimate $\hat{\Delta}$ of Δ is formulated by the following

$$(7) \qquad \begin{array}{ll} \hat{\Delta} = \hat{\Delta}_{12} & \text{when } U_{23} \geq u_\alpha \\ \hat{\Delta} = \xi \hat{\Delta}_{12} + (1-\xi)\hat{\Delta}_{13} & \text{when } U_{23} < u_\alpha \end{array}$$

, where $\qquad u_\alpha = \frac{1}{2} z_\alpha / \sqrt{N_{23}} \sqrt{12\lambda_{23}(1-\lambda_{23})}$, $m_i + m_j = N_{ij}$, $\lambda_{ij} = m_i / N_{ij}$,

and $1-\Phi(z_\alpha) = \alpha$, $\Phi(x)$ is the standard normal *c.d.f.* $N(0, 1)$. We shall intend to derive the asymptotic mean value and mean square error about Δ of the estimate $\hat{\Delta}$ (7) and moreover to evaluate the asymptotic efficiency relative to the "*never pool*" estimate $\hat{\Delta}_{12}$. As the first step, consider the asymptotic joint distribution of U_{23} and $\hat{\Delta}_{12}$. The following are so easy to show that their proofs are omitted.

$$\mu_{23} = E(U_{23}) = \int_{-\infty}^{\infty} F(x)\ dF(x-\delta)$$

$$\sigma_{23}^2 = Var(\sqrt{N_{23}} U_{23}) \sim (r_{23} - \mu_{23}^2)/\lambda_{23} + (q_{23} - \mu_{23}^2)/(1-\lambda_{23})$$

$$q_{23} = \int_{-\infty}^{\infty} F^2(x)\ dF(x-\delta), \quad r_{23} = \int \{1 - F(x-\delta)\}^2\ dF(x)$$

, where $\sim$ means asymptotic equality.
By the results of Hodges and Lehmann [1]

(8) $$Pr(\hat{\Delta}_{12}<a)=Pr(U_{12}(X,\ Y-a)<\mu_0)$$

, where μ_0 is the expected value of $U_{12}(X,Y)$ under $\Delta=0$, that is $\frac{1}{2}$, then we may get the identity

$$G(u,\ a)=Pr[\sqrt{N_{23}}(U_{23}-\mu_{23})/\sigma_{23}<u,\ \sqrt{N_{12}}\ (\hat{\Delta}_{12}-\Delta)<a]$$
$$=P_r[\sqrt{N_{23}}\ (U_{23}-\mu_{23})/\sigma_{23}<u,\ \sqrt{N_{12}}\ \{U_{12}\ (X,Y-\Delta-\frac{a}{\sqrt{N_{21}}})-\mu_{12}'\}/\sigma_{12}'$$
$$<\sqrt{N_{12}}\ (\frac{1}{2}-\mu_{12}')/\sigma_{12}']$$

, where

$$\mu_{12}'=EU_{12}(X,Y-\Delta-\frac{a}{\sqrt{N_{12}}})\sim\frac{1}{2}-a\int_{-\infty}^{\infty}f^2(x)dx/\sqrt{N_{12}}$$

$$\sigma_{12}'^2=Var\ U_{12}(X,\ Y-\Delta-\frac{a}{\sqrt{N_{12}}})\sim 1/12\lambda_{12}(1-\lambda_{12})$$

and then

$$\sqrt{N_{12}}\ (\frac{1}{2}-\mu_{12}')/\sigma_{12}'\sim a\sqrt{12\lambda_{12}(1-\lambda_{12})}\int_{-\infty}^{\infty}f^2(x)dx\ \ (=ak_{12}).$$

Since it is well-known that the joint *c.d.f.* of U-statistics is asymptotically normal, we may find the asymptotic expression

(9) $$G(u,a)=\int_{-\infty}^{u}\int_{-\infty}^{ak_{12}}g(x,y\,;\ 0,Q\,;\ 1,1\,;\ \rho_{12})\ dx\ dy$$

, where $g(x,y\,;\ \mu_x,\mu_y\,;\ \sigma_x^2,\sigma_y^2\,;\ \rho_{xy})$ denote the bivariate normal density function with means μ_x, μ_y, variances σ_x^2, σ_y^2 and correlation ρ_{xy} and

(10) $$\rho_{12}=[\int_{-\infty}^{\infty}F(x)\{1-F(x-\delta)\}\ dF(x)-\mu_{12}'\mu_{23}]/\sqrt{\lambda_{23}(1-\lambda_{12})}\ \sigma_{12}'\ \sigma_{23}.$$

