A Bivariate Analogue of Pooling of Data

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A BIVARIATE ANALOGUE OF POOLING OF DATA

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§ 1. Summary and introduction

This paper attempts to give some extensions by the consideration of pooling data of a bivariate population, and shows us certain formulae and some properties of the inference procedures.

The principle and methodological aspects of pooling data used in this paper have been discussed by Bancroft (1), Kitagawa (1) Bennett (1) and various other authors. And recently Asano (1), one of the authors of this paper, proposed also seven types of the inference procedures to be of use in practices on the background of biometrical and pharmaceutical researches.

These have been, however, mainly developed in case when the observations were obtained from the univariate populations, and, so far as the authors of this paper are awake to, the results and the properties caused by pooling data have not been given in case when the observations were drawn from certain multivariate populations.

Under the consideration of certain practical necessity, the inference problems through this paper are discussed for bivariate population. And here the authors of this paper shall note that the inferences of a mean vector for more general multivariate can be also expressed by the similar formulae as Section 2 and that while the inferences of a generalized variance and a variance-covariance matrix cannot be expressed explicitly by the similar formulae because of the complexity of the fundamental distribution of the statistic considered for us, but may be able to express by similar properties as Section 3 and 4.

In conclusion, the authors wish to express their heartiest thanks to Prof. T. Kitagawa for his kind advice and valuable suggestions and criticism in connection with this work.

2. Pooling of sample mean vectors

2.1. Type 1. (The inference of population mean vector with "known" population dispersion matrix)

Let O_{N_1}: $(\boldsymbol{x}_1^{(1)}, \boldsymbol{x}_2^{(1)}, \cdots, \boldsymbol{x}_{N_1}^{(1)})$ be a random sample of N_1 vector observations from a bivariate non-degenerate normal population $N[\boldsymbol{\mu}^{(1)}, \sum]$ and let O_{N_2}: $(\boldsymbol{x}_1^{(2)}, \boldsymbol{x}_2^{(2)}, \cdots, \boldsymbol{x}_{N_2}^{(2)})$ be another rondom sample of N_2 from some bivariate normal population $N[\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}]$. The values of these two common population dispersion matrices are known to us, but the populations have not necessarily the same population mean vector. The distinction between two populations may be regarded however as hypothetical. Let us suppose that we may pool the two sample mean vectors and form an estimate vector of the assumed same population mean vectors simultaneously in case when testing the hypothesis that $\boldsymbol{\mu}^{(1)}=\boldsymbol{\mu}^{(2)}$ shows that the hypothesis cannot be rejected.

Our rule of inference procedure is as follows:

(i) Let $\bar{\boldsymbol{x}}^{(i)}$ be sample mean vector defined by O_{Ni}, $(i=1, 2)$.

(ii) Let the statistic $\boldsymbol{U}'\boldsymbol{U}$ be defined by

$$\boldsymbol{U}'\boldsymbol{U}=(\bar{\boldsymbol{x}}^{(1)}-\bar{\boldsymbol{x}}^{(2)})'\left(\frac{1}{N_1}+\frac{1}{N_2}\right)^{-1}\boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{x}}^{(1)}-\bar{\boldsymbol{x}}^{(2)})$$

where $\quad \bar{\boldsymbol{x}}^{(i)}=\sum_{j=1}^{Ni}\boldsymbol{x}_j^{(i)}/N_i$.

(iii) Then let us define the statistic $\bar{\bar{\boldsymbol{x}}}$ in the following way.

(a) $\bar{\bar{\boldsymbol{x}}}=(N_1\bar{\boldsymbol{x}}^{(1)}+N_2\bar{\boldsymbol{x}}^{(2)})/(N_1+N_2)$ if $\boldsymbol{U}'\boldsymbol{U}\leq\chi_2^2(\alpha)$,

(b) $\bar{\bar{\boldsymbol{x}}}=\bar{\boldsymbol{x}}^{(1)}$ if $\boldsymbol{U}'\boldsymbol{U}>\chi_2^2(\alpha)$,

where $\chi_2^2(\alpha)$ means the significance value of χ^2-distribution with significance level α in case when the degrees of freedom is equal to 2.

Theorem 1.1. *The distribution function of $\bar{\bar{\boldsymbol{x}}}$ is given by*

$$(1.1)\qquad Pr.\{\bar{\bar{\boldsymbol{x}}}<\boldsymbol{u}\}^*=\int\limits_{r_1<\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}\left(u_1-\frac{N_1\mu_1{}^{(1)}+N_2\mu_1{}^{(2)}}{N_1+N_2}\right)}\frac{1}{\sqrt{2\pi}}exp\left[-\frac{r_1^2}{2}\right]\cdot\left[1-\Phi\left\{\frac{\rho}{1-\rho^2}r_1-\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}\right.\right.$$

$$\left.\left.\left(u_2-\frac{N_1\mu_2^{(1)}+N_2\mu_2^{(2)}}{N_1+N_2}\right)\right\}\right]dr_1\times\iint\limits_{(s_1+a_1)^2+\left(s_2+\frac{\rho a_1-a_2}{\sqrt{1-\rho^2}}\right)^2<\chi_2{}^2(\alpha)}\frac{1}{2\pi}exp\left[-\frac{1}{2}(s_1^2+s_2^2)\right]ds_1ds_2$$

$$+\iint\limits_{(s_1+a_1)^2\left(s_2+\frac{\rho u_1-a_2}{\sqrt{1-\rho^2}}\right)\geq\chi_2{}^2(\alpha)}\frac{1}{2\pi}exp\left[-\frac{1}{2}(s_1^2+s_2^2)\right]\left[\int\limits_{r_1<\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}(u_1-\mu_1^{(1)})+\sqrt{\frac{N_2}{N_1}}s_1}\frac{1}{\sqrt{2\pi}}exp\left[-\frac{r_1^2}{2}\right]\cdot\left[1-\Phi\left\{\frac{\rho}{\sqrt{1-\rho^2}}r_1\right.\right.\right.$$

$$\left.\left.\left.-\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}(u_2-\mu_2^{(1)})-\sqrt{\frac{N_2}{N_1}}\left(\frac{\rho s_1}{\sqrt{1-\rho^2}}-s_2\right)\right\}dr_1\right]ds_1ds_2.$$

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* Note : For the sake of convenience, throughout this paper, the notation of $\Pr\{\bar{\boldsymbol{x}}<\boldsymbol{u}\}$ or $\Pr\{\hat{\boldsymbol{\Sigma}}<\boldsymbol{u}\}$ shows at the same time the probability that each element of the vector $\bar{\boldsymbol{x}}$ or the matrix $\hat{\boldsymbol{\Sigma}}$ does not over a corresponding element of the vector or the matrix $\boldsymbol{u}$ namely, the distribution functions of the respective elements of $\bar{\boldsymbol{x}}$ or $\hat{\boldsymbol{\Sigma}}$ simultaneously.

