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MEDIAN-UNBIASED ESTIMATORS[1]

By

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§ 1. **Summary.** The property of median-unbiasedness is discussed in relation to other probability properties of point estimators. Median-unbiased estimators are given for the parameters of normal, binomial, and Poisson distributions, for which they are shown to be approximately equal to the commonly used mean-unbiased estimators, except with extreme parameter values or very small sample sizes.

§ 2. **Unbiasedness and other properties of point estimators.** An estimator $\theta=\theta^*(x)$ of a real-valued parameter θ is called ***median-unbiased*** if $\text{Prob}\,[\theta^*(X)<\theta \mid \theta]=\text{Prob}\,[\theta^*(X)>\theta \mid \theta]$ for each θ; that is, if for each θ, the median of the estimator's distribution is θ. For any estimator with continuous distributions, this condition takes the form $\text{Prob}\,[\theta^*(X)\leqq\theta \mid \theta]=1/2$ for each θ.

A central place in estimation theory has long been occupied by the property or "criterion" of ***mean-unbiasedness*** (usually called simply "unbiasedness", a designation which would risk ambiguity in the present discussion): θ^* is mean-unbiased if $E\,[\theta^*(X) \mid \theta]=\theta$ for each θ. But as Savage has remarked ([1], p. 244) "it is now widely agreed that a serious reason to prefer unbiased estimates seems never to have been proposed." Recent discussions of these properties of estimators may be found in [2]; [3], pp. 10–12, 22, 83, 174; [4], pp. 2.1, 3.13–3.15, and in further references cited therein. Some principal points of these discussions and some complementary comments are given in the following paragraphs:

(a) If $\theta^*(x)$ is a median-unbiased estimator of θ, then any strictly monotone function $g(\theta)$ has the median-unbiased estimator $g(\theta^*(x))$. This property, which is not shared by mean-unbiased estimators, seems particularly natural and convenient in contexts of application where an estimated value may have to be substituted for θ in several different formulae, not all linear.

(b) Mean-unbiasedness is sometimes incompatible with considerations of precision of estimators which seem more basic. For example, even if a parameter is known to have a positive value, each mean-unbiased estimator may have to assume negative values with positive probability. In such

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cases the obvious improvement of replacing negative estimates by zero destroys mean-unbiasedness, but such improvement does not affect the property of median-unbiasedness.

(c) In the important areas of linear estimation (regression) theory and multivariate analysis, with normality of error-distributions assumed, the classical "best unbiased" estimators are normally distributed, and hence are median-unbiased as well as mean-unbiased. Conversely, in these problems we may consider as an alternative to the classical criterion "best unbiased", the *criterion* "uniformly best median-unbiased". Both criteria are satisfied uniquely by the classical estimators, and so are mathematically equivalent under the assumptions mentioned. Hence one is free to choose which, if either, is preferable as an expression of the notion of a good estimator; and if one criterion is chosen, the other becomes a mathematically-entailed property. In this sense, the criterion of mean-unbiasedness need not be regarded as essentially bound up with the general reasonableness and usefulness of the classical estimators. (The latter comment does not apply to the important theory and techniques of linear estimation when normality of errors is not assumed.) The same comments apply to asymptotic estimation theory, where asymptotically normal and "asymptotically unbiased" estimators play a basic role; here "asymptotically median-unbiased" is equivalent to "asymptotically mean-unbiased", with respect to all asymptotically normal estimators.

Thus within the scope of normal-error estimation theory, the customary reference to mean-unbiasedness, which seems to stem from Gauss' fundamental work, can be replaced whenever desired by a reference to median-unbiasedness without affecting the essential content of the theory. It is interesting that an earlier formulation of the problem of point-estimation, that of Laplace in 1774 ([5], p. 636), took median-unbiasedness as the criterion of location of an estimator's distribution, along with minimization of mean absolute-error as the criterion of concentration; Laplace's results included that the sample mean (i. e. the classical estimator) is best in terms of his criteria in the case of normal errors but in no other case. (The writer is indebted to Dr. Churchill Eisenhart for references to Laplace's results.)

