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ON THE STOCHASTIC DYNAMIC PROGRAMMING FOR AN INDEPENDENT, STATIONARY, STOCHASTIC PROCESS.

By

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§ 1. The Conception of the Stochastic Dynamic Programming. In his recent study on the dynamic programming, Bellman [4] (p. 81) expresses the essence of the dynamic programming by the following five conditions, considering a system.

a. In each case we have a (physical) system characterized at any stage by a small set of parameters, the *state variables*.

b. At each stage of either process we have a choice of a number of decisions.

c. The effect of a decision is a transformation of the state variables.

d. The past history of the system is of no importance in determining future actions.

e. The purpose of the process is to maximize some function of the state variables.

The purpose of this work is to discuss some stochastic aspects of successive process of strategies in connection with the dynamic programming on the basis of the above fundamental concepts.

In this section, let us consider the outline of our stochastic dynamic programming. At first, it is necessary to build a system. Let us build our system by the following notations.

1.1. Notations.

(i) $x^{(i)}(i=1, 2,\cdots, s)$; this $x^{(i)}$ is the i-th strategic variable. $x_t^{(i)}$ $(i=1, 2,\cdots, s$; $t=0, 1, 2,\cdots)$ is a value of the i-th strategic variable at the time t. The time O denotes the present time. For the sake of our convenience, let $\boldsymbol{x}=(x^{(1)}, x^{(2)},\cdots, x^{(s)})$ be a vector which is composed of strategic variables, $x^{(i)}$'s, and let $\boldsymbol{x}_t$ be a vector which is composed of $x_t^{(i)}$'s $(i=1, 2,\cdots s)$.

(ii) $c_t^{(i)}$ $(i=1, 2,\cdots, s$; $t=0, 1, 2,\cdots)$; this $c_t^{(i)}$ is a profit which is earned by unit level of the i-th strategic variable $x_t^{(i)}$ at the time t. For the sake of our convenience, let $\boldsymbol{c}_t=(c_t^{(1)}, c_t^{(2)},\cdots, c_t^{(s)})$ be a vector which is composed of $c_t^{(i)}$'s ($i=1, 2,\cdots, s$). And let $C_t^{(i)}$ be a random variable of which $c_t^{(i)}$ is a realization, and let $\boldsymbol{C}_t=(C_t^{(1)}\ C_t^{(2)},\cdots,\ C_t^{(s)})$ be a random vector which is composed of random variables $C_t^{(i)}$'s ($i=1, 2,\cdots, s$).

(iii) g_t; this g_t is a total profit $(\boldsymbol{c}_t, \boldsymbol{x}_t)=\sum_{i=1}^{s} c_t^{(i)}.\ x_t^{(i)}$ which is obtained by the activity level $\boldsymbol{x}_t$.

(iv) $b^{(j)}$ ($j=0, 1, 2,\cdots, m$); this $b^{(j)}$ is the j-th state variable which denotes a quantity of the j-th limited resource, and $b_t^{(j)}$ ($j=0, 1, 2,\cdots, m$; $t=0, 1, 2,\cdots$) is a value of the variable $b^{(j)}$ at the time t. Specially, let $b^{(0)}$ be a variable which denotes the amount of money (or more generally denotes the liquid capital). For the sake of our convenience, let $\boldsymbol{b}$ be a vector which is composed of $b^{(j)}$'s, and let $\boldsymbol{b}_t$ be a vector which is composed of $b_t^{(j)}$'s ($j=0, 1, 2,\cdots, m$).

1.2. System Building. Adapting the above notations, let us build a system which is defined by the following five conditions.

Condition 1. A stochastic process $\{\boldsymbol{C}_0, \boldsymbol{C}_1, \boldsymbol{C}_2,\cdots\}$ is an independent, stationary, stochastic process. Each random vector $\boldsymbol{C}_t$ realizes at the time $t+1$. In what follows, let $\boldsymbol{\alpha}=(\alpha^{(1)}, \alpha^{(2)},\cdots, \alpha^{(k)})$ be a vector which is composed of characteristic values of the distribution function which associates to the random vector $\boldsymbol{C}_0$ (and also to the random vectors $\boldsymbol{C}_1, \boldsymbol{C}_2,\cdots$).

Condition 2. Each strategic variable $x^{(i)}$ ($i=1, 2,\cdots, s$) is a function of state variables $b^{(j)}$'s ($j=0, 1, 2,\cdots, m$) and $\alpha^{(l)}$'s ($l=1, 2,\cdots, k$). In what follows, let us denote these functions as follows,

$$x^{(i)}=F^{(i)}(\boldsymbol{b}\,;\,\boldsymbol{\alpha}) \qquad (i=1, 2,\cdots, s) \tag{1.1}$$

and call these the *strategic function.* For the sake of simplicity, let us denote our strategic vector by $\boldsymbol{x}=\boldsymbol{F}(\boldsymbol{b}\,;\,\boldsymbol{\alpha})$.

Condition 3. (i) We can obtain complete informations on the values $b_0^{(j)}$'s ($j=0, 1, 2,\cdots, m$) at the time 0.

(ii) If we carry out our activity by the level $\boldsymbol{x}_0$ at the time 0, we can obtain the profit, which may be negative as the loss,

$$g_0=\sum_{i=1}^{s} c_0^{(i)}.\ x_0^{(i)}=(\boldsymbol{c}_0, \boldsymbol{x}_0) \tag{1.2}$$

at the time 1.

(iii) The profit g_0 is invested into the liquid capital $b_0^{(0)}$ at the time 1, but other limited resources are invariable. That is

$$b_1^{(0)}=b_0^{(0)}+g_0, \quad b_0^{(j)}=b_1^{(j)} \quad (j=1, 2,\cdots, m). \tag{1.3}$$

Condition 4. (i) When the process reaches the time t ($t=1, 2,\cdots$), we can know the value $b_t^{(0)}$.

(ii) If we carry out our activity by the level $\boldsymbol{x}_t$ at the time t, we can obtain the profit, which may be negative as the loss,

$$(1.4)\qquad g_t=\sum_{i=1}^{s} c_t^{(i)}\cdot x_t^{(i)}=(\boldsymbol{c}_t,\ \boldsymbol{x}_t)\qquad (t=1,\ 2,\cdots)$$

at the time $t+1$.

(iii) The profit g_t is invested into the liquid capital $b_t^{(0)}$ at the time $t+1$, but other limited resources are invariable. That is,

$$(1.5)\qquad b_{t+1}^{(0)}=b_t^{(0)}+g_t,\quad b_1^{(j)}=b_2^{(j)}=\cdots\quad (j=1,\ 2,\cdots).$$

Our purpose of this work consists in choice of the optimal strategic function $F(\boldsymbol{b},\ \boldsymbol{\alpha})$ in above system.