Proof . Now let us put for a moment

$$(1.2)\qquad \boldsymbol{y}^{(i)\prime}=(y_1^{(i)}, y_2^{(i)})=\left(\frac{\bar{x}_1^{(i)}-\mu_1^{(i)}}{\sqrt{\sigma_{11}/N_i}}, \frac{\bar{x}_2^{(i)}-\mu_2^{(i)}}{\sqrt{\sigma_{22}/N_i}}\right), \quad (i=1,2),$$

and by using the following orthogonal transformations

$$(1.3)\qquad \boldsymbol{p}=\sqrt{\frac{N_1N_2}{N_1+N_2}}\left(-\frac{\boldsymbol{y}^{(1)}}{\sqrt{N_1}}+\frac{\boldsymbol{y}^{(2)}}{\sqrt{N_2}}\right), \quad \boldsymbol{q}=\sqrt{\frac{N_1N_2}{N_1+N_2}}\left(\frac{\boldsymbol{y}^{(1)}}{\sqrt{N_2}}+\frac{\boldsymbol{y}^{(2)}}{\sqrt{N_1}}\right),$$

the joint elementary probability of $\boldsymbol{p}$ and $\boldsymbol{q}$ may be given by

$$(1.4)\qquad h(\boldsymbol{p}',\boldsymbol{q}')=\frac{1}{(2\pi)^2(1-\rho^2)}exp\Big[-\frac{1}{2(1-\rho^2)}\{(p_1^2-2\rho p_1p_2+p_2^2)$$

$$+(q_1^2-2\rho q_1q_2+q_2^2)\}\Big].$$

Now let the sample space be divided into two distinct sets D_1 and D_2 which are defined as domains $\boldsymbol{U}'\boldsymbol{U}\leq\chi_2^2(\alpha)$ and $\boldsymbol{U}'\boldsymbol{U}>\chi_2^2(\alpha)$ respectively. Then we may and shall decompose the probability (1.1) into two parts,

$$Pr.\{\bar{\boldsymbol{x}}<\boldsymbol{u}\}=Pr\left\{\frac{N_1\bar{\boldsymbol{x}}^{(1)}+N_2\bar{\boldsymbol{x}}^{(2)}}{N_1+N_2}<\boldsymbol{u}, D_1\right\}+Pr.\{\bar{\boldsymbol{x}}^{(1)}<\boldsymbol{u}, D_2\}.$$

On the other hand, in view of (1.2) and (1.3), we may be obtained that $(N_1\bar{\boldsymbol{x}}^{(1)}+N_2\bar{\boldsymbol{x}}^{(2)})/(N_1+N_2)=\boldsymbol{E}+\boldsymbol{F}\boldsymbol{q}$ and

$\bar{\boldsymbol{x}}^{(1)}=\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}-\boldsymbol{G}\boldsymbol{F}\boldsymbol{p}$ where $\boldsymbol{E}=(N_1\boldsymbol{\mu}^{(1)}+N_2\boldsymbol{\mu}^{(2)})/(N_1+N_2)$,

$\boldsymbol{F}=\begin{pmatrix}\sqrt{\sigma_{11}} & 0\\ 0 & \sqrt{\sigma_{22}}\end{pmatrix}\Big/\sqrt{N_1+N_2}$ and $G=\sqrt{N_2/N_1}$.

Then the first term of (1,2) may be given by

$$(1.5)\qquad Pr\left\{\frac{N_1\bar{\boldsymbol{x}}^{(1)}+N_2\bar{\boldsymbol{x}}^{(2)}}{N_1+N_2}<\boldsymbol{u}, D_1\right\}=\underset{\substack{\boldsymbol{q}<\boldsymbol{F}^{-1}(\boldsymbol{u}-\boldsymbol{E})\\ (\boldsymbol{p}+\boldsymbol{a})'\boldsymbol{R}^{-1}(\boldsymbol{p}+\boldsymbol{u})<\chi_2^2(\alpha)}}{\iiint} h(\boldsymbol{p}',\boldsymbol{q}')\prod_{i=1}^{2}dp_idq_i$$

where $\boldsymbol{a}'=\left(\sqrt{\frac{N_1N_2}{N_1+N_2}}\frac{\mu^{(2)}-\mu^{(1)}}{\sqrt{\sigma_{11}}}, \sqrt{\frac{N_1N_2}{N_1+N_2}}\frac{\mu_2^{(2)}-\mu_2^{(1)}}{\sqrt{\sigma_{22}}}\right)$ and $\boldsymbol{R}=\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$. And, by using the transformation $\boldsymbol{p}=\boldsymbol{L}\boldsymbol{s}$ and $\boldsymbol{q}=\boldsymbol{L}\boldsymbol{r}$, where $\boldsymbol{L}=\begin{pmatrix}1 & \rho\\ \sigma & -\sqrt{1-\rho^2}\end{pmatrix}$, $\boldsymbol{s}=\begin{pmatrix}s_1\\ s_2\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}r_1\\ r_2\end{pmatrix}$, the probability may be written in the following way:

$$(1.6)\qquad Pr.\left\{\frac{N_1\bar{\boldsymbol{x}}^{(1)}+N_2\bar{\boldsymbol{x}}^{(2)}}{N_1+N_2}<\boldsymbol{u}, D_1\right\}$$

$$=\iint_{\omega 1}\frac{1}{2\pi}exp\Big[-\frac{1}{2}(r_1^2+r_2^2)\Big]dr_1dr_2\iint_{\omega 2}\frac{1}{2\pi}exp\Big[-\frac{1}{2}(s_1^2+s_2^2)\Big]ds_1ds_2$$

$$\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}\left(u_2-\frac{N_1\mu_1^{(1)}+N_2\mu_1^{(2)}}{N_1+N_2}\right)$$

$$=\int\limits_{-\infty}\frac{1}{\sqrt{2\pi}}exp\left[-\frac{r_1^2}{2}\right]\cdot\left[1-\Phi\left\{\frac{\rho}{1-\rho^2}r_1-\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}\left(u_2-\frac{N_1\mu_2^{(1)}+N_2\mu_2^{(2)}}{N_1+N_2}\right)\right\}\right]dr_1$$

$$\cdot\iint\limits_{(s_1+a_1)^2+\left(s_2+\frac{\rho a_1-a_2}{\sqrt{1-\rho^2}}\right)^2<\chi_2^2(\alpha)}\frac{1}{2\pi}exp\left[-\frac{1}{2}(s_1^2+s_2^2)\right]ds_1ds_2,$$

where we put that $\omega_1=\left\{r_1<\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}\left(u_1-\frac{N_1\mu_1^{(1)}+N_2\mu_1^{(2)}}{N_1+N_2}\right),\ r_2>\frac{\rho}{1-\rho^2}\ r_1\ -\right.$ $\left.\frac{\sqrt{N_2+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}\left(u_2-\frac{N_1\mu_2^{(1)}-N_2\mu_2^{(2)}}{N_1+N_2}\right)\right\}$, $\omega_2=\{(\boldsymbol{Ls}+\boldsymbol{a})'\boldsymbol{R}^{-1}(\boldsymbol{Ls}+\boldsymbol{a})<\chi_2^2(\alpha)\}$.

This gives us now easily the first term of the left-hand side of (1.1) by the transformation $v_1=s_1+a_1$ and $v_2=s_2+\frac{\rho a_1-a_2}{\sqrt{1-\rho^2}}$.