The asymptotic joint density of $\sqrt{N_{23}}\ (U_{23}-\mu_{23})$ and $\hat{\Delta}_{12}$ may be obtained from (9) as follows,

(11) $$g(x,\ y\,;\ 0,\ \Delta\,;\ \sigma_{23}^2,\ 1/N_{12}k_{12}^2\,;\ \rho_{12}).$$

By the similar techniques, though slightly complicated, we may find the asymptotic joint density of $\sqrt{N_{23}}\ (U_{23}-\mu_{23})$ and $\xi\hat{\Delta}_{12}+(1-\xi)\hat{\Delta}_{13}$,

(12) $$g(x,\ y\,;\ 0,\ \xi\Delta+(1-\xi)(\Delta+\delta)\,;\ \sigma_{23}^2,\ \sigma_{1.23}^2\,;\ \rho_{1.23})$$

, where

$$(13)\quad \sigma_{1.23}^{2} = \left[\frac{\xi^2}{N_{12}\lambda_{12}(1-\lambda_{12})} + \frac{(1-\xi)^2}{N_{13}\lambda_{13}(1-\lambda_{13})} + \frac{2\xi(1-\xi)}{m_1}\right] \Big/ 12\left(\int f^2 dx\right)^2$$

$$\rho_{1.23} = \left[\frac{\xi}{\lambda_{23}}\left\{\int_{-\infty}^{\infty} F(x)\{1-F(x-\delta)\}\, dF(x) - \mu_{23}\mu_{12}'\right\}\right.$$

$$\left. + \frac{1-\xi}{1-\lambda_{23}}\left\{\int_{-\infty}^{\infty} F(x)F(x+\delta)\, dF(x) - \mu_{23}\mu_{13}'\right\}\right] \Big/ \sqrt{N_{23}}\,\sigma_{23}\sigma_{1.23}\left(\int f^2 dx\right).$$

From the definition (7) of $\hat{\varDelta}$ and (11), (12), the asymptotic probability of $\hat{\varDelta}$ may be expressed as follows,

$$Pr(\hat{\varDelta}<a) = Pr[\sqrt{N_{23}}(U_{23}-\mu_{23})/\sigma_{23} \geqq v_\alpha,\ \hat{\varDelta}_{12}<a]$$

$$+ Pr[\sqrt{N_{23}}(U_{23}-\mu_{23})/\sigma_{23}<v_\alpha,\ \xi\hat{\varDelta}_{12}+(1-\xi)\hat{\varDelta}_{13}<a]$$

$$= \int_{v_\alpha}^{\infty}\int_{-\infty}^{a} g(x,\ y;\ 0,\ \varDelta;\ 1,\ 1/N_{12}k_{12}^2;\ \rho_{12})\ dx\ dy$$

$$+ \int_{-\infty}^{v_\alpha}\int_{-\infty}^{a} g(x,\ y;\ 0,\ \varDelta+(1-\xi)\ \delta;\ 1,\ \sigma_{1.23}^2;\ \rho_{1.23})\ dx\ dy$$

, where $v_\alpha = \sqrt{N_{23}}(u_\alpha-\mu_{23})/\sigma_{23}$.
The probability above leads to the asymptotic density $w(a)$ of $\hat{\varDelta}$

$$(14)\quad w(a) = \int_{v_\alpha}^{\infty} g(x,\ a;\ 0,\ \varDelta;\ 1,\ 1/N_{12}k_{12}^2;\ \rho_{12})\ dx$$

$$+ \int_{-\infty}^{v_\alpha} g(x,\ a;\ 0,\ \varDelta+(1-\xi)\delta;\ 1,\ \sigma_{1.23}^2;\ \rho_{1.23})\ dx.$$

From (14),

$$E(\hat{\varDelta}) = \int_{-\infty}^{\infty} a\ da\left[\int_{v_\alpha}^{\infty} g(x,\ a;\ 0,\ \varDelta;\ 1,\ 1/N_{12}k_{12}^2;\ \rho_{12})\ dx\right.$$

$$\left. + \int_{-\infty}^{v_\alpha} g(x,\ a;\ 0,\ \varDelta+(1-\xi)\delta;\ 1,\ \sigma_{1.23}^2;\ \rho_{1.23})\ dx\right].$$

After some computations, we may get the mean value

$$(15)\qquad \mathrm{E}(\hat{\varDelta}) = \varDelta + (1-\xi)\delta\Phi(v_\alpha) + \left(\frac{\rho_{12}}{\sqrt{N_{12}}k_{12}} - \sigma_{1.23}\rho_{1.23}\right)\varphi(v_\alpha)$$