Now, if we choose a strategic function $F(\boldsymbol{b},\ \boldsymbol{\alpha})$, then $b_1^{(0)}$, $x_1^{(i)}$'s ($i=1,\ 2,\cdots,\ s$), g_1; $b_2^{(0)}$, $x_2^{(i)}$'s, g_2; $\cdots$ are expressed by functions of b_0, c_0, $c_1,\cdots$. And then, we can forecast future values of $b_t^{(0)}$'s, $x_t^{(i)}$'s and g_t 's ($t=1,\ 2,\ \cdots;\ i=1,\ 2,\ \cdots,\ s$) associating random variables respectively at the time 0. In what follows, let the large characters, $B_t^{(0)}$'s, $X_t^{(i)}$'s and G_t 's ($t=1,\ 2,\ \cdots;\ i=1,\ 2,\ \cdots,\ s$), be random variables of which $b_t^{(0)}$'s, $x_t^{(i)}$'s and g_t 's are realizations respectively.

1.3. Choice of the Strategic Function. When we choose our strategic function, we must decide our fundamental attitude in the stochastic dynamic programming. Let us assume as follows.

Assumption 1. *The purpose of our stochastic dynamic programming consists in effective profit earning under incomplete informations, throughout an infinite earning process.*

Now, if we try to choose our strategic function on the basis of the above fundamental attitude, we are necessary to form a functional which evaluates the degree of effectiveness of our strategic function. The functional is subject to b_0 and $\boldsymbol{\alpha}$. In what follows, let us denote by $U(\boldsymbol{b}_0,\boldsymbol{\alpha};F)$ the valuation on our strategic function F, and name the functional $U(\boldsymbol{b}_0,\ \alpha;\ F)$ the ***ultimate objective functional.***

"What functional can be our ultimate objective functional" is our important problem of this work. It is desirable to derive necessary and sufficient conditions for our ultimate objective functional, but it seems difficult to obtain the conditions. In this work, let us choose the ultimate objective functional which satisfies the following two conditions. There, $s_n=g_0+g_1+\cdots g_{n-1}$ and $S_n=G_0+G_1+\cdots+G_{n-1}$.

Condition 1. (Condition for Discrimination) $U(\boldsymbol{b}_0,\ \boldsymbol{\alpha};\ F)$ is an one valued functional of $\boldsymbol{b}_0$, $\boldsymbol{\alpha}$ and $F(\boldsymbol{b},\ \boldsymbol{\alpha})$, and is not only one value in all domain of F.

Condition 2. (Condition for Effectiveness) We can choose a suitable function $\varphi(s_n,\ n)$ which is monotonously non-decreasing function of s_n and is monotonously non-increasing function of n, so that

(i) $U(\boldsymbol{b}_0,\ \boldsymbol{\alpha};\ F)$ may be φ-converged in probability by S_n and n, i.e.

$$(1.6)\qquad \lim_{n\to\infty} P\{\,|\varphi(S_n,\ n)-U(\boldsymbol{b}_0,\ \boldsymbol{\alpha}\,;\ F)\,|>\varepsilon\}=0$$

for any small positive number ε, or so that

(ii) $U(\boldsymbol{b}_0, \boldsymbol{\alpha}; F)$ may be $(\varphi, \boldsymbol{u})$-converged in probability by S_n and $\boldsymbol{n}$, i. e.

$$(1.7)\qquad \lim_{n\to\infty} P\{\,\Big|\int_{\frac{dp\{\varphi(s_n,\ n)\}}{d\{\varphi(s_n,\ n)\}}>\delta} u^{(n)}\{\varphi(s_n,\ n)\}.\ dp\{\varphi(s_n,\ n)\}-U(\boldsymbol{b}_0,\ \boldsymbol{\alpha}\,;\ F)\,\Big|>\varepsilon\}=0$$

for any small positive number ε and δ. Here, $u^{(n)}\{\varphi(s_n, \boldsymbol{n})\}$ is a suitably selected valuation function on $\varphi(s_n, \boldsymbol{n})$ and $\boldsymbol{n}$.

Note 1. If we can selected a function $\varphi(s_n, \boldsymbol{n})$ on which $\varphi(S_n, \boldsymbol{n})$ converges in probability to a fixed value on each F, let us select a φ-converged, ultimate, objective function. And if we cannot select a φ-converged, ultimate, objective function, but if we can select a function $\varphi(s_n, \boldsymbol{n})$ on which $\varphi(S_n, \boldsymbol{n})$ converges in probability to a random variable on each F, let us select a $(\varphi, \boldsymbol{u})$-converged, ultimate, objective function. The former may be regarded as a special case of the later.

Now, in this work, some functions $\varphi(s_n, \boldsymbol{n})=s_n$ or $\varphi(s_n, \boldsymbol{n})=s_n/\boldsymbol{n}$ or $\varphi(s_n, \boldsymbol{n})=\sqrt[n]{\{b_0^{(0)}+s_n\}/b_0^{(0)}}$ will be adopted to choose the ultimate objective functional, setting aside that these functions are optimal or not.

After we choose an ultimate objective functional $U(\boldsymbol{b}_0, \boldsymbol{\alpha}; F)$, we must choose our strategic function F which maximizes the ultimate objective functional. Let us call the chosen strategic function F to be ***φ-optimal*** if $U(\boldsymbol{b}_0, \boldsymbol{\alpha}; F)$ is a functional which is φ-converged in probability, and call it to be ***(φ, u)-optimal*** if $U(\boldsymbol{b}_0, \boldsymbol{\alpha}; F)$ is a functional which is $(\varphi, \boldsymbol{u})$-converged in probability. And after we choose the φ-optimal or $(\varphi, \boldsymbol{u})$-optimal strategic function F, we can decide the φ-optimal or $(\varphi, \boldsymbol{u})$-optimal present strategy $\boldsymbol{x}_0=\boldsymbol{F}(\boldsymbol{b}_0, \boldsymbol{\alpha})$.

In some system, even if we try to choose directly our φ-optimal or $(\varphi, \boldsymbol{u})$-optimal strategic function on the basis of the above process (1.1)$\sim$(1.5), we shall be involved various complexities. In such a system, we must simplify our strategy making process, neglecting the various complexities of our considerations and calculations. The simplification is our consent to miss. In what follows, let us call " our subjective decision " which solves a part of strategy making which is left from our pursuing process on the basis of the process (1.1)$\sim$(1.5), the *subjective policy*.

Let us consider some simple examples on our strategy making.

§ 2. The programming in the Game by the Bernoulli Trial.

2.1. System Building. The game by the Bernoulli trial is pursued in a system which is defined by the following conditions.