The second term of (1.2) may be similarly given as follows:

$$(1.7)\qquad Pr.\{\bar{\boldsymbol{x}}^{(1)}<\boldsymbol{u}, D_2\}=\iiiint\limits_{\omega_4}\frac{1}{(2\pi)^2(1-\rho^2)}exp\left[-\frac{1}{2(1-\rho^2)}\{(p_1^2-2\rho p_1p_2+p_2^2)\right.$$

$$\left.+(q_1^2-2\rho q_1q_2+q_2^2)\}\right]\cdot\prod_{i=1}^{2}dp_idq_i$$

$$=\iiiint\limits_{\omega_5}\frac{1}{(2\pi)^2}exp\left[-\frac{1}{2}(W_1^2+W_2^2+r_1^2+r_2^2)\right]\prod_{i=1}^{2}dW_idr$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}exp\left[-\frac{1}{2}(W_1^2+W_2^2)\right]\left[\int\limits_{r_1<\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}(u_1-\mu_1^{(1)})+\sqrt{\frac{N_2}{N_1}}W_1}\frac{1}{\sqrt{2\pi}}exp\left[-\frac{r_1^2}{2}\right]\left[1-\Phi\left\{\frac{\rho}{\sqrt{1-\rho^2}}r_1\right.\right.\right.$$

$$\left.\left.\left.-\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}(u_2-\mu_2^{(1)})-\sqrt{\frac{N_2}{N_1}}\left(\frac{\rho W_1}{\sqrt{1-\rho^2}}-W_2\right)\right\}dr_1\right]dW_1dW_2$$

where we put that

$$\omega_4=\{\boldsymbol{q}<\boldsymbol{F}^{-1}(\boldsymbol{u}-\boldsymbol{\mu}^{(1)})+\boldsymbol{Gp},\ (\boldsymbol{p}+\boldsymbol{a})\boldsymbol{R}^{-1}(\boldsymbol{p}+\boldsymbol{a})\geq\chi_2^2(\alpha)\},$$

$$\omega_5=\left\{r_1<\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}(u_1-\mu_1^{(1)})+\sqrt{\frac{N_2}{N_1}}W_1,\right.$$

$$\left.r_2>\frac{\rho}{\sqrt{1-\rho^2}}r_1-\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}(u_2-\mu_2^{(1)})-\sqrt{\frac{N_2}{N_1}}\left(\frac{\rho}{\sqrt{1-\rho^2}}W_1-W_2\right)\right\}$$

Thus we obtain the second term of the left-hand side of (1.1). In combination of (1.6) and (1.7), we may obtain (1.1) to be proved.

Theorem 1.2. *The mean vector* $E\{\bar{\bar{x}}\}$ *and the mean square deviation M.S.D.* $\{\bar{\bar{x}}\}$ *of the estimate* $\bar{\bar{x}}$ *are given by*

$$E\{\bar{\bar{\boldsymbol{x}}}\}=\boldsymbol{\mu}^{(1)}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})Pr.\{D_1\}-\boldsymbol{GF}I_{D_2}\{\boldsymbol{p}\} \tag{1.8}$$

and

$$M.S.D.\{\bar{\bar{\boldsymbol{x}}}\}=\frac{1}{N_1+N_2}\boldsymbol{\Sigma}+\left(\frac{N_2}{N_1+N_2}\right)^2(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'Pr.\{D_1\} \tag{1.9}$$
$$+\frac{N_2}{N_1}I_{D_2}\{\boldsymbol{F}\boldsymbol{p}\boldsymbol{p}'\boldsymbol{F}'\},$$

where we put

$$Pr.\{D_1\}=\underset{(\boldsymbol{p}+\boldsymbol{a})'\boldsymbol{R}^{-1}(\boldsymbol{p}+\boldsymbol{a})\le\chi_2{}^2(\alpha)}{\iint}\frac{1}{2\pi\sqrt{|\boldsymbol{R}|}}\,exp\left[-\frac{1}{2}\boldsymbol{p}\boldsymbol{R}^{-1}\boldsymbol{p}\right]dp_1dp_2 \tag{1.10}$$

$$I_{D_2}\{\boldsymbol{p}\}=\underset{(\boldsymbol{p}+\boldsymbol{a})'\boldsymbol{R}^{-1}(\boldsymbol{p}+\boldsymbol{a})>\chi_2{}^2(\alpha)}{\iint}\boldsymbol{p}\frac{1}{2\pi\sqrt{|\boldsymbol{R}|}}\,exp\left[-\frac{1}{2}\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}\right]dp_1dp_2. \tag{1.11}$$

Proof: Making use of the notation in the enunciation and proof of theorem 1.1, we may write

$$E(\bar{\bar{\boldsymbol{x}}})=\underset{D_1}{\iiiint}(\boldsymbol{E}+\boldsymbol{F}\boldsymbol{q})\frac{1}{(2\pi)^2|\boldsymbol{R}|}exp\left[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\right]\prod_{i=1}^{2}dp_idq_i \tag{1.12}$$
$$+\underset{D_2}{\iiiint}(\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}-\boldsymbol{GF}\boldsymbol{p})\frac{1}{(2\pi)^2|\boldsymbol{R}|}exp\left[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\right]\prod_{i=1}^{2}dp_idq_i$$
$$=\underset{D_1}{\iiiint}\left\{\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\right\}\frac{1}{(2\pi)^2|\boldsymbol{R}|}exp\left[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\right]$$
$$\prod_{i=1}^{2}dp_idq_i$$
$$+\underset{D_2}{\iiiint}(\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}-\boldsymbol{GF}\boldsymbol{p})\frac{1}{(2\pi)^2|\boldsymbol{R}|}exp\left[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\right]\prod_{i=1}^{2}dp_idq_i.$$

Thus the mean (1.8) is obtained and the M.S.D.$\{\bar{\bar{x}}\}$ is obtained from a following relation:

$$E\{\bar{\bar{\boldsymbol{x}}}\bar{\bar{\boldsymbol{x}}}'\}=\underset{D_1}{\iiiint}\Big[\boldsymbol{\mu}^{(1)}\boldsymbol{\mu}^{(1)\prime}+\boldsymbol{F}\boldsymbol{q}\boldsymbol{q}'\boldsymbol{F}'+\frac{N_2}{N_1+N_2}\{\boldsymbol{\mu}^{(1)}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'+(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\boldsymbol{\mu}^{(1)\prime}\} \tag{1.13}$$
$$+\left(\frac{N_2}{N_1+N_2}\right)^2(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'+(\boldsymbol{\mu}^{(1)}\boldsymbol{q}'\boldsymbol{F}'+\boldsymbol{F}\boldsymbol{q}\boldsymbol{\mu}^{(1)\prime})$$

$$+\frac{N_2}{N_1+N_2}\{\boldsymbol{F}\boldsymbol{q}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'+(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\boldsymbol{q}'\boldsymbol{F}'\}\Big]\frac{1}{(2\pi)^2|\boldsymbol{R}|}exp\Big[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}$$

$$+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\Big]\prod_{i=1}^{2}dp_i dq_i$$

$$+\iiiint_{D_2}\{\boldsymbol{\mu}^{(1)}\boldsymbol{\mu}^{(1)\prime}+\boldsymbol{F}\boldsymbol{q}\boldsymbol{q}'\boldsymbol{F}'+G^2\boldsymbol{F}\boldsymbol{p}\boldsymbol{p}'\boldsymbol{F}'+(\boldsymbol{\mu}^{(1)}\boldsymbol{q}'\boldsymbol{F}'+\boldsymbol{F}\boldsymbol{q}\boldsymbol{\mu}^{(1)\prime})-G(\boldsymbol{\mu}^{(1)}\boldsymbol{p}\boldsymbol{F}'$$

$$+\boldsymbol{F}\boldsymbol{p}\boldsymbol{\mu}^{(1)\prime})-G(\boldsymbol{F}\boldsymbol{q}\boldsymbol{p}'\boldsymbol{F}'+\boldsymbol{F}\boldsymbol{p}\boldsymbol{q}'\boldsymbol{F}')\}\frac{1}{(2\pi)^2|\boldsymbol{R}|}exp\Big[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\Big]$$