(d) It is interesting to try to account for the fact that the criterion of mean-unbiasedness came to occupy a central position in the development of estimation theory, when in retrospect no very clear grounds can be found to support that position. It seems likely that attention was focused on mean-unbiasedness as a consequence of the interest which developed during the seventeenth century in problems of making fair ("unbiased") economic valuations ("estimates") of articles or commodities being bought and sold ([6]). For such purposes, if precise valuations are not available, equity requires that any method of valuation should at least in the long run give each party fair value, and hence should be mean-unbiased. Consideration of such problems was complicated by the simultaneous early consideration of problems of estimation involving observational errors in surveying and astronomical work. An interesting contribution to the rather obscure but heated controversial discussion of such problems was made by Galileo (c. f. [6]), who stressed the distinction between "estimation" in the sense of fair valuation, for which mean-unbiasedness seemed relevant and appropriate, and "estimation" in the sense of the astronomer and surveyor, for whom errors of over- and underestimation could not necessarily be claimed to be even comparable. The latter non-comparability entails the lack of inherent meaningfulness of "mean-unbiasedness". Galileo's comments seem to have merited more attention than they received.

(e) For many typical purposes of estimation, several writers have proposed that a confidence limit or interval at a uniquely chosen confidence level can with advantage be replaced by use of a nested set of confidence limits or intervals ([7], [8], [9], [10]). In such "omnibus" estimates, the center of the nested set of limits or intervals is a point estimator with the property of median-unbiasedness. (A median-unbiased estimator is, formally, an upper (and also a lower) 50 per cent confidence limit estimator.)

(f) All of the considerations of the present paper are based on probability distribution properties of point estimators, in the sense of [10], for example, and are not particularly relevant to the more basic issues of statistical inference treated, for example, in [11].

§ 3. **Best median-unbiased estimators in simple standard problems.** The simplest standard problems of estimation are those in which the family of densities $f(x, \theta)$, $\theta \varepsilon \Omega$, satisfies the monotone likeliheod ratio condition ([3], p. 68). Under this condition, it was shown in [10] that a uniformly-best median-unbiased estimator exists (and is admissible). For example, in the problem of estimating the mean of a normal distribution (with known or unknown variance), on the basis of n independent observations $y_1, \cdots y_n$, the classical estimator $\hat{\theta}=\bar{y}$ is the uniformly-best median-unbiased estimator when Ω is the real line. Other examples are discussed in the following paragraphs.

In general, if t is a sufficient statistic for the real-valued parameter θ, with the monotone likelihood ratio property, and if its cumulative distribution $F(t, \theta)$is continuous in t and θ, then the equation $F(t, \tilde{\theta})=1/2$ defines implicitly the uniformly-best median-unbiased estimator $\tilde{\theta}=\tilde{\theta}(t)$. Thus a table of $F(t, \theta)$ as a function of t, for each θ, or the equivalent, is the basis for determination of such estimates.

If $F(t, \theta)$ is not continuous in t, such estimators can nevertheless be constructed by use of auxiliary randomization variables. However, for reasons such as those discussed in [11], Sect. 3, many statisticians feel that use of such randomization is well avoided for most typical purposes of estimation. The criterion of median-unbiasedness cannot be met exactly without randomization when t has discrete distributions; but the criterion may be considered most nearly satisfied by a non-randomized (admissible) estimator $\tilde{\theta}(t)$ with "minimum median-bias", defined implicitly, for each t, by the equation

$$\text{Prob}(T<t \mid \tilde{\theta}) = \text{Prob}(T>t \mid \tilde{\theta}).$$

Such equations are conveniently solved for $\tilde{\theta}(t)$ by use of suitable tables of $F\ (t,\ \theta)$; some examples are given below.