Condition 1. A stochastic process $\{C_0, C_1, C_2,\cdots\}$ is an independent, stationary, stochastic process. Each random variable C_t ($t=0, 1, 2, \cdots$) is realized on 1 or -1 by probabilities p or $q=1-p$ respectively at the time $t+1$. If we participate in the bet at the time t and win the bet, the realization of C_t is equal to 1, and if we lose the bet, the realization of C_t is equal to -1.

Condition 2. According to the rule of game, we can participate in each bet in the game as long as our funds for the bet is equal to or is larger then 1. Our strategy at the time t decides whether "we participate in the bet at that time" or "we do not participate in the bet". If our funds $b_t^{(0)}$ is larger than a suitable positive integer k, our strategy decides to participate in the bet at the time t, and if it is equal to or is smaller than the integer k, our strategy decides not to participate in the bet at the time t. We can express our strategic function as follows.

$$(2.1) \quad x=F(b^{(0)}, p)=\begin{cases} 1, \text{ twhen } b^{(0)}>k\geqq 0 \text{ (we participate in the bet)} \\ 0, \text{ when } 0\leqq b^{(0)}\leqq k \text{ (we do not participate in the bet)} \end{cases}$$

Condition 3. (i) $b_0^{(0)}$ is a positive integer, and we know the amount of $b_0^{(0)}$.

(ii) If we carry out our strategy x_0 at the time 0, we can obtain the profit

$$(2.2) \qquad g_0=c_0 . x_0$$

at the time 1.

(iii) The profit g_0 is invested into the initial funds $b_0^{(0)}$ at the time 1. That is,

$$(2.3) \qquad b_1^{(0)}=b_0^{(0)}+g_0$$

Condition 4. (i) When the process reaches the time t ($t=1, 2, \cdots$), we can know the amount of funds $b_t^{(0)}$.

(ii) If we carry out our strategy x_t at the time t, we can obtain the profit

$$(2.4) \qquad g_t=c_t . x_t \quad (t=1, 2, \cdots)$$

at the time $t+1$.

(iii) The profit g_t is invested into the funds $b_t^{(0)}$ at the time $t+1$. That is,

$$(2.5) \qquad b_{t+1}^{(0)}=b_t^{(0)}+g_t \quad (t=1, 2, \cdots).$$

Note 1. If our strategic function decides not to participate in a bet at a stage in our game, we cannot obtain any profit and do not suffer any loss at any subsequent bets in the game. That is, not to participate in the game according to our strategic function, is also to leave the game forever.

2.2. Choice of the Strategic Function. Let us consider how to choose our strategic function in the following two special cases, considering whether the stochastic process $\{G_0, G_1, \cdots\}$ is an independent, stationary, stochastic process or it is not.

The case 1 (the case of independent process)

In our system, if we lose all our immediate funds at the time t, we must leave the game at that moment, and then $G_t=G_{t+1}=\cdots=0$. If $b_0^{(0)}$ is bounded and $q>0$, random profits G_t's$(t=0, 1, 2, \cdots)$ cannot always be independent with each other. But if we have or are given other funds by $b_0^{(0)}$ to participate in other games, in a system which is defined by the conditions 1~4 in the section 2.1, then it is of no important to care for the finiteness of our immediate funds for the game. That is, we must decide so that either the strategic function (2.1) may always be equal to 1 or it may always be equal to 0, in all successive games. Then, k is equal to 0 or is not smaller than $b_0^{(0)}$ in all successive games. If we denote the profit at the j-th bet in the i-th game by G_{ij}, the sequence of random profits $\{G_{10}, G_{11}, G_{12}, \cdots, G_{20}, G_{21}, \cdots\}$ is an independent, stationary, stochastic process. And all values $E(G_{ij})$'s are equal to a constant which is determined by $b_0^{(0)}$, p and F.

Let n_m be the total number of bets in the first m games. According to the "law of large numbers", we can admit the following expression

$$(2.6) \qquad \lim_{n_m\to\infty} P\{|\sum_i\sum_j G_{ij}/n_m-E(G_{10})|>\varepsilon\}=0$$

for any small positive number ε. That is, $U(b_0^{(0)}, p; F)=E(G_{10})$ is our ultimate objective fuuctional.

Since $E(G_{10})=E(C_{10})\cdot x_{10}=(p-q)\cdot x_{10}$, our strategic function (2.1) is determined as follows.

i) If $p>1/2$, then $k=0$ (that is, we participate in each bet as long as we have positive funds).

ii) If $p\leqq 1/2$, then $k\geqq b_0^{(0)}$ (that is, we do not participate in the game).

The case 2 (the case of non-independent process).

Next, if our funds is only funds for all games, we cannot participate in the immediate and any other games forever after we lose all our funds. In such a case, we cannot repeat failure and participation. We have only one chance to leave the game according to the decision of our strategic function. The stochastic process $\{G_0, G_1, G_2, \cdots\}$ does not be independent, stationary, stochastic process. If we intend to choose our strategic function (i.e. the number k), we must directly pursue relative relations among G_t's $(t=0, 1, 2, \cdots)$.

At first, let us calculate the probability by which we leave the game by the loss r. r may be $b_0^{(0)}-k$ which is determined by our strategy or may be $b_0^{(0)}$ which is determined by the rule of the game, in the system which is defined in the section 2.1.

Let i_n (in some cases, this i_n will be denoted by simple i) be the number of "win's" in the period from the time 0 to the time $n-1$, and let $j_n = n-i_n$ (in some cases, this j_n will be denoted by simple j) be the number of "loss'es" in the same period. Then, we can participate in the bet at the time t, if and only if the relations $j_n-i_n<r$'s ($n=0, 1, 2, \cdots, t-1$) are satisfied at all times n's ($n=0, 1, 2, \cdots, t-1$). Feller [7] (p. 282~) considered the random walk of our funds by the difference equation, in a system which is based on the Bernoulli trial. But, for the sake of strictness, let us directly pursue the random walk of our funds.

Let $E(i, j)$ (i and j are positive integer) be the event

(i) "we can participate in the game at all times t's at which $t=0, 1, 2, \cdots, n-1$, without suffering the loss r, and

(ii) we win at i times and lose at $j=n-i$ times".

And let $P(i, j)$ be the probability by which the event $E(i, j)$ will realize. We can admit the following theorem.

Theorem 2.1. 1) $P(i, j)={}_nC_i\cdot p^i\cdot q^j$, *if* $j<r$, *or* $j=r$ *and* $i=0$.

2) $P(i, j)=({}_nC_i-{}_nC_{j-r})\cdot p^i\cdot q^j$, *if* $r\leqq j\leqq r+i-1$.

3) $P(i, j)=({}_nC_i-2{}_{n-1}C_{i-1})\cdot p^i\cdot q^j$, *if* $j=r+i$ *and* $j\geqq 1$.