$$\prod_{i=1}^{2}dp_i dq_i.$$

Then substituting the following relations, $\boldsymbol{I}_{D_1+D_2}\{\boldsymbol{q}\boldsymbol{q}'\}=\begin{pmatrix}1\rho\\ \rho 1\end{pmatrix}$, $\boldsymbol{I}_{D_1+D_2}\{\boldsymbol{q}\}=\boldsymbol{I}_{D_1}\{\boldsymbol{q}\}=\begin{pmatrix}0\\0\end{pmatrix}$ and $\boldsymbol{I}_{D_2}\{\boldsymbol{q}\boldsymbol{p}'\}=\boldsymbol{I}_{D_2}\{\boldsymbol{p}\boldsymbol{q}'\}=\begin{pmatrix}n0\\00\end{pmatrix}$ for (1.13), we may obtain that

$$E\{\bar{\bar{\boldsymbol{x}}}\bar{\bar{\boldsymbol{x}}}'\}=\boldsymbol{\mu}^{(1)}\boldsymbol{\mu}^{(1)\prime}+\frac{1}{N_1+N_2}\boldsymbol{\Sigma}+\frac{N_2}{N_1+N_2}\{\boldsymbol{\mu}^{(1)}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'+(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\boldsymbol{\mu}^{(1)\prime}\}Pr.\{D_1\}$$

$$+\Big(\frac{N_2}{N_1+N_2}\Big)^2(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'Pr.\{D_1\}+\frac{N_2}{N_1}\boldsymbol{I}_{D_2}\{\boldsymbol{F}\boldsymbol{p}\boldsymbol{p}'\boldsymbol{F}'\}$$

$$-\sqrt{\frac{N_2}{N_1}}\boldsymbol{I}_{D_2}\{\boldsymbol{\mu}^{(1)}\boldsymbol{p}'\boldsymbol{F}'+\boldsymbol{F}\boldsymbol{p}\boldsymbol{\mu}^{(1)\prime}\}.$$

Thus the mean square deviation M.S.D. $\{\bar{\bar{\boldsymbol{x}}}\}$ of the estimate $\bar{\bar{\boldsymbol{x}}}$ may be otatined in the following manner.

$$\begin{aligned}M.S.D.\{\bar{\bar{\boldsymbol{x}}}\}&=E\{(\bar{\bar{\boldsymbol{x}}}-\boldsymbol{\mu}^{(1)})(\bar{\bar{\boldsymbol{x}}}-\boldsymbol{\mu}^{(1)})'\}\\&=E\{\bar{\bar{\boldsymbol{x}}}\bar{\bar{\boldsymbol{x}}}'\}-\boldsymbol{\mu}^{(1)}E\{\bar{\bar{\boldsymbol{x}}}'\}-E\{\bar{\bar{\boldsymbol{x}}}\}\boldsymbol{\mu}^{(1)\prime}+\boldsymbol{\mu}^{(1)}\boldsymbol{\mu}^{(1)\prime}\\&=\frac{1}{N_1+N_2}\boldsymbol{\Sigma}+\Big(\frac{N_2}{N_1+N_2}\Big)^2(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'Pr.\{D_1\}\\&\quad+\frac{N_2}{N_1}\boldsymbol{I}_{D_2}\{\boldsymbol{F}\boldsymbol{p}\boldsymbol{p}'\boldsymbol{F}'\}.\end{aligned}$$

Corollary 1.1. *Specially when $\boldsymbol{\mu}^{(2)}=\boldsymbol{\mu}^{(1)}$. $E\{\bar{\bar{\boldsymbol{x}}}\}=\boldsymbol{\mu}^{(1)}$, that is, $\bar{\bar{\boldsymbol{x}}}$ is an unbiassed estimate of $\boldsymbol{\mu}^{(1)}$, but the variance components and the absolute value of covariance components of the dispersion matrix are greater than $\sigma_{ii}/(N_1+N_2)$ and $|\sigma_{ij}|/(N_1+N_2)$ and less than σ_{ii}/N_1 and $|\sigma_{ij}|/N_1$, respectively, $(i,j=1,2)$.*

Corollary 1.2. *Specially when N_1 is very large compared with N_2, that is $N_1 \gg N_2$, the bias of the estimate of $\bar{\bar{\boldsymbol{x}}}$ becomes small, and the dis-*

persion becomes $\sum/(N_1+N_2)$ *nearly.*

Corollary 1.3. *Specially when* $\boldsymbol{\mu}^{(1)}=\boldsymbol{\mu}^{(2)}$, *the variance component of the estimate* $\bar{\bar{\boldsymbol{x}}}$ *in case of bivariate pooling of data becomes smaller than the variance of the corresponding estimate in case of univariate pooling of data. The difference is given by*

$$\frac{N_2\sigma_{ii}}{(N_1+N_2)N_1}[I_{D_2}\{p_i^2\}-I_{D_2*}\{p_i^2\}]<0,\ \ (i=1,2)$$

where

$$I_{D_2}\{p_i^2\}=\underset{\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}>\chi_2{}^2(\alpha)}{\iint} p_i^2\frac{1}{2\pi\sqrt{|\boldsymbol{R}|}}exp\Big[-\frac{1}{2}\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}\Big]\prod_{i=1}^{2}dp_i,$$

$$I_{D_2*}\{p_i^2\}=\underset{|p_i|>\sqrt{\chi_1{}^2(\alpha)}}{\int} p_i^2\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}p_i^2}\,dp_i.$$

Corollary 1.4. *In case when the dispersion matrices of two bivariate normal populations are distinct but these two correlation coefficients is assumed to be common, the similar results given by the similar inference rule of this section are obtained.*

Otherwise if these correlation coefficients are assumed to be unequal. the similar results may not be expressed explicitly.

2.2. Type 2. (The case with " unknown " population dispersion matrix in Type 1)

Let O_{N_1}: $(x_1^{(1)}, x_2^{(1)}, \cdots, x_{N_1}^{(1)})$ be a random sample of N_1 vector observations from a bivariate non-degenerate normal population $N[\boldsymbol{\mu}^{(1)}, \sum]$ and let O_{N_2}: $(x_1^{(2)}, x_2^{(2)}, \cdots, x_{N_2}^{(2)})$ be another random sample of N_2 from some bivariate normal population $N[\boldsymbol{\mu}^{(2)}, \sum]$. In this section, the values of these two common population dispersion matrices are assumed to be unknown to us, but the populations have not necessarily the same population mean vector. Our attempt of the present inference procedure is the same in Type 1.