By the similar computations,

$$M.S.E.\ (\hat{\Delta})=E(\hat{\Delta}-\Delta)^2$$

(16) $$=\frac{1}{N_{12}k_{12}^2}\{1-\Phi(v_\alpha)\}+\sigma_{1.23}^2\Phi(v_\alpha)+\left(\frac{\rho_{12}^2}{N_{12}k_{12}^2}-\sigma_{1.23}^2\rho_{1.23}^2\right)v_\alpha\varphi(v_\alpha)$$

$$+\delta^2(1-\xi)^2\Phi(v_\alpha)-2\delta(1-\xi)\sigma_{1.23}\rho_{1.23}\varphi(v_\alpha).$$

We suppose $\delta=0$ and determine the value of ξ minimizing the mean square error (16). From the equation $d[M.S.E.(\hat{\Delta})]/d\xi=0$, we may easily get the identity

$$\{\Phi(v_\alpha)-v_\alpha\varphi(v_\alpha)\}(\xi-\lambda_{23})=0$$

and we get $\xi=\lambda_{23}$.

Though $\xi=\lambda_{23}$ does not give the exact minimum value of the mean square error of $\hat{\Delta}$ in the case of $\delta\neq0$, we adopt $\xi=\lambda_{23}$ for sufficiently small $\delta>0$. Lastly in the case where δ is sufficiently small, say $\delta=o(1/\sqrt{N})$, we shall compute the asymptotic efficiency of $\hat{\Delta}$. It follows from (10) and (13) that

$$\rho_{12}=-\sqrt{\lambda_{12}(1-\lambda_{23})}\ ,\ \rho_{1.23}=o(1/\sqrt{N}).$$

Then from (16)

(17) $$M.S.E.\ (\hat{\Delta})=[1-\lambda_{12}(1-\lambda_{23})\{\Phi(v_\alpha)-v_\alpha\varphi(v_\alpha)\}]/12(\int f^2dx)^2N_{12}.$$

Let the asymptotic efficiency of Δ with regard to the "*never pool*" estimate Δ_{12}, which is defined by the reciprocal ratio of asymptotic variances , be denoted by $e_{s,n}$, then we get the following

(18) $$e_{s,n}=[1-\lambda_{12}(1-\lambda_{23})\{\Phi(v_\alpha)-v_\alpha\varphi(v_\alpha)\}]^{-1}.$$

In the case of $\lambda_{ij}=1/2$, we get the Table I

TABLE I

v	1.65	1.30	1.00	.5	0	—.5
e	1.243	1.205	1.176	1.148	1.143	1.138

§ 3. **An estimate about the median θ.** As stated in section 1, we assume that

$$Pr(X\leq x)=F(x-\theta),\ Pr(Y\leq y)=F(y-\theta-\delta)$$

and $F(x)$ is symmetric about zero. In order to formulate the estimate $\hat{\theta}$ of θ, we first define

$$(19)\qquad \begin{aligned} U_1(X) &= \binom{m_1+1}{2}^{-1} \sum_{1\leq i\leq j\leq m_1} \psi(X_i, X_j) \\ U_2(Y) &= \binom{m_2+1}{2}^{-1} \sum_{1\leq i\leq j\leq m_2} \psi(Y_i, Y_j) \end{aligned}$$

, where $\psi(x, x')=1$ or 0 if $x+x'>0$ or otherwise and

$$(20)\qquad \theta_2 = med\left[\frac{1}{2}(Y_i+Y_j)\right],\ 1\leq i\leq j\leq m_2$$

is the median of $\binom{m_2+1}{2}$ average $(Y_i+Y_j)/2$.

The preliminary test is performed by Mann-Whitney statistic U_{12} (4). We define an estimate $\hat{\theta}$ of θ as follows,

$$(21)\qquad \begin{aligned} \hat{\theta} &= \hat{\theta}_1 & \text{when } U_{12}\geq u_\alpha \\ \hat{\theta} &= \lambda_{12}\hat{\theta}_1+(1-\lambda_{12})\,\hat{\theta} & \text{when } U_{12}<u_\alpha \end{aligned}$$

, where $u_\alpha=\frac{1}{2}+z_\alpha/\sqrt{N_{12}}\sqrt{12\lambda_{12}(1-\lambda_{12})}$.