4) $P(i, j)=0$, *if* $r+i+1\leqq j$.

[Proof] It is evident that the assertions 1) and 4) are true. Let us prove the assertion 2) by the double mathematical induction on i and j, in the following way.

[A] Let us verify that the assertion 2) is true, if $j=r$.

In this case, if and only if $c_0=c_1=\cdots=c_{r-1}=-1$, we must leave the game. Then, the probability by which we must leave the game till we reach $E(i, r)$ is equal to $p^i.q^r$. We can admit the following relation.

$$(2.7)\qquad P(i, r)=({}_nC_i-1)\cdot p^i\cdot b^r=({}_nC_i-{}_nC_0)\cdot p^i\cdot q^r\ \ (i+r=n)$$

which proves the assertion 2) in this particular case.

[B] Let us verify that if the assertion 2) is true on all i's and j's in the domain of $i\geqq l$ and $r\leqq j\leqq r+l-1$ (l is a positive integer), the assertion 2) is also true on all i's in the domain of $i\geqq l+1$ and $j=r+l$. The restriction on i arises from the restriction on i's and j's in the assertion 2).

[B-1] At first, let us consider the case of $i=l+1$ and $j=r+l$.

Now, we can assume two paths by which we reach $E(l+1, r+l)$. The first is the path by which we reach $E(l, r+l)$ and win the next bet, and the second is the path by which we reach $E(l+1, r+l-1)$ and lose the next bet. But, if we reach $E(l, r+l)$, the relation $(r+l)-l=r$ satisfies, and we cannot participate in the game at the next time. That is, we can reach $E(l+1, r+l)$ by only the second path. We can admit the following expression.

$$(2.8)\quad \begin{aligned} P(l+1, r+l) &= P(l+1, r+l-1)\cdot q = ({}_{n-1}C_{l+1} - {}_{n-1}C_{l-1})\cdot p^{l+1}\cdot q^{r+l} \\ &= ({}_nC_{l+1} - {}_nC_l)\cdot p^{l+1}\cdot q^{r+l} \quad (n=r+2l+1) \end{aligned}$$

We have proved the assertion 2) in the case of $i=l+1$ and $j=r+l$.

[B–2] Next, we must verify that the assertion 2) is true on $i=l+m+1$ (m is a positive integer) and $j=r+l$, if the assertion is true on all i's in the domain of $i\leqq l+m$ and $j=r+l$. If we pursue the paths by which we reach $E(l+m+1, r+l)$, like as we have done in the case of [B–1], we can verify that the assertion 2) is true in this particular case.

According to the consequence of [B–1] and [B–2], we can admit the assertion [B]. And according to the consequence of [A] and [B], we can verify the assertion 2).

Next, let us verify the assertion 3).

If $j=r+i$ and $i\geqq 1$, we can reach $E(i, j)$ by only path by which we reach $E(i, j-1)$ and lose the next bet. Since $j=r+i$, i.e. $j-1=r+i-1$, we can admit the following expression by the assertion 2).

$$(2.9)\qquad P(i, j-1) = ({}_{n-1}C_i - {}_{n-1}C_{j-r-1})\cdot p^i\cdot q^{j-1} \quad (i+j=n)$$

Then, we can also admit the following expression, if $j=r+i$ and $i\geqq 1$.

$$(2.10)\quad P(i, j) = ({}_{n-1}G_i - {}_{n-1}C_{j-r-1})\cdot p^i\cdot q^j = ({}_nC_i - 2{}_{n-1}C_{i-1})\cdot p^i\cdot q^j$$

After all, our assertion has been proved. Q.E.D.

Now, let us put $P_n = \sum_{\substack{i+j\leqq n\\ j-i=r}} P(i, j)$. According to the definition of $P(i, j)$, we can admit the following expression.

$$(2.11)\qquad P_n = \sum_{\substack{i+j\leqq n\\ j-i=r}} P(i, j) = 1 - \sum_{\substack{i+j=n\\ j-i<r}} P(i, j) = 1 - \sum_{\frac{n-r}{2}<i} P(i, j)$$

And we can obtain the following theorem which coincides with the Feller's assertion [7] (p. 287).

Theorem 2.2. *If $p>0$ and $q>0$, then*

$$(2.12)\qquad \lim_{n\to\infty} P_n = \begin{cases} 1 & \text{if } p\leqq 1/2 \\ (q/p)^r & \text{if } p>1/2\ . \end{cases}$$

Proof. we are going to consider the case in which n is sufficiently large and $1>p>0$, then it is sufficient to pursue the earning process in only case of $r\leqq j\leqq r+i-1$. According to the theorem 2.1, we can admit the following expression.

$$(2.13)\qquad P(i,\ j) = {}_nC_i\cdot p^i\cdot q^j - {}_nC_{j-r}\cdot p^{i+r}\cdot q^{j-r}\cdot\left(\frac{q}{p}\right)^r \quad (i+j=n)$$

Since n is sufficiently large, and both p and q are positive and these are independent of n, we can approximate the binomial distribution by the normal one. We can admit the following expression, by the expressions (2.11) and (2.13).

$$(2.14)\qquad \begin{aligned} P_n &= 1-\frac{1}{\sqrt{2\pi npq}}\int_{\frac{n-r}{2}}^{\infty}\left[exp\left\{-\frac{(i-np)^2}{2npq}\right\}-exp\left\{-\frac{(i+r-np)^2}{2npq}\right\}\cdot\left(\frac{q}{p}\right)^r\right]\cdot di \\ &= 1-\frac{1}{\sqrt{2\pi}}\int_{\frac{n-2np-r}{2\sqrt{npq}}}^{\infty} e^{-\frac{y^2}{2}}\cdot dy+\frac{1}{\sqrt{2\pi}}\int_{\frac{n-2np+r}{2\sqrt{npq}}}^{\infty} e^{-\frac{y^2}{2}}\cdot dy\cdot\left(\frac{q}{p}\right)^r \end{aligned}$$

That is, we can easily admit the theorem 2.2. Q.E.D.

Further, according to the theorem 2.2, we can obtain the following theorem.

Theorem 2.3. *When a game is carried out in the system which is built in the section* 2.1.