Now our rule of inference procedure is as follows:

(i) Let $\bar{\boldsymbol{x}}^{(i)}$ be sample mean vector defined by O_{Ni}, $(i=1,2)$, and let $\boldsymbol{S}$ be the unbiassed estimate of common population dispersion given by

$$\boldsymbol{S}=(N_1+N_2-2)^{-1}\Big\{\sum_{i=1}^{N_1}(\boldsymbol{x}_i^{(1)}-\bar{\boldsymbol{x}}^{(1)})(\boldsymbol{x}_i^{(1)}-\bar{\boldsymbol{x}}^{(1)})'+\sum_{j=1}^{N_2}(\boldsymbol{x}_j^{(2)}-\bar{\boldsymbol{x}}^{(2)})(\boldsymbol{x}_j^{(2)}-\bar{\boldsymbol{x}}^{(2)})'\}.\Big\}$$

(ii) Let the statistic T^2 be defined by

$$T^2=\frac{N_1N_2}{N_1+N_2}(\bar{\boldsymbol{x}}^{(2)}-\bar{\boldsymbol{x}}^{(1)})'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}^{(2)}-\bar{\boldsymbol{x}}^{(1)}).$$

(iii) Then let us define the statistic $\bar{\bar{\boldsymbol{x}}}$ in the following way:

(a) $\bar{\boldsymbol{x}}=(N_1\bar{\boldsymbol{x}}^{(1)}+N_2\bar{\boldsymbol{x}}^{(2)})/(N_1+N_2)$, if $\frac{N_1+N_2-3}{2(N_1+N_2-2)}T^2\leq F_{2,N_1+N_2-3}(\alpha)$,

(b) $\bar{\boldsymbol{x}}=\bar{\boldsymbol{x}}^{(1)}$, if $\frac{N_1+N_2-3}{2(N_1+N_2-2)}T^2>F_{2,N_1+N_2-3}(\alpha)$,

where $F_{2,N_1+N_2-3}(\alpha)$ denotes α-percent point of F-distribution with the pair of degrees of freedom $(2, N_1+N_2-3)$.

Theorem 2.1. *The distribution function of* $\bar{\boldsymbol{x}}$ *is given by*

$$
(2.1)\quad Pr.\{\bar{\boldsymbol{x}}<\boldsymbol{u}\}=\int\limits_{r_1<\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}\left(u_1-\frac{N_1\mu_1^{(1)}+N_2\mu_1^{(2)}}{N_1+N_2}\right)}\left[1-\Phi\left\{\frac{\rho}{1-\rho^2}r_1-\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}\left(u_2-\frac{N_1\mu_2^{(1)}+N_2\mu_2^{(2)}}{N_1+N_2}\right)\right\}\right]
$$

$$
\cdot\frac{1}{\sqrt{2\pi}}exp\left[-\frac{r_1^2}{2}\right]dr_1
$$

$$
\cdot\iiint\limits_{D_1}\frac{1}{2\pi}exp\left[-\frac{1}{2}(s_1^2+s_2^2)\right]\frac{(\chi^2/2)^{\frac{N_1+N_2-3}{2}-1}e^{-\frac{\chi^2}{2}}}{2\Gamma\left(\frac{N_1+N_2-3}{2}\right)}ds_1ds_2d\chi^2
$$

$$
+\iint\limits_{D_2}\left[\int\limits_{r_1<\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}(u_1-\mu_1^{(1)})+\sqrt{\frac{N_2}{N_1}}s_1}\left[1-\Phi\left\{\frac{\rho}{\sqrt{1-\rho^2}}r_1-\frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}(u_2-\mu_2^{(1)})-\sqrt{\frac{N_2}{N_1}}\left(\frac{\rho}{\sqrt{1-\rho^2}}s_1-s_2\right)\right\}\right]\right.
$$

$$
\left.\cdot\frac{1}{\sqrt{2\pi}}exp\left[-\frac{r_1^2}{2}\right]dr_1\right]\cdot\frac{1}{2\pi}exp\left[-\frac{1}{2}(s_1^2+s_2^2)\right]\frac{(\chi^2/2)^{\frac{N_1+N_2-3}{2}-1}e^{-\frac{\chi^2}{2}}}{2\Gamma\left(\frac{N_1+N_2-3}{2}\right)}ds_1ds_2d\chi^2,
$$

where sets D_1 *and* D_2 *are defined as domains* $\left\{(s_1+a_1)^2+\left(s_2+\frac{\rho a_1-a_2}{\sqrt{1-\rho^2}}\right)^2\right\}/\chi^2\leq\frac{2}{N_1+N_2-3}F_{2,N_1+N_2-3}(\alpha)$ *and* $\left\{(s_1+a_1)^2+\left(s_2+\frac{\rho a_1-a_2}{\sqrt{1-\rho^2}}\right)^2\right\}/\chi^2>\frac{2}{N_1+N_2-3}F_{2,N_1+N_2-3}(\alpha)$ *respectively.*

Proof: Let us use the notation in the enunciation of this section and proof of Theorem 1.1.

Let

$$
\boldsymbol{B}=\begin{pmatrix}\frac{1}{\sqrt{\sigma_{11}}} & 0\\ -\frac{\sqrt{\sigma_{11}}\rho}{\sqrt{|\boldsymbol{\Sigma}|}} & \frac{\sqrt{\sigma_{11}}}{\sqrt{|\boldsymbol{\Sigma}|}}\end{pmatrix},\quad \boldsymbol{Y}=\sqrt{\frac{N_1N_2}{N_1+N_2}}(\bar{\boldsymbol{x}}^{(2)}-\bar{\boldsymbol{x}}^{(1)}),\quad \boldsymbol{Y}^*=\begin{pmatrix}Y_1^*\\ Y_2^*\end{pmatrix}=\boldsymbol{BY},\ \boldsymbol{S}^*=\boldsymbol{BSB}',
$$

$$
\boldsymbol{Q}=\begin{pmatrix}\frac{Y_1^*}{\sqrt{\boldsymbol{Y}^{*\prime}\boldsymbol{Y}^*}} & \frac{Y_2^*}{\sqrt{\boldsymbol{Y}^{*\prime}\boldsymbol{Y}^*}}\\ \frac{Y_2^*}{\sqrt{\boldsymbol{Y}^{*\prime}\boldsymbol{Y}^*}} & \frac{Y_1^*}{\sqrt{\boldsymbol{Y}^{*\prime}\boldsymbol{Y}^*}}\end{pmatrix},\quad \boldsymbol{V}=\boldsymbol{QY}^*,\ \boldsymbol{C}=\boldsymbol{Q}(N_1+N_2-2)\boldsymbol{S}^*\boldsymbol{Q}',
$$

then $B\sum B'=I$, and Q is an orthogonal matrix. Y is distributed according to $N\left[\sqrt{\frac{N_1N_2}{N_1+N_2}}(\mu^{(2)}-\mu^{(1)}),\sum\right]$ and $(N_1+N_2-2)S$ is distributed as $\sum_{\beta=1}^{N_1+N_2-2}Z_\beta Z_\beta'$ with the Z_β independent, each with distribution $N[O,\sum]$, and Z_β is independent of $\bar{x}^{(1)}$, $\bar{x}^{(2)}$, thereby is independent of p and q. $(N_1+N_2-2)S^*$ is distributed as $\sum_{\gamma=1}^{N_1+N_2-2}Z_\beta^* Z_\beta^{*\prime}=\sum_\beta BZ_\beta(BZ_\beta)'$ with the Z_β^* independent, each with distribution $N[O,I]$, and Z_β^* is independent of Y^*. p, and q. The conditional distribution of C given Q is that of $\sum_\beta W_\beta W_\beta'$ where conditionally theW_β defined by QZ_β^* are independent, each with distribution $N[O,I]$.

If we put that $C=\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}$, then $c_{11\cdot2}=c_{11}-c_{12}c_{22}^{-1}c_{21}$ is conditionally distributed as χ^2 with N_1+N_2-3 degrees of freedom. Since the conditional distribution of $c_{11\cdot2}$ does not depend on Q it is unconditionally distributed as χ^2, and is independent of p, q.