By the similar considerations and computations as section 2, we may find the following values

$$(22)\qquad \begin{aligned} \mu_{12} &= EU_{12} = \int_{-\infty}^{\infty} F(x)\ dF(x-\delta) \\ \sigma_{12}^2 &= Var(\sqrt{N_{12}}U_{12}) \sim (r_{12}-\mu_{12}^2)/\lambda_{12}+(q_{12}-\mu_{12}^2)/(1-\lambda_{12}^2) \\ \mu_i &= EU_i\left(X-\theta-\frac{a}{\sqrt{m_i}}\right) \sim \frac{1}{2}-2a\int f^2dx/\sqrt{m_i} \\ \sigma_i^2 &= Var\ U_i\left(X-\theta-\frac{a}{\sqrt{m_i}}\right) \sim 1/3m_i \quad ,\quad i=1,2 \\ \rho &= \text{corr.}\left(U_{12}, U_1(X-\theta-\frac{a}{\sqrt{m_1}})\right) = \sqrt{3}\ [2\int_{-\infty}^{\infty} F(x)\{1-F(x-\delta)\} \\ &\qquad dF(x)-\mu_{12}]/\sqrt{\lambda_{12}}\sigma_{12} \\ \rho' &= \text{corr.}\left(U_{12}, U_2(Y-\theta-\delta-\frac{a}{\sqrt{m_2}})\right) = \sqrt{3}\ [2\int_{-\infty}^{\infty} F(x)F(x+\delta)dF(x)-\mu_{12}] \\ &\qquad /\sqrt{1-\lambda_{12}}\sigma_{12}\ . \end{aligned}$$

In fact, denoting $Z=X-\theta-a/\sqrt{m_1}$,

$$\sigma_1^2 = \binom{m_1+1}{2}^{-2}[(2m_1-4)\binom{m_1}{2}\{E\psi(Z_1, Z_2)\psi(Z_1, Z_2)+E\psi(Z_1, Z_1)\cdot\psi(Z_2, Z_3)\}$$

(23)

$$+\binom{m_1-1}{2}^2 E\psi(Z_1, Z_2)\psi(Z_2, Z_3)] - \mu_1^2 + o(1/m_1).$$

And

$$E\psi(Z_1, Z_2)\psi(Z_1, Z_3) = \int\limits_{\substack{x_1+x_2>2a/\sqrt{m_1}\\ x_1+x_3>2a/\sqrt{m_1}}} dF(x_1)dF(x_2)dF(x_3)$$

$$= \frac{1}{3} - 4a\int_{-\infty}^{\infty} fFdF/\sqrt{m_1} + o(1/\sqrt{m_1}).$$

Similarly, we get

$$E\psi(Z_1, Z_2)\psi(Z_3, Z_4) = \frac{1}{4} - 2a\int_{-\infty}^{\infty} f^2dx/\sqrt{m_1} + o(1/\sqrt{m_1})$$

$$E\psi(Z_1, Z_1)\psi(Z_2, Z_3) = \frac{1}{4} - a\left[\int_{-\infty}^{\infty} f^2\, dx + f(0)\right]/2\sqrt{m_1} + o(1/\sqrt{m_1}).$$

Substituting these values into (23), then we find $\sigma_1^2 \sim 1/3m_1$. The other results in (22) may be similarly computed. By using the works of Hodges and Lehmann

(24) $$Pr(\hat{\theta}_1 < a) = Pr(U_1(X-a) < \mu_0)$$

where μ_0 is the expected value of $U_1(X)$ under $\theta = 0$, that is, $\mu_0 = 1/2$, we shall derive the asymptotic joint density of U_{12} and $\hat{\theta}_1$.

$$Pr[\sqrt{N_{12}}\ (U_{12}-\mu_{12}) < u,\ \sqrt{m_1}\ (\hat{\theta}_1 - \theta) < a]$$

$$= Pr\left[\sqrt{N_{12}}(U_{12}-\mu_{12}) < u,\ \sqrt{m_1}\{U_1(X-\theta-\frac{a}{\sqrt{m_1}}) - \mu_1\} < \sqrt{m_1}(\frac{1}{2}-\mu_1)\right]$$

and then $\sqrt{m_1}\left(\frac{1}{2}-\mu_1\right) \sim 2a\int_{-\infty}^{\infty} f^2dx\ (= ak)$ from (22).