[A] *if $p\leq 1/2$, we can admit the following expression for any small positive number ε,*

$$(2.15)\qquad \lim_{n\to\infty} P\left\{\left|\sum_{i=0}^{n-1} G_i+(b_0^{(0)}-k)\right|>\varepsilon\right\}=0$$

[B] *and if $p>1/2$, we can admit the following expressions for any small positive number ε.*

$$(2.16)\qquad \lim_{n\to\infty} P\left\{-\varepsilon<\sum_{i=0}^{n-1} G_i/n<0\right\}=\left(\frac{q}{p}\right)^{b_0^{(0)}-k}$$

$$(2.17)\qquad \lim_{n\to\infty} P\left\{\left|\sum_{i=0}^{n-1} G_i/n-(p-q)\right|<\varepsilon\right\}=1-\left(\frac{q}{p}\right)^{b_0^{(0)}-k}$$

Proof. The assertion [A] is immediate result of the theorem 2.2. And the expression (2.16) in the assertion [B] is also immediate result of the theorem 2.2. Let us verify the assertion (2.17).

Now, let us put $r_n=\sum_{i=0}^{n-1} g_i/n$ and $R_n=\sum_{i=0}^{n-1} G_i/n$. Like as we have derived the expression (2.14), we can also derive the following expression (2.18) by the expression (2.11) and (2.13), considering $r_n=(i-j)/n=2i/n-1$.

(2.18)

$$P\{|R_n-(p-q)|<\varepsilon\}=\frac{1}{\sqrt{2\pi npq}}\int_{\substack{(p-q)-\varepsilon<r_n\\<(p-q)+\varepsilon}}\left\{e^{-\frac{(i-np)^2}{2npq}}-e^{\frac{-(i+b_0^{(0)}-k-np)^2}{2npq}}\cdot\left(\frac{q}{p}\right)^{b_0^{(0)}-k}\right\}\cdot di$$

$$=\frac{1}{\sqrt{\frac{8\pi pq}{n}}}\int_{-\varepsilon}^{\varepsilon}\left\{e^{-\frac{n\cdot r_n^2}{8pq}}-e^{\frac{-n\cdot(r_n-(b_0^{(0)}-k)/n)^2}{8pq}}\cdot\left(\frac{q}{p}\right)^{b_0^{(0)}-k}\right\}\cdot dr_n \quad \left(i=\frac{n\cdot(r_n+1)}{2}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\frac{n\varepsilon}{2\sqrt{pq}}}^{\frac{n\varepsilon}{2\sqrt{pq}}}e^{-\frac{t^2}{2}}-e^{-\frac{1}{2}\cdot\left(t-\frac{b_0^{(0)}-k}{2\sqrt{npq}}\right)^2}\cdot\left(\frac{q}{p}\right)^{b_0^{(0)}-k}\cdot dt \quad \left(t=\frac{r_n}{\sqrt{4pq/n}}\right)$$

When $n\to\infty$, the expression (2.18) eonverges to the value $1-(p/q)^{b_0^{(0)}-k}$. That is we have proved the assertion (2.17). Q.E.D.

Note 2. The probabilities (2.12), (2.16) and (2.17) are not limitations of relative frequency distribution.

Now, according to the theorem 2.3, when $p\leqq 1/2$, we can obtain a φ-converged, ultimate, objective functional

$$U(b_0^{(0)},\ p;\ k)=\begin{cases}-(b_0^{(0)}-k), & \text{if } b_0^{(0)}>k\\ 0, & \text{if } b_0^{(0)}\leqq k\end{cases} \tag{2.19}$$

after we put $\varphi(s_n,\ n)=s_n=\sum_{t=0}^{n-1}g_t$.

Our φ-optimal strategic function which maximizes the functional (2.19) is defined by k which is equal to or is larger than $b_0^{(0)}$.

Next, let $\bar{R}(b_0^{(0)},\ p;\ k)$ be a random variable which realizes on $(p\text{-}q)$ by a probability $(q/p)^{b_0^{(0)}-k}$ and realizes on 0 by the probability $1-(q/p)^{b_0^{(0)}-k}$, and let us consider the case of $p>1/2$.

If $p>1/2$, even if $n\to\infty$, the random variable $\varphi(S_n,\ n)=S_n/n=\sum_{i=0}^{n-1}G_i/n$ does not always converges in probability to a value on each k, and it converges in probability to a random variable $\bar{R}(b_0^{(0)},\ p;\ k)$ on each k. We must choose a $(\varphi,\ u)$-converged, ultimate, objective functional, after we introduce a suitable subjective valuation function $u^{(n)}(\varphi(s_n,\ n))$.

For example, let us make the probability by which we must leave the game by the loss of all our funds smaller than β. Then, after we calculate the number r which satisfies the following relation

$$\left(\frac{q}{p}\right)^{\tilde{r}}=\beta \quad \text{i.e.} \quad \frac{-\log\beta}{\log p-\log q}=\tilde{r}, \tag{2.20}$$

we can express our subjective valuation function $u^{(n)}(\varphi(s_n,\ n))$ as follows.

$$(2.21)\qquad u^{(n)}(\varphi(s_n, n)) = \begin{cases} s_n/n, & \text{if } b_0^{(0)}+s_n \geqq \tilde{r} \\ -M, & \text{if } b_0^{(0)}+s_n < \tilde{r} \end{cases}$$

Here, M is a sufficiently large number.

We can choose our (φ, u)-converged, ultimate, objective functional as follows

$$(2.22)\qquad \begin{aligned} U(b_0^{(0)}, p; k) &= \lim_{n\to\infty} \int_{dp(\varphi)/d\varphi>\delta} u^{(n)}(\varphi).dp(\varphi) \\ &= \begin{cases} (p-q)\cdot\left\{1-\left(\frac{q}{p}\right)^{b_0^{(0)}-k}\right\}, & \text{if } k \geqq \tilde{r} \\ -M\cdot\left(\frac{q}{p}\right)^{b_0^{(0)}-k} + (p-q)\cdot\left\{1-\left(\frac{q}{p}\right)^{b_0^{(0)}-k}\right\}, & \text{if } k < \tilde{r} \end{cases} \end{aligned}$$

after we choose an any small positive number δ.

After all, if $p>1/2$, the (φ, u)-optimal, strategic function (2.1) is defined by $k=r$ which maximizes the (φ, u)-converged, ultimate, objective function (2.22).

§ 3. The Programming on Some Special Linear Models.

3.1. Introduction.

The linear model is restricted by the conditions

$$(3.1)\qquad (A^{(0)}, x)=b^{(0)},\ (A^{(1)}, x)=b^{(1)}, \cdots,\ (A^{(m)}, x)=b^{(m)},\ x \geqq 0$$

in which the activity level x is separable (that is, infinitely divisible) and we can forecast to obtain the profit

$$(3.2)\qquad (c, x) = \sum_{i=1}^{s} c^{(i)}\cdot x^{(i)}$$

by the activity. Where c (the vector of profits per unit activity level) is a predicted vector which is independent of the activity level.

The linear model can express many sorts of activities. But, in this work, let us assume that the model expresses a firm. In order to build our system on the linear model, let us explain characters of limited resources $b^{(j)}$'s $(j=0, 1, 2, \cdots, m)$ more precisely.