On the other hand, if we put that $Q=\begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}$, then $V_1=\sum q_{1i}Y_i^*=\sqrt{Y^{*\prime}Y^*}$, and putting $\mu^{(2)}=\mu^{(1)}$, V_1^2 given by $p'R^{-1}p$ or by $Y^{*\prime}Y^*$ has a χ^2-distribution with 2 degrees of freedom. Therefore if $\mu^{(2)}=\mu^{(1)}$, $T^2(N_1+N_2-3)/2(N_1+N_2-2)$ is distributed as a F with 2 and N_1+N_2-3 degrees of freedom.

The joint probability density of p, q and χ^2 given by $c_{11\cdot2}$ may be given by

$$h(p',q')f_{N_1+N_2-3}(\chi^2)=\frac{1}{(2\pi)^2|R|}exp\left[-\frac{1}{2}(p'R^{-1}p+q'R^{-1}q)\cdot\frac{(\chi^2/2)^{\frac{N_1+N_2-3}{2}-1}e^{-\frac{\chi^2}{2}}}{2\Gamma\left(\frac{N_1+N_2-3}{2}\right)}\right].$$

Now let the sample space be devided into two mutually distinct sets D_1 and D_2 which are defined as, $(N_1+N_2-3)p'R^{-1}p/2\chi^2\leq F_{2,N_1+N_2-3}(\alpha)$ and $(N_1+N_2-3)p'R^{-1}p/2\chi^2>F_{2,N_1+N_2-3}(\alpha)$ respectively. Then

$$(2.2)\qquad Pr.\{\bar{x}<u\}=Pr.\left\{\frac{N_1\bar{x}^{(1)}+N_2\bar{x}^{(2)}}{N_1+N_2}<u,D_1\right\}+Pr.\{\bar{x}^{(1)}<u,D_2\}.$$

The first term of the right-hand side of (2.2) may be

$$Pr\left\{\frac{N_1\bar{x}^{(1)}+N_2\bar{x}^{(2)}}{N_1+N_2}<u,D_1\right\}=\underset{q<F^{-1}(u-E),\,D_1}{\int\int\int\int}h(p',q')f_{N_1+N_2-3}(\chi^2)(\prod_{i=1}^{2}dp_idq_i)d\chi^2,$$

and by using the transformation (1.5), the probability may be given by

$$Pr.\left\{\frac{N_1\bar{x}^{(1)}+N_2\bar{x}^{(2)}}{N_1+N_2}<u,D_1\right\}=\underset{\substack{Lr<F^{-1}(u-E)\\ (N_1+N_2-3)s'Is/2\chi^2\leq F_{2,N_1+N_2-3}(\alpha)}}{\int\int\int\int\int}\frac{1}{(2\pi)^2}exp\left[-\frac{1}{2}(r'Ir+s'Is)f_{N_1+N_2-3}(\chi^2)\right.$$

$$(\prod_{i=1}^{2}ds_i)d\chi^2$$

$$= \int\int\int\int_{\substack{r_1 < \frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}\left(u_1 - \frac{N_1\mu_1^{(1)}+N_2\mu_1^{(2)}}{N_1+N_2}\right) \\ (N_1+N_2-3)s'Is/2\chi^2 \leq F_{2,N_1+N_2-3}(\alpha)}} \left[1-\Phi\left\{\frac{\rho}{1-\rho^2}r_1 - \frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}\left(u_2 - \frac{N_1\mu_2^{(1)}+N_2\mu_2^{(2)}}{N_1+N_2}\right)\right\}\right]\left(\frac{1}{2\pi}\right)^{3/2}$$

$$exp\left[-\frac{1}{2}(r_1^2 + s'Is)\right] f_{N_1+N_2-3}(\chi^2)\,dr_1 \cdot \left(\prod_{i=1}^{2} ds_i\right) d\chi^2,$$

and putting that $v_1 = s_1 + a_1$ and $V_2 = s_2 + \frac{\rho a_1 - a_2}{\sqrt{1-\rho^2}}$, we may obtain

$$\text{(2.3)} \qquad Pr\left\{\frac{N_1\bar{x}^{(1)} + N_2\bar{x}^{(2)}}{N_1+N_2} < u, D_1\right\}$$

$$= \int\int\int\int_{\substack{r_1 < \frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{11}}}\left(u_1 - \frac{N_1\mu_1^{(1)}+N_2\mu_1^{(2)}}{N_1+N_2}\right) \\ (N_1+N_2-3)(v_1^2+v_2^2)/2\chi^2 \leq F_{2,N_1+N_2-3}(\alpha)}} \left[1-\Phi\left\{\frac{\rho}{1-\rho^2}r_1 - \frac{\sqrt{N_1+N_2}}{\sqrt{\sigma_{22}}\sqrt{1-\rho^2}}\left(u_2 - \frac{N_1\mu_2^{(1)}+N_2\mu_2^{(2)}}{N_1+N_2}\right)\right\}\right]$$

$$\cdot\left(\frac{1}{2\pi}\right)^{3/2} exp\left[-\frac{1}{2}\left\{r_1^2 + (v_1 - a_1)^2 + \left(v_2 - \frac{\rho\, a_1 - a_2}{\sqrt{1-\rho^2}}\right)^2\right\}\right] f_{N_1+N_2-3}(\chi^2)\,dr_1\left(\prod_{i=1}^{2} dv_i\right) d\chi^2$$

The second term of the right-hand side of (2.2) may be similarly

$$\text{(2.4)} \quad Pr.\{\bar{x}^{(1)} < u, D_2\} = \int\int\int\int_{\substack{q < F^{-1}(u-\mu^{(1)})+Gp \\ p'R^{-1}p(N_1+N_2-3)/2\chi^2 > F_{2,N_1+N_2-3}(\alpha)}} h(p', q') f_{N_1+N_2-3}(\chi^2)\left(\prod_{i=1}^{2} dp_i dq_i\right) d\chi^2$$

$$= \int\int\int\int_{\substack{Lr < F^{-1}(u-\mu^{(1)})+Gp \\ (N_1+N_2-3)s'Is/2\chi^2 > F_{2,N_1+N_2-3}(\alpha)}} \frac{1}{(2\pi)^2} exp\left[-\frac{1}{2}(r'Ir + s'Is)\right] f_{N_1+N_2-3}(\chi^2)\left(\prod_{i=1}^{2} dp_i dq_i\right) d\chi^2.$$

In combination of (2.3) and (2.4), we shall reach (2.1) to be proved.