3.1 Normal variance or standard deviation. In the problem of estimation of the variance σ^2 of a normal distribution (with known or unknown mean), on the basis of N independent observations $y_1, \cdots y_n$, the best median-unbiased estimator is

$$\tilde{\sigma}^2 = s^2 k_n^2, \text{ where } k_n^2 = \frac{n}{\chi^2_{n,0.5}},$$

and similarly for the normal standard deviation σ,

$$\tilde{\sigma} = s k_n,$$

where s^2 is the usual mean-unbiased estimator of σ^2 based on n degrees of freedom. Here $\chi^2_{n,0.5}$ denotes the median of the chi-square distribution with n degrees of freedom; $n=N$ and $s^2 = \frac{1}{N}\sum_i^N (y_i-\mu)^2$ if $\mu = E(y)$ is known; and $n=N-1$ and $s^2 = \frac{1}{n}\sum_i^N (y_i - \bar{y})^2$ if μ is unknown.

Table 1 gives values of k_n^2 and k_n which can be used to compute $\tilde{\sigma}^2$ or $\tilde{\sigma}$ from values of the classical estimates s^2 or s. Since $1<k_n<1.02$ for $n \geqq 18$ and $1<k_n<1.06$ for $n \geqq 6$, this modification of the classical estimators is a quantitatively minor one except for small n, and for many purposes it will suffice to take the approximate values $\tilde{\sigma}^2 \doteq s^2$ and $\tilde{\sigma} \doteq s$ except for small n.

If the criteria of admissibility and median-unbiasedness are adopted for this problem we see that the classical estimates s^2 and s are justified as very convenient and close approximations to the best estimates except for small n. The magnitudes of the median-bias of the classical estimators,

$B(\sigma^2,\ s^2) \equiv B(\sigma,\ s) = \text{Prob}\{s^2 > \sigma^2 \mid \sigma^2\} - \text{Prob}\{s^2 < \sigma^2 \mid \sigma^2\}$, are independent of σ; they are shown in Table 2 for various n. For example, for $n=70$, s^2 underestimates σ^2 with probability. $0.5115 = 0.50 + (0.023)/2$.

3.2 Poisson mean. Let $x = (y_1, \cdots y_n)$ be a sample of n independent

TABLE 1. *Constants for computing best median-unbiased estimators of the variance or standard deviation of a normal distribution, from values of the classical estimators.*

$\tilde{\sigma}^2 = k_n^2 s^2$ *and* $\tilde{\sigma} = k_n s$, *where* $s^2 = \left[\frac{\sum_{i=1}^{n+1}(x_i - \bar{x})^2}{n}\right]$ *if* $\mu = E(X)$ *is unknown, and* $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$ *if* μ *is known.*

n	k_n^2	k_n	n	k_n^2	k_n
1	2.198	1.483	20	1.034	1.017
2	1.443	1.201	21	1.033	1.016
3	1.268	1.126	22	1.031	1.015
4	1.192	1.092	23	1.030	1.015
5	1.149	1.072	24	1.028	1.014
6	1.122	1.059	25	1.027	1.014
7	1.103	1.050	26	1.026	1.013
8	1.089	1.044	27	1.025	1.013
9	1.079	1.039	28	1.024	1.012
10	1.070	1.035	29	1.023	1.012
11	1.064	1.031	30	1.023	1.011
12	1.058	1.029	40	1.017	1.008
13	1.054	1.026	50	1.013	1.007
14	1.050	1.025	60	1.011	1.006
15	1.046	1.023	70	1.010	1.005
16	1.043	1.021	80	1.008	1.004
17	1.041	1.020	90	1.007	1.004
18	1.038	1.019	100	1.007	1.003
19	1.036	1.018	1000	1.001	1.000

TABLE 2. *Median-bias of classical unbiased estimator* s^2 *of the variance* σ^2 *of a normal distribution.*
Bias $B(s^2, \sigma^2) = Prob\{s^2 > \sigma^2\} - Prob\{s^2 < \sigma^2\} = 2[Prob\{s^2 > \sigma^2\} - 1/2]$.