Thus the asymptotic normality of $\sqrt{N_{12}}\ (U_{12}-\mu_{12})$ and $\sqrt{m_1}\ (\hat{\theta}_1 - \theta)$ may be established from the fact of the asymptotic normality of U-statistics. The joint density may be written as

(25) $$g(x, y;\ 0, 0;\ \sigma_{12}^2, 1/3k^2;\ \rho).$$

Similarly, we get the asymptotic joint density of $\sqrt{N_{12}}(U_{12}-\mu_{12})$ and $\sqrt{N_{12}}$ $\{\lambda_{12}(\hat{\theta}_1 - \theta) + (1-\lambda_{12})(\hat{\theta}_2 - \theta - \delta)\}$ as follows,

$$(26)\qquad g(x,\ y\ ;\ 0,\ 0\ ;\ \sigma_{12}^2,\ 1/3k^2\ ;\ \rho').$$

From (21) and (25), (26), we may derive the asymptotic distribution of $\hat{\theta}$,

$$Pr\ (\theta<a) = \int_{v_\alpha}^{\infty}\int_{-\infty}^{\sqrt{m_1}(a-\theta)} g(x,\ y\ ;\ 0,\ 0\ ;\ 1,\ 1/3k^2\ ;\ \rho)\ dx\ dy$$

$$+\int_{-\infty}^{v_\alpha}\int_{-\infty}^{\sqrt{N_{12}}(a-\theta-(1-\lambda_{12})\delta)} g(x,\ y\ ;\ 0,\ 0\ ;\ 1,\ 1/3k^2\ ;\ \rho')\ dx\ dy$$

, where $v_\alpha=\sqrt{N_{12}}\ (u_\alpha-\mu_{12\cdot})\sigma_{12}$.
The asymptotic density $w(a)$ of $\hat{\theta}$ may be expressed as

$$(27)\qquad w(a)=\int_{v_\alpha}^{\infty} g(x,\ a\ ;\ 0,\ \theta\ ;\ 1,\ 1/3k^2m_1\ ;\ \rho)\ dx$$

$$+\int_{-\infty}^{v_\alpha} g(x,\ a\ ;\ 0,\ \theta+(1-\lambda_{12})\delta\ ;\ 1,\ 1/3k^2N_{12}\ ;\ \rho')\ dx.$$

By the similar computations as section 2, we get the following

$$(28)\qquad E(\hat{\theta})=\theta+(1-\lambda_{12})\delta\Phi(v_\alpha)+\varphi(v_\alpha)\left(\frac{\rho}{\sqrt{m_1}}-\frac{\rho'}{\sqrt{N_{12}}}\right)/\sqrt{3}\ k$$

$$(29)\qquad M.S.E.\ (\hat{\theta})=E(\hat{\theta}-\theta)^2$$

$$=\{1-\Phi(v_\alpha)\}/3k^2m_1+\Phi(v_\alpha)/3k^2N_{12}+v_\alpha\varphi(v_\alpha)\left(\frac{\rho^2}{m_1}-\frac{\rho'^2}{N_{12}}\right)/3k^2$$

$$+(1-\lambda_{12})^2\delta^2\Phi(v_\alpha)-2(1-\lambda_{12})\delta\rho'\varphi(v_\alpha)/\sqrt{3N_{12}}\ k.$$

For sufficiently small δ, say $\delta=o(1/\sqrt{N_{12}})$, we get $\rho=-\sqrt{1-\lambda_{12}}$ and $\rho'=o(1/\sqrt{N_{12}})$ from (22) and then

$$(29)'\qquad M.S.E.\ (\hat{\theta})=\{1-\Phi(v_\alpha)\}/3k^2m_1+\Phi(v_\alpha)/3k^2N_{12}$$

$$+(1-\lambda_{12})v_\alpha\varphi(v_\alpha)/3k^2m_1$$

, where $v_\alpha=z_\alpha/\sqrt{12\lambda_{12}(1-\lambda_{12})}$.
Thus the asymptotic efficiency $e_{s,n}$ of $\hat{\theta}$ relative to the "*never pool*" estimate $\hat{\theta}_1$ may be given by (30),

$$(30)\qquad e_{s,n}=[1-(1-\lambda_{12})\{\Phi(v_\alpha)-v_\alpha\varphi(v_\alpha)\}]^{-1}.$$

When $m_1 = m_2$, we get the following Table II for $\theta_{s,n}$,

TABLE II

v	1.65	1.30	1.00	.5	0	−.5
e	1.642	1.516	1.428	1.347	1.333	1.320

References

[1] HODGES, J. L. Jr. and LEHMANN, E. L. *Estimates of location based on rank tests* Ann. Math. Statist. (1963) 34 598-611.

[2] TAMURA, R. *Nonparametric inferences with a preliminary test.* (to appear in Bull. Math. Statist.)

SHIMANE UNIVERSITY