Theorem 2.2. *The mean vector $E\{\bar{\bar{x}}\}$ and the mean square deviation M.S.D.$\{\bar{x}\}$ of the estimate $\bar{\bar{x}}$ are given by*

$$\text{(2.5)} \qquad E\{\bar{\bar{x}}\} = \mu^{(1)} + \frac{N_2}{N_1+N_2}(\mu^{(2)} - \mu^{(1)})Pr.\{D_1\} - GFI_{D_2}\{p\},$$

and

$$\text{(2.6)} \qquad M.S.D.\{\bar{\bar{x}}\} = \frac{1}{N_1+N_2}\Sigma + \left(\frac{N_2}{N_1+N_2}\right)^2 (\mu^{(2)} - \mu^{(1)})(\mu^{(2)} - \mu^{(1)})'\, Pr.\{D_1\}$$

$$+ \frac{N_2}{N_1} I_{D_2}\{Fpp'F'\},$$

where we put

$$(2.7)\qquad Pr.\{D_1\}=\underset{(N_1+N_2-3)\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}/2\chi^2\leq F_{2,N_1+N_2-3}(\alpha)}{\iiint}\frac{1}{2\pi\sqrt{|\boldsymbol{R}|}}exp\Big[-\frac{1}{2}\,\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}\Big]\frac{(\chi^2/2)^{\frac{N_1+N_2-3}{2}-1}e^{-\frac{\chi^2}{2}}}{2\Gamma\left(\frac{N_1+N_2-3}{2}\right)}dp_1dp_2d\chi^2$$

Proof: Making use of the notation in the enunciation and proof of Theorem 2.1, we may write

$$E\{\bar{\boldsymbol{x}}\}=\int\iiint\!\!\int_{D_1}\Big\{\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\Big\}\frac{1}{(2\pi)^2\sqrt{|\boldsymbol{R}|}}exp\Big[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}$$

$$+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\Big]\frac{(\chi^2/2)^{\frac{N_1+N_2-3}{2}-1}e^{-\frac{\chi^2}{2}}}{2\Gamma\left(\frac{N_1+N_2-3}{2}\right)}(\prod_{i=1}^{2}dp_idq_i)d\chi^2$$

$$+\iiiint\!\!\int_{D_2}(\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}-\boldsymbol{G}\boldsymbol{F}\boldsymbol{p})\frac{1}{(2\pi)^2\sqrt{|\boldsymbol{R}|}}exp\Big[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\Big]$$

$$\frac{(\chi^2/2)^{\frac{N_1+N_2-3}{2}-1}e^{-\frac{\chi^2}{2}}}{2\Gamma\left(\frac{N_1+N_2-3}{2}\right)}\Big(\prod_{i=1}^{2}dp_idq_i\Big)\,d\chi^2$$

$$=\boldsymbol{\mu}^{(1)}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})Pr.\{D_1\}-\boldsymbol{G}\boldsymbol{F}I_{D_2}\{\boldsymbol{p}\}.$$

Thus the mean (2.5) is obtained and the mean square de viation M.S.D. may be obtained from a following relation;

$$(2.8)\quad E\{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}'\}$$

$$=\iint\!\!\iint_{D_1}\!\!\int\Big\{\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\Big\}\Big\{\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\Big\}'$$

$$\frac{1}{(2\pi)^2\sqrt{|\boldsymbol{R}|}}exp\Big[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\Big]f_{N_1+N_2-3}(\chi^2)\,(^2\textstyle\prod_{i=1}dp_idq_i)d\chi^2$$

$$+\iint\!\!\iint_{D_2}\!\!\int\Big\{\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\Big\}\Big\{\boldsymbol{\mu}^{(1)}+\boldsymbol{F}\boldsymbol{q}+\frac{N_2}{N_1+N_2}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\Big\}'$$

$$\frac{1}{(2\pi)^2\sqrt{|\boldsymbol{R}|}}exp\Big[-\frac{1}{2}(\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}+\boldsymbol{q}'\boldsymbol{R}^{-1}\boldsymbol{q})\Big]f_{N_1+N_2-3}(x^2)\,(^2\textstyle\prod_{i=1}dp_idq_i)d\chi^2.$$

Then substituting the following relations

$$(2.9)\qquad I_{D_1+D_2}\{\boldsymbol{q}\boldsymbol{q}'\}=\begin{pmatrix}1&\rho\\\rho&1\end{pmatrix},\quad I_{D_1+D_2}\{\boldsymbol{q}\}=I_{D_1}\{\boldsymbol{q}\}=\begin{pmatrix}0\\0\end{pmatrix},$$

$$I_{D_2}\{\boldsymbol{p}\boldsymbol{q}'\}=I_{D_2}\{\boldsymbol{q}\boldsymbol{p}'\}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

for (2.8), we may obtain that

$$(2.10)\quad E\{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}'\}=\boldsymbol{\mu}^{(1)}\boldsymbol{\mu}^{(1)\prime}+\frac{1}{N_1+N_2}\boldsymbol{\Sigma}+\frac{N_2}{N_1+N_2}\{\boldsymbol{\mu}^{(1)}(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'$$

$$+(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})\boldsymbol{\mu}^{(1)\prime}\}\,Pr.\{D_1\}+\left(\frac{N_2}{N_1+N_2}\right)^2(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'Pr.\{D_1\}$$

$$+\frac{N_2}{N_1}I_{D_2}\{\boldsymbol{F}\boldsymbol{p}\boldsymbol{p}'\boldsymbol{F}'\}-\sqrt{\frac{N_2}{N_1}}\;I_{D_2}\{\boldsymbol{\mu}^{(1)}\boldsymbol{p}'\boldsymbol{F}'+\boldsymbol{F}\boldsymbol{p}\boldsymbol{\mu}^{(1)\prime}\}.$$

Hence the $M.S.D.\{\bar{\boldsymbol{x}}\}$ may be obtained conclusionally by

$$(2.11)\quad M.S.D.\{\bar{\boldsymbol{x}}\}=E\{(\bar{\boldsymbol{x}}-\boldsymbol{\mu}^{(1)})(\bar{\boldsymbol{x}}-\boldsymbol{\mu}^{(1)})'\}$$

$$=E\{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}'\}-\boldsymbol{\mu}^{(1)}E\{\bar{\boldsymbol{x}}'\}-E\{\bar{\boldsymbol{x}}\}\boldsymbol{\mu}^{(1)\prime}+\boldsymbol{\mu}^{(1)}\boldsymbol{\mu}^{(1)\prime}$$

$$=\frac{1}{N_1+N_2}\boldsymbol{\Sigma}+\left(\frac{N_2}{N_1+N_2}\right)^2(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})(\boldsymbol{\mu}^{(2)}-\boldsymbol{\mu}^{(1)})'Pr.\{D_1\}+\frac{N_2}{N_1}I_{D_2}\{\boldsymbol{F}\boldsymbol{p}\boldsymbol{p}'\boldsymbol{F}'\}.$$

Corollary 2.1. *Specially under the same assumption as corollary* 1.1 *or corollary* 1.2, *the same results are obtained correspondently.*

Corollary 2.2. *Specially when* $\boldsymbol{\mu}^{(2)}=\boldsymbol{\mu}^{(1)}$, *the variance component of the estimate* $\bar{\boldsymbol{x}}$ *in case of bivariate pooling of data becomes smaller than the variance of the corresponding estimate in case of univariate pooling of data. The difference is given by*

$$(2.12)\quad \frac{N_2\sigma_{ii}}{(N_1+N_2)N_1}[I_{D_2}\{p_i^2\}-I_{D_2^*}\{p_i^2\}]<0,\quad (i=1,2)$$

where

$$I_{D_2}\{p_i^2\}=\underset{\frac{\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}}{\chi^2}\cdot\frac{N_1+N_2-3}{2}>F_{2,N_1+N_2-3}(\alpha)}{\int\int\int} p_i^2\frac{1}{2\pi\sqrt{|\boldsymbol{R}|}}exp\left[-\frac{1}{2}\,\boldsymbol{p}'\boldsymbol{R}^{-1}\boldsymbol{p}\right]\cdot\frac{(\chi^2/2)^{\frac{N_1+N_2-3}{2}-1}e^{-\frac{\chi^2}{2}}}{2\Gamma\frac{N_1+N_2-3}{2}}dp_1dp_2d\chi^2,$$

$$I_{D_2^*}\{p_i^2\}=\underset{\sqrt{N_1+N_2}\cdot\frac{\left|p_i+\sqrt{\frac{N_1N_2}{N_1+N_2}}\frac{\mu_i(2)-\mu_i(1)}{\sigma}\right|}{\sqrt{2V}}>t_{N_1+N_2-2}(\alpha)}{\int\int} p_i^2\frac{1}{\sqrt{2\pi}}e^{-\frac{p_i^2}{2}}\frac{V^{\frac{N_1+N_2}{2}-2}e^{-V}}{\Gamma\left(\frac{N_1+N_2-2}{2}\right)}\,dp_idV.$$