n	$1/2\, B(s^2, \sigma^2)$	n	$1/2\, B(s^2, \sigma^2)$
1	−0.183	10	−0.060
2	−0.132	15	−0.049
3	−0.108	20	−0.042
4	−0.094	30	−0.034
5	−0.084	40	−0.030
6	−0.077	50	−0.027
7	−0.071	60	−0.024
8	−0.067	70	−0.023
9	−0.063		

observations from a Poisson distribution with unknown mean θ, $0<\theta<\infty$; then the sufficient statistic $z=\sum_{i=1}^{n} y_i$ has the Poisson distribution with mean $n\theta$,

$$h(z, \theta) = \text{Prob}\{Z=z \mid \theta\} = e^{-n\theta}\frac{(n\theta)^z}{z!}, \quad z=0, 1, 2\cdots,$$ which has the

monotone likelihood ratio property. For each $z=0, 1, 2\cdots$, $\tilde{\theta}(z)$ is the value of θ for which

$$\text{Prob}\{Z<z \mid \theta=\tilde{\theta}(z)\} = \text{Prob}\{Z>z \mid \theta=\tilde{\theta}(z)\}.$$

Such values of $\tilde{\theta}(z)$ are easily determined by use of tables of the Poisson distribution, and are given in Table 3. A single such table suffices for all sample sizes n, since the distribution of z depends on $n\theta$ but not on n and θ separately; hence the table gives, for each z, the value of $n\tilde{\theta}(z)$, which is to be divided by the sample size n occurring in any particular application.

TABLE 3. *Minimum-median-bias estimator $\tilde{\theta}$ of Poisson mean θ.*
$z=\sum_{i=1}^{n} y_i=$ *sample total* $=n\hat{\theta}$, *where* $\hat{\theta}=$ *classical estimate* $=z/n$.

z	$n\tilde{\theta}(z)$
0	0
1	1.146
2	2.156
3	3.159
4	4.161
5	5.162
·	
·	
·	

z	$n\tilde{\theta}(z)$
9	9.166
10	10.165
·	
·	
·	
14	14.165
·	
20	20.17
25	25.17

It will be seen that $\tilde{\theta}(z)$ differs only slightly from the classical estimator $\hat{\theta}(z)=z/n$: for $0 \leq z \leq 25$,

$$z \equiv n\hat{\theta}(z) \leq n\tilde{\theta}(z) < z+0.2.$$

Table 4 compares the median-bias functions of $\tilde{\theta}(z)$ and $\hat{\theta}(z)$; again the differences are slight. Thus for most purposes the classical estimator $\hat{\theta}(z)$ will serve as a convenient and close approximation to $\tilde{\theta}(z)$. These comparisons provide a new justification for the classical mean-unbiased estimator, having approximately minimum median-bias among non-randomized admissible estimators.

3.3 Binomial mean. Let $x=(y_1, \cdots y_n)$ be a sample of n independent Bernoulli observations, $\text{Prob}\{Y_i=1 \mid \theta\}=\theta$, $\text{Prob}\{Y_i=0 \mid \theta\}=1-\theta$, with θ unknown, $0<\theta<1$. Then the sufficient statistic

$$z=\sum_{i=1}^{n} y_i \text{ has the binomial distribution}$$

$$h(z, \theta)=\text{Prob}\{Z=z \mid \theta\}=\binom{n}{z}\theta^z(1-\theta)^{n-z}, \quad z=0, 1, \cdots n.$$

For each sample size n, an estimator $\tilde{\theta}(z)$ can be determined, from tables of the binomial distribution, which has minimum median-bias in the above sense. Table 5 gives such estimates $\tilde{\theta}(z)$, for sample sizes $n=3, 5, 10,$ and 20, in comparison with the classical estimates $\hat{\theta}(z)=\frac{z}{n}$. For $n=2$, $\hat{\theta}(z)\equiv$

TABLE 4. *Median-bias of classical estimator* $\hat{\theta}=\sum_{i=1}^{n} y_i/n$ *of the mean* θ *of a Poisson distribution compared with that of estimator* $\tilde{\theta}$ *of table* 3. *For each sample size* n, *Prob* $\{\hat{\theta}>\theta \mid \theta\}-$ *Prob* $\{\hat{\theta}<\theta \mid \theta\}=B(n\theta, n\hat{\theta})$, *and Prob* $\{\tilde{\theta}>\theta \mid \theta\}-$ *Prob* $\{\tilde{\theta}<\theta \mid \theta\}=B(n\theta, n\tilde{\theta})$.