We can classify all sorts of limited resources to two classes. Let us call a sort of limited resources *liquid capital*, when any part of these can be sole to buy any other sorts of necessary limited resources at any moment, and call each limited resource except liquid capital *fixed resource.*

Now, even if the activity is restricted by many sorts of liquid capitals, when we make our strategy, we can amalgamate these vavious restrictions to only one restriction of liquid capital which is expressed in term of total amount of liquid capital, since the liquid capitals can freely and mutually be exchanged through the intermediation of money. That is, let $l^{(j)}$ be the price of the j-th liquid capital, we can amalgamate various restriction to the following only one restriction.

$$(3.3)\quad ((l^{(0)}\cdot \boldsymbol{A}^{(0)}+l^{(1)}\cdot \boldsymbol{A}^{(1)}+\cdots+l^{(k)}\cdot \boldsymbol{A}^{(k)})\cdot \boldsymbol{x})=l^{(0)}\cdot b^{(0)}+l^{(1)}\cdot b^{(1)}+\cdots+l^{(k)}\cdot b^{(k)}$$

In this work, let only $b^{(0)}$ be the liquid capital and other resources are fixed resources.

3. 2. System Building.

Let us build a system which is defined by the following conditions.

Condition 1. An independent, stationary, stochastic process $\{\boldsymbol{C}_0, \boldsymbol{C}_1, \cdots\}$ is given. Each random vector $\boldsymbol{C}_t$ $(t=0, 1, 2, \cdots)$ realizes at the time $t+1$. $\boldsymbol{\alpha}$ is a vector which is composed of characteristic values of $\boldsymbol{C}_0$ (and also $\boldsymbol{C}_1, \boldsymbol{C}_2, \cdots$).

Condition 2. Strategic variables $x^{(i)}$'s $(i=1, 2, \cdots, s)$ are restricted by the condition (3. 1), and are functions of limited resources $b^{(j)}$'s $(j=0, 1, 2, \cdots, m)$ and $\boldsymbol{\alpha}$. For the sake of simplicity, let us denote the strategic vector by $\boldsymbol{x}=\boldsymbol{F}(\boldsymbol{b}\,;\,\boldsymbol{\alpha})$.

Condition 3. The vectors which are composed of coefficients of production in (3. 1), i. e. $\boldsymbol{A}^{(0)}, \boldsymbol{A}^{(1)}, \cdots, \boldsymbol{A}^{(m)}$, do not change throughout all period in our concern. We can know these values at all times.

Condition 4. (i) The limited resources except liquid capital, i. e. $b_t^{(1)}, b_t^{(2)}, \cdots, b_t^{(m)}$ $(t=0, 1, 2, \cdots)$ do not change throughout all our concerning periods. That is

$$(3.4)\qquad b_0^{(j)}=b_1^{(j)}=b_2^{(j)}=\cdots\quad (j=1, 2, \cdots)\,.$$

These values $b_t^{(j)}$'s $(t=0, 1, 2, \cdots;\ j=1, 2, \cdots, m)$ are known at all times t 's. And we know the amount of liquid capital $b_0^{(0)}$ at the time 0.

(ii) If we carry out our activity by the level $\boldsymbol{x}_0$ at the time 0, we can obtain the profit, which may be negative as the loss,

$$(3.5)\qquad g_0=(\boldsymbol{c}_0, \boldsymbol{x}_0)$$

at the time 1.

(iii) The profit g_0 is invested into the liquid capital $b_0^{(0)}$ at the time 1. That is

$$(3.6)\qquad b_1^{(0)}=b_0^{(0)}+g_0.$$

Condition 5. (i) When the process reaches the time t $(t=1, 2, \cdots)$, we can also know the amount of the liquid capital $b_t^{(0)}$.

(ii) If we carry out our activity by the level $\boldsymbol{x}_t$ at the time t, we can obtain the profit, which may be negative as the loss,

$$g_t = (\boldsymbol{c}_t, \boldsymbol{x}_t) \tag{3.7}$$

at the time $t+1$.

(iii) The profit g_t is invested into only liquid capital $b_t^{(0)}$ $(t=1, 2, \cdots)$. That is,

$$b_{t+1}^{(0)} = b_t^{(0)} + g_t \quad (t=1, 2, \cdots)\,. \tag{3.8}$$

3.3. Choice of the Strategic Function 1. Now, if the sequence of random profits at the successive times 0, 1, 2, ⋯, i.e. $\{G_0, G_1, \cdots\}$, is forecasted to be an independent, stationary, stochastic, process, let us call the earning process the ***independent, stationary, earning process.***

When a system builds an independent, stationary, earning process, our programming process is very simple. That is, in such a system, if G_i's have an expected value and a finite variance, we can admit the following expression for any small positive number ε by the "law of large numbers".

$$\lim_{n\to\infty} \mathrm{P}\{\,|\sum_{t=0}^{n-1} G_t/n - \sum_{t=0}^{n-1} E(G_t)/n\,| > \varepsilon\} = 0 \tag{3.9}$$

That is, we can obtain φ-converged, ultimate, objective functional

$$\boldsymbol{U}(b_0^{(0)}, \alpha\,;\, F) = \lim_{n\to\infty} \sum_{t=0}^{n-1} E(G_t)/n \tag{3.10}$$

after we put $\varphi(s_n, n) = \sum_{t=0}^{n-1} g_t/n$. And since G_t's $(t=0, 1, 2, \cdots)$ are independent with each other, we can obtain the φ-optimal strategic function, $F(b, \alpha)$, as follows.

Let us calculate the activity level $\bar{x}$ which maximizes the expected profit $E(G) = E\{(C_0, x)\}$ under the condition (3.1). If the variance of $(C_0, \bar{x})$ is bounded, we can put $\bar{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{b}, \boldsymbol{\alpha})$, as our φ-optimal strategic function.

We can see an actual example of system which builds an independent, stationary, earning process. At first, we can obtain the following theorem.

Theorem 3.1. *In the system which is defined by the conditions* 1~4 *in the section* 3.2, *if and only if the following two conditions A and B are satisfied, the system builds an independent, stationary, earning process.*

Condition A. If the random variable $X_t^{(i)}(t=1, 2, \cdots)$ *of which the realization is a future strategic value* $x_t^{(i)}$ *does not independent of the profits* $G_0, G_1, \cdots, G_{n-1}$, *all* $C_t^{(i)}$*'s* $(t=0, 1, 2'\cdots)$ *must be equal to* 0.

Condition B. If $C_t^{(i)}$*'s* $(t=0, 1, 2, \cdots)$ *do not always be equal to* 0, *all* $x_t^{(i)}$*'s* $(t=0, 1, 2, \cdots)$ *must be equal with each other, independently of* g_0,

$g_1, \cdots, g_{t-1}$.