$n\theta$	$1/2\,B(n\theta, n\hat{\theta})$	$1/2\,B(n\theta, n\tilde{\theta})$
0.0	0	Same, except where values are given.
0.0+	−0.500	
0.005	−0.495	
0.01	−0.490	
0.05	−0.451	
0.10	−0.405	
0.15	−0.361	
0.20	−0.319	
0.25	−0.279	
0.30	−0.141	
0.40	−0.170	
0.50	−0.107	
0.60	−0.048	
0.70	+0.003	
0.80	+0.051	
0.90	+0.093	
1.0	−0.236	+0.132
1.1	−0.199	
1.2	−0.163	
1.3	−0.127	
1.4	−0.092	
1.5	−0.058	
1.6	−0.025	
1.7	+0.007	
1.8	+0.037	
1.9	+0.066	
2.0	−0.177	+0.094
2.2	−0.123	
2.4	−0.070	
2.6	−0.018	
2.8	+0.031	
3.0	−0.147	+0.077
3.2	−0.103	
3.4	−0.058	
3.6	−0.015	
3.8	+0.027	
4.0	−0.129	+0.067
4.2	−0.090	
4.4	−0.051	
4.6	−0.013	
4.8	+0.024	
5.0	−0.116	+0.060
5.2	−0.080	
5.4	−0.046	
5.6	−0.012	
5.8	+0.022	
6.0	−0.106	+0.054
6.5	−0.027	
7.0	−0.099	+0.050
7.5	−0.025	
8.0	−0.093	+0.047
8.5	−0.023	
9.0	−0.087	+0.044
10.0	−0.083	+0.042
11.0	−0.079	+0.040
14.0	−0.070	+0.036
15.0	−0.068	+0.034
⋮		
19.0	−0.061	+0.031
⋮		
20.0	−0.059	+0.030
⋮		
21.0	−0.058	+0.029
25.0	−0.053	+0.027
30.0	−0.048	+0.024
40.0	−0.042	+0.021
50.0	−0.038	+0.019
70.0	−0.032	+0.016
100.0	−0.027	+0.013

$\tilde{\theta}(z)$. It is seen that $|\tilde{\theta}(z)-\hat{\theta}(z)|\leq 0.016$ for $n\geq 5$, and $|\tilde{\theta}(z)-\hat{\theta}(z)|\leq 0.007$ for $n\geq 20$. For $n=10$, the median-bias of the classical estimator is compared with that of $\tilde{\theta}$ in Table 6. For $n=20$, the same comparison is given in Table 7. Again all of the differences are slight.

Thus if the estimator $\tilde{\theta}(z)$ is adopted on the criterion of having minimum median-bias among non-randomized admissible estimators, for many purposes the classical mean-unbiased estimator $\hat{\theta}$ will serve as a convenient

TABLE 5. *Minimum-median-bias estimator* $\tilde{\theta}$ *of the binomial parameter* θ, *compared with the classical estimator* $\hat{\theta}$.

$$z=\sum_{i=1}^{n} y_i, \quad \hat{\theta}=z/n.$$

$n=3$

z	$\hat{\theta}$	$\tilde{\theta}$
0	0	0
1	0.3	0.347
2	0.6	0.653
3	1.0	1.0

$n=5$

z	$\hat{\theta}$	$\tilde{\theta}$
0	0	0
1	0.2	0.216
2	0.4	0.406
3	0.6	0.594
4	0.8	0.784
5	1.0	1.0

$n=10$

z	$\hat{\theta}$	$\tilde{\theta}$
0	0	0
1	0.1	0.111
2	0.2	0.209
3	0.3	0.306
4	0.4	0.403
5	0.5	0.5
6	0.6	0.597
7	0.7	0.694
8	0.8	0.791
9	0.9	0.889
10	1.0	1.0

$n=20$

z	$\hat{\theta}$	$\tilde{\theta}$
0	0	0
1	0.05	0.057
2	0.10	0.106
3	0.15	0.156
4	0.20	0.205
5	0.25	0.254
6	0.30	0.303
7	0.35	0.352
8	0.40	0.402
9	0.45	0.451
10	0.50	0.500
11	0.55	0.549
12	0.60	0.598
13	0.65	0.648
14	0.70	0.697
15	0.75	0.746
16	0.80	0.795
17	0.85	0.845
18	0.90	0.894
19	0.95	0.943
20	1.00	1.000

TABLE 6. *Median-bias of classical estimator* $\hat{\theta}=z/n$ *of a binomial parameter* θ, *for sample size* $n=10$, *compared with the estimator* $\tilde{\theta}$ *of table* 5.

$$B(\theta, \hat{\theta})=Prob\{\hat{\theta}>\theta \mid \theta\}-Prob\{\hat{\theta}<\theta \mid \theta\},$$

$$B(\theta, \tilde{\theta})=Prob\{\tilde{\theta}>\theta \mid \theta\}-Prob\{\tilde{\theta}<\theta \mid \theta\}.$$

θ	$1/2\, B(\theta, \hat{\theta})$	$1/2\, B(\theta, \tilde{\theta})$
0.00	0.0	Same, except where value given.
0.00+	−0.5	
0.01	−0.404	
0.02	−0.317	
0.03	−0.237	
0.04	−0 165	
0.05	−0.099	
0.06	−0.039	
0.07	+0.016	

θ	$1/2\, B(\theta, \hat{\theta})$	$1/2\, B(\theta, \tilde{\theta})$
0.08	+0.066	
0.09	+0.111	
0.10	−0.236	+0.151
0.11	−0.197	+0.188
0.12	−0.158	
0.13	−0.120	
0.14	−0.082	
0.15	−0.044	
0.16	−0.008	
0.17	+0.027	

θ	$1/2B(\theta,\ \hat{\theta})$	$1/2\ B(\theta,\ \tilde{\theta})$
0.18	+0.061	
0.19	+0.093	
0.20	−0.178	+0.124
0.21	−0.147	
0.22	−0.117	
0.23	−0.086	
0.24	−0.056	
0.25	−0.026	
0.26	+0.004	
0.27	+0.034	
0.28	+0.062	
0.29	+0.090	
0.30	−0.150	+0.117
0.31	−0.123	
0.32	−0.096	
0.33	−0.068	
0.34	−0.041	

θ	$1/2\ B(\theta,\ \hat{\theta})$	$1/2\ B(\theta,\ \tilde{\theta})$
0.35	−0.014	
0.36	+0.013	
0.37	+0.040	
0.38	+0.066	
0.39	+0.092	
0.40	−0.133	+0.118
0.41	−0.108	
0.42	−0.082	
0.43	−0.056	
0.44	−0.030	
0.45	−0.044	
0.46	+0.022	
0.47	+0.047	
0.48	+0.073	
0.49	+0.098	
0.50	0.0	

TABLE 7. *Median-bias of classical estimator* $\hat{\theta}=z/n$ *of a binomial parameter* θ, *for sample size* $n=20$, *compared with the estimator* $\tilde{\theta}$ *of table* 5.

$$B(\theta,\ \hat{\theta})=Prob\{\hat{\theta}>\theta\mid\theta\}-Prob\{\hat{\theta}<\theta\mid\theta\},$$
$$B(\theta,\ \tilde{\theta})=Prob\{\tilde{\theta}>\theta\mid\theta\}-Prob\{\tilde{\theta}<\theta\mid\theta\}.$$