We can see an example of system which satisfies the conditions in the theorem 3.1, as follows.

Example 1. Let $x_t^{(s)}$ be the activity level which only holds liquid capital, and let $x_t^{(i)}$'s ($i=1, 2, \cdots, s-1$) be the activity levels which surely use positive quantities of some fixed resources to be carried out by positive level. And let $\tilde{x}_t$ be a vector which is composed of $x_t^{(i)}$'s ($i=1, 2, \cdots, s-1$) and $\tilde{C}_t$ be a random vector which is composed of $C_t^{(i)}$'s ($i=1, 2, \cdots, s-1$), and $\tilde{A}^{(j)}$'s ($j=1, 2, \cdots, m$) be vectors which are composed of coefficients of production of $\tilde{x}_t$ for $b^{(j)}$'s ($j=1, 2, \cdots, m$) respectively.

We can easily admit that if the following conditions a) and b) are satisfied, the system which is defined by the conditions 1~4 in the section 3.2 satisfies the above conditions A and B in the theorem 3.1.

Condition a. $C_t^{(s)}=0$ ($t=0, 1, 2, \cdots$)

Condition b. We can decide the activity level $\bar{x}$ which maximizes the expected profit $E\{(\tilde{C}_0, \tilde{x})\}$ under the conditions

$$(3.11) \quad (\tilde{A}^{(1)}, \tilde{x})=b^{(1)}, \ (\tilde{A}^{(2)}, \tilde{x})=b^{(2)}, \cdots, \ (\tilde{A}^{(m)}, \tilde{x})=b^{(m)}, \ \tilde{x}\geqq 0,$$

and further the following relations are admited.

$$(3.12) \quad (\tilde{A}^{(0)}, \bar{x})\leqq b_0^{(0)}, \ P\{b_0^{(0)}+\sum_{t=0}^{n-1}(\tilde{C}_t, \bar{x})<(\tilde{A}^{(0)}, \bar{x})\}=0 \quad (n=1, 2, \cdots)$$

3.4. Choice of the Optimal Strategic Function 2. Let us express $G_t/B_t^{(0)}$ by H_t. If the stochastic process $\{H_0, H_1, H_2, \cdots\}$ is an independent, stationary, stochastic process in a system, let us call our earning process the *proportional earning process.*

When a system builds a proportional earning process, the total profit till the time n, i.e. $\sum_{t=0}^{n-1} G_t$, is expressed as follows.

$$(3.13) \quad \sum_{t=0}^{n-1} G_t=B_n^{(0)}-b_0^{(0)}=b_0^{(0)}\cdot\{(1+H_0)\cdot(1+H_1)\cdots(1+H_{n-1})-1\}$$

Let m be a expected value of H_t's ($t=0, 1, 2, \cdots$), and σ^2 be the variance of these. The expected value of $\sum_{t=0}^{n-1} G_t$, say E_n, and the variance of it, say V_n, are expressed by the following values respectively.

$$(3.14) \quad \begin{aligned} E_n &= b_0^{(0)}\cdot\{(1+m)^n-1\} \\ V_n &= \{b_0^{(0)}\cdot(1+m)^n\}^2\cdot\left\{\left(1+\frac{\sigma^2}{(1+m)^2}\right)^n-1\right\} \end{aligned}$$

We can see some examples in which the variances $\lim_{n\to\infty} V_n$ and $\lim_{n\to\infty} V_n/n^2$

do not be equal to 0, but are positive infinite. And then, both $\lim_{n\to\infty}\sum_{t=0}^{n-1} G_t$ and $\lim_{n\to\infty}\sum_{t=0}^{n-1} G_t/n$ do not converge in probability to any values and to any random variables. We must choose our φ-converged, ultimate, objective functional by a function $\varphi(s_n, n)$ which differs from $s_n=\sum_{t=0}^{n-1} G_t$ and $s_n/n=\sum_{t=0}^{n-1} G_t/n$.

Now, when $p\{(1+H_t)>0\}=1$, we can consider $L_t=\log(1+H_t)$. According to the definition of the proportional earning process, the stochastic process $\{H_0, H_1, H_2, \cdots\}$ is an independent, stationary, stochastic process. And then, the stochastic process $\{L_0, L_1, L_2, \cdots\}$ is also an independent, stationary, stochastic process. If the random variable L_0 (and then $L_1, L_2, \cdots$) has a finite expected value and a finite variance, we can admit the following expression

$$\lim_{n\to\infty} P\{\,|\sum_{t=0}^{n-1} L_t/n-\sum_{t=0}^{n-1} E(L_t)/n\,|>\varepsilon\}=0 \tag{3.15}$$

for any small positive number ε. And further, we can admit the following expression.

$$\frac{\sum_{t=0}^{n-1} L_t}{n}=\log\sqrt[n]{\frac{B_n^{(0)}}{b_0^{(0)}}}=\log\sqrt[n]{\frac{b_0^{(0)}+\sum_{t=0}^{n-1} G_t}{b_0^{(0)}}} \tag{3.16}$$

After we put $\varphi(s_n, n)=(b_0^{(0)}+\sum_{t=0}^{n-1} g_t)/b_0^{(0)}$, we can admit that $\lim_{n\to\infty}\sum_{t=0}^{n-1} E(L_t)/n$ is a φ-converged, ultimate, objective functional in the proportional earning process. Since L_t's ($t=0, 1, 2, \cdots$) are independent with each other, we can obtain the φ-optimal strategic function $F(b, \alpha)$ as follows.

Let us calculate the vector $\bar{\boldsymbol{k}}$ which maximizes the expected value

$$E(L)=E\{\log(1+(\boldsymbol{C}_0, \boldsymbol{k}))\} \quad (\boldsymbol{k}=\boldsymbol{x}/b^{(0)}) \tag{3.17}$$

under the condition (3.1). If the variance of $L=\log(1+(\boldsymbol{C}_0, \bar{\boldsymbol{k}}))$ is bounded, we can admit that $\boldsymbol{x}=\bar{\boldsymbol{k}}\cdot b^{(0)}$ is our φ-optimal strategic function.

Example 2. If $b^{(1)}=b^{(2)}=\cdots=b^{(m)}=0$ in the condition (3.1), we can obtain

$$(\boldsymbol{A}^{(0)}, \boldsymbol{x})=b^{(0)},\ (\boldsymbol{A}^{(1)}, \boldsymbol{x})=(\boldsymbol{A}^{(2)}, \boldsymbol{x})=\cdots=(\boldsymbol{A}^{(m)}, \boldsymbol{x})=0,\ \boldsymbol{x}\geqq 0. \tag{3.18}$$

In a system which is defined on such a linear model, we can admit that our effective strategic function is expressed by $\boldsymbol{x}=\boldsymbol{k}\cdot b^{(0)}$. Then the system builds an independent, stationary, earning process, and $G_t/B_t^{(0)}=(\boldsymbol{C}_t, \boldsymbol{X}_t)/B_t^{(0)}=(\boldsymbol{C}_t, \boldsymbol{k})$.