θ	$1/2\ B(\theta,\ \hat{\theta})$	$1/2\ B(\theta,\ \tilde{\theta})$
0.00	0.0	Same, except where values given.
0.00+	−0.5	
0.01	−0.318	
0.02	−0.168	
0.03	−0.044	
0.04	+0.058	
0.05	−0.236	+0.142
0.06	−0.161	+0.210
0.07	−0.087	
0.08	−0.017	
0.09	+0.048	
0.10	−0.177	+0.108
0.11	−0.120	
0.12	−0.063	
0.13	−0.008	
0.14	+0.045	
0.15	−0.148	+0.095
0.16	−0.099	
0.17	−0.050	
0.18	−0.003	
0.19	+0.044	
0.20	−0.130	+0.089
0.21	−0.086	
0.22	−0.042	
0.23	+0.001	
0.24	+0.044	

θ	$1/2\ B(\theta,\ \hat{\theta})$	$1/2\ B(\theta,\ \tilde{\theta})$
0.25	−0.117	+0.085
0.26	−0.077	
0.27	−0.036	
0.28	+0.005	
0.29	+0.045	
0.30	−0.108	+0.084
0.31	−0.070	
0.32	−0.031	
0.33	+0.008	
0.34	+0.046	
0.35	−0.101	+0.083
0.36	−0.064	
0.37	−0.027	
0.38	+0.011	
0.39	+0.048	
0.40	−0.096	+0.084
0.41	−0.059	
0.42	−0.023	
0.43	+0.014	
0.44	+0.050	
0.45	−0.091	+0.086
0.46	−0.056	
0.47	−0.020	
0.48	+0·017	
0.49	+0.053	
0.50	0.0	

and close approximation to $\tilde{\theta}$.

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References

[1] L. J. Savage: *The Foundations of Statistics*, John Wiley and Sons, New York (1954).

[2] H. R. van der Vaart: *Some extensions of the idea of bias*, Annals of Mathematical Statistics, Vol. 32 (1961), 436-47.

[3] E. L. Lehmann: *Testing Statistical Hypotheses*, John Wiley and Sons, New York (1956).

[4] E. L. Lehmann: *Notes on the Theory of Estimation*, Associated Students Store, University of California, Berkeley (1950), (mimeographed).

[5] Laplace: *Determiner le milieu que l'on doit prendre entre trois observations données d'un même phénomène*, Part V (pp. 634-44) of *Mémoire sur la probabilité des causes pour les évènements*, Memoire de Mathematique et de Physique, presenté à l'Académie Royal des Sciences, par divers savans, Vol. IV (1774), (Paris).

[6] C. M. Walsh: *The Problem of Estimation.* A Seventeenth-Century controversy and its bearing on modern statistical questions, especially index-numbers, P. S. King and Son, London (1921).

[7] J. W. Tukey: *Standard confidence points*, Memorandum Report 26, Statistical Research Group, Princeton University, July 26 (1949), (unpublished).

[8] D. R. Cox: *Some problems connected with statistical inference*, Annals of Mathematical Statistics, Vol. 29 (1958), 357-72.

[9] A. Birnbaum: *Confidence curves : an omnibus technique for estimation and testing hypotheses*, Journal of the American Statistical Association, Vol. 56 (1961), 246-249.

[10] A. Birnbaum: *A unified theory of estimation. I*, Annals of Mathematical Statistics, Vol. 32 (1961), 112-135.

[11] A. Birnbaum: *On the foundations of statistical inference*, Journal of the American Statistical Association, Vol. 57 (1962), 269-306, with discussion, 307-326.

[12] C. Eisenhart and C. S. Martin: *The relative frequencies with which certain estimators of the standard deviation of a normal population tend to underestimate its value*, abstract, presented at a meeting of the Institute of Mathematical Statistics, Madison, Wisconsin, Sept. 7-10 (1948).