Note 3. Since the expression (3.17) is a concave function of $\boldsymbol{k}$, we can obtain unique solution. This assertion has been shown by Kuhn and Tucker [14] (p. 481~492).

3.5. An Approximate Solution

At last, let us assume for $(\boldsymbol{C}_0, \boldsymbol{k})$ to be so small that we can decide to neglect the terms which exceed the 2nd degree in the expantion of (3.17), by our subjective policy. Then, we can approximate the function (3.17) by

$$(3.19) \qquad (\boldsymbol{m}_0, \boldsymbol{k}) - 1/2 \cdot \boldsymbol{k}' \cdot (\boldsymbol{M} + \textstyle\sum) \cdot \boldsymbol{k}.$$

Here, $\boldsymbol{m}_0$ is the expected vector of $\boldsymbol{C}_0$, and $\boldsymbol{M}$ is the squar matrix in which the ij-component is $m_0^{(i)} \cdot m_0^{(i)}$ ($m_0^{(i)}$ is the i-th component of the vector $\boldsymbol{m}_0$), and $\sum$ is the variance covariance matrix of the random vector $\boldsymbol{C}_0$.

Note 4. Since the expression (3.19) is also concave function of $\boldsymbol{k}$, we can obtain unique solution. And further, the solution is derived by the quadratic programming.

Next, let us consider an actual simple example which is defined on the restriction $x+y=1$. x is the level of our real activity and y is the level of the stock of money. Let C be a random variable which expresses the profit which is earned by the activity x, and r be the interest of money.

According to our previous considerations, the level x is that which maximizes

$$(3.20) \quad E\{(C \cdot x + r(1-x)) - \frac{1}{2} \cdot (C \cdot x + r(1-x))^2 + \frac{1}{3} \cdot (C \cdot x + r(1-x))^3\}$$

if we neglect the terms which exceed the third degree of $(C \cdot x - r(1-x))$. Of course, x must not be larger than a suitable positive number γ, so that we may admit the approximate expression (3.20). That is, if we put $E(C) = m$, $E\{(C-m)^2\} = \mu_2$ and $E\{(C-m)^3\} = \mu_3$, x is min (γ, ξ). There, ξ is a solution of the following equation which maximizes the expression (3.20).

$$(3.21) \qquad a \cdot \xi^2 - 2b \cdot \xi + c = 0$$

Here, $a = (m-r)^3 + 3 \cdot \mu_2 \cdot (m-r) + \mu_3$, $b = (1/2 - r) \cdot \{(m-r)^2 + \mu_2\}$,
$c = (m-r) \cdot (1 - r + r^2)$.

We can obtain the following table of $\xi(m, \mu_2, \mu_3/\mu_2)$ which maximizes (3.20).

Here, if $1 \geqq \xi \geqq 0$, $r = 0.03$, and if $\xi > 1$, $r = 0.06$. And the activity level x is $\min\{\gamma, \xi\ (m, \mu_2, \mu_3/\mu_2)\}$.

Table 1. This table is that of ξ $(m, \mu_2, \mu_3,/\mu_2)$.

$\sqrt[3]{\mu_3}/\sqrt[2]{\mu_2}$	0.00	0.00	0.00	0.40	0.40	0.40	0.80	0.80	0.80
m \ $\sqrt{\mu_2}$	0.10	0.20	0.30	0.10	0.20	0.30	0.10	0.20	0.30
0.04	1.00	0.26	0.12	1.00	0.26	0.12	1.00	0.27	0.12
0.05	1.00	0.53	0.23	1.00	0.53	0.23	1.00	0.57	0.24
0.06	1.00	0.82	0.36	1.00	0.83	0.37	1.00	0.93	0.38
0.07	1.10	1.00	0.47	1.11	1.00	0.49	1.13	1.00	0.52
0.08	2.47	1.00	0.62	2.53	1.00	0.63	∞	1.00	0.72
0.09	∞	1.00	0.77	∞	1.00	0.79	∞	1.01	1.00

3. 6. On the Gibrat's Law. We know an important statistic law " the Gibrat's law " which arises from the " effect of proportionarity ". We can admit the following theorem 3. 2 which relates with the " Gibrat's law ".

Let us consider a family of infinite firms. Let $b_t^{(0i)}$ $(i=1, 2, \cdots; t=0, 1, 2, \cdots)$ be the liquid capital of i-th firm at the time t, and $k_t^{(i)} \cdot b_t^{(0i)}$ be the profit which is earned at the time $t+1$, by the activity of $i-$th firm at the time t. $K_t^{(i)}$ denotes the random variable of which $k_t^{(i)}$ is a realization.

Theorem 3. 2. *Let us assume that*

1. *all firms in a family of infinite firms have same liquid capital at the initial time* 0, *and*

2. *each firm in the family is in a system which builds a proportional earning process by the strategy of each entreprenour, and*

3. *all probability distributions which associate to random variables* $K_t^{(i)}$*'s* $(t=0, 1, 2, \cdots; i=1, 2, \cdots)$ *are same on all* i*'s and* t*'s, and have a finite expected value and a finite variance, and further, these random variables* $K_t^{(i)}$*'s are independent with each other.*

Then. if $n\rightarrow\infty$, *the relative frequency distribution which is formed by liquid capitals of all firms at the time* n *converges in probability to the log-normal distribution.*

Note 5. The above assumption 3 shows that

a. the limitations of relative frequency distribution which are formed by the sequence $k_0^{(i)}, k_1^{(i)}, k_2^{(i)}, \cdots$ are same for all i's $(i=1, 2, 3, \cdots)$, and

b. the limitations of relative frequency distrbution which are formed by the sequence $k_t^{(1)}, k_t^{(2)}, k_t^{(3)}, \cdots$ are same for all t's $(t=0, 1, 2, \cdots)$, and

c. above two limited relative frequency distributions which are formed in a and b, coincide with the probability distribution which associates to the random variable $K_0^{(1)}$.

Note 6. In some cases, we recognize the probability distribution as the limitation of relative frequency distribution which is built by a infinite, independent, stationary, random sequence. When our forecast is built being associated by a sort of probability distribution, our considerations of this work will give a sort of foundation to choose our optimal stratey under incomplete informations. Recently, the choice of the optimal strategy under incomplete informations is studied on the basis of utility functions, [8], [9], etc. It seems effective to apply not only such utility functions but the above stochastic dynamic programming.

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