§ 3. Pooling of sample dispersion matfix

Let O_{N_1}: $(\boldsymbol{x}_1^{(1)},\boldsymbol{x}_2^{(1)},\cdots,\boldsymbol{x}_{N_1}^{(1)})$ be a random sample of N_1 from a normal population $N[\boldsymbol{\mu}^{(1)},\boldsymbol{\Sigma}^{(1)}]$, and let O_{N_2}: $(\boldsymbol{x}_1^{(2)}\boldsymbol{x}_2^{(2)},\cdots,\boldsymbol{x}_{N_2}^{(2)})$ be another random sample of N_2 from some normal population $N[\boldsymbol{\mu}^{(2)},\boldsymbol{\Sigma}^{(2)}]$. The population dispersion matrices $\boldsymbol{\Sigma}^{(1)}$ and $\boldsymbol{\Sigma}^{(2)}$ may or may not be distinct. Now let us assume that we are in a situation of estimating the $\boldsymbol{\Sigma}^{(1)}$ on the basis of pooling data in order to make better the precision of a estimation of $\boldsymbol{\Sigma}^{(1)}$.

And we shall pool the corresponding elements of the two sample dispersion matrices $\hat{\Sigma}^{(1)}$ and $\hat{\Sigma}^{(2)}$ respectively and form an estimate of respective element of $\Sigma^{(1)}$ in case when testing of the hypothesis that $|\Sigma^{(1)}|=|\Sigma^{(2)}|$ against the alternative $|\Sigma^{(1)}|<|\Sigma^{(2)}|$ leads us to the decision that the hypothesis $|\Sigma^{(1)}|=|\Sigma^{(2)}|$ cannot be rejected.

Our rule of inference procedure is as follows:

(i) Let $A^{(1)}$ and $A^{(2)}$ be matrices of sample sums of squares and cross products about the sample mean defined by

$$A^{(i)}=\sum_{\alpha=1}^{N_i}(x_\alpha^{(i)}-\bar{x}^{(i)})(x_\alpha^{(i)}-\bar{x}^{(i)})', \qquad (i=1,2),$$

where $\bar{x}^{(i)}=\sum_{\alpha=1}^{N_i} x_\alpha^{(i)}/N_i$.

(ii) Now let us introduce the estiuate $\hat{\hat{\Sigma}}^{(1)}$ of $\Sigma^{(1)}$ defined in the following way.

(a) $\hat{\hat{\Sigma}}^{(1)}=(N_1\hat{\Sigma}^{(1)}+N_2\hat{\Sigma}^{(2)})/(N_1+N_2)$, if $|A^{(2)}|^{1/2}/|A^{(1)}|^{1/2}\leq\lambda_\alpha$,

(b) $\hat{\hat{\Sigma}}^{(1)}=\hat{\Sigma}^{(1)}$, if $|A^{(2)}|^{1/2}/|A^{(1)}|^{1/2}>\lambda_\alpha$,

where we put that $\hat{\Sigma}^{(i)}=A^{(i)}/N_i=\begin{pmatrix}\hat{\sigma}_{11}^{(i)} & \hat{\sigma}_{12}^{(i)}\\ \hat{\sigma}_{21}^{(i)} & \hat{\sigma}_{22}^{(i)}\end{pmatrix}$, $(i=1,2)$, and the non-negative switching constant prescribed λ_α denotes a value of α-percent point of F-distribution with the pair $(2N_2-4, 2N_1\text{-}4)$ of degrees of freedom.

Here we note that the test or switching criterion of our test procedure is substituted for an ordinary but troublesome test criterion $(N_1+N_2)^{N_1+N_2}\prod_{i=1}^{2}|A^{(i)}|^{N_i/2}/N_1^{N_1}N_2^{N_2}|A^{(1)}+A^{(2)}|^{(N_1+N_2)/2}$ and now our concern is the results of an inference procedure caused by the present test criterion.

Theorem 3.1. *The distribution function of* $\hat{\hat{\Sigma}}^{(1)}$ *is given by*

$$(3.1)\quad Pr.\{\hat{\hat{\Sigma}}^{(1)}<u\}=\int\cdots\int_{D_1} f(\hat{\sigma}_{11}^{(1)},\hat{\sigma}_{22}^{(1)},\hat{\sigma}_{11}^{(2)},r_1,r_2,\lambda)d\hat{\sigma}_{11}^{(1)}d\hat{\sigma}_{22}^{(1)}d\hat{\sigma}_{11}^{(2)}dr_1dr_2d\lambda$$
$$+\int\cdots\int_{D_1} f(\hat{\sigma}_{11}^{(2)},\hat{\sigma}_{22}^{(1)},\hat{\sigma}_{11}^{(2)},r_1,r_2,\lambda)d\hat{\sigma}_{11}^{(1)}d\hat{\sigma}_{22}^{(1)}d\hat{\sigma}_{11}^{(2)}dr_1dr_2d\lambda,$$

where we put that $u=\begin{pmatrix}u_{11} & u_{12}\\ u_{12} & u_{22}\end{pmatrix}$ *and*

$$(3.2)\quad f(\hat{\sigma}_{11}^{(1)},\hat{\sigma}_{22}^{(1)},\hat{\sigma}_{11}^{(2)},r_1,r_2,\lambda)=\prod_{i=1}^{2}\frac{N_i^{N_i1}}{4\pi\Gamma(N_i-2)\{\sigma_{11}^{(i)}\sigma_{22}^{(i)}(1-\rho_i^2)\}^{\frac{N_i-1}{2}}}\sum_{\nu_i}$$
$$\frac{1}{\nu_i!}\left\{\frac{N_i\rho_i\sigma_{11}^{(i)}\sigma_{22}^{(i)}}{\sigma_{11}^{(i)}\sigma_{22}^{(i)}(1-\rho_i^2)}\right\}^{\nu_i}$$
$$\cdot(1-r^2)^{\frac{N_1-4}{2}+\nu_2}r_1^{\nu_1}\cdot(1-r_2^2)^{\frac{N_2-4}{2}-\nu_2}r_2^{\nu_2}\cdot\hat{\sigma}_{11}^{(1)\,\nu_1+\nu_2-1}e^{-\frac{N_1\hat{\sigma}_{11}^{(1)}}{2\sigma_{11}^{(1)}(1-\rho_1^2)}}$$

$$\cdot \hat{\sigma}_{11}^{(2)-1} e^{-\frac{N_2 \hat{\sigma}_{11}^{(2)}}{2\sigma_{11}^{(2)}(1-\rho_2^2)}} \cdot \hat{\sigma}_{22}^{(1)\,\nu_1+\nu_2-1} e^{-\frac{N_1 \hat{\sigma}_{22}^{(1)}}{2\sigma_{22}^{(1)}(1-\rho_1^2)}}$$

$$\cdot (\lambda^2)^{\nu_2-1} e^{-\frac{N_2}{2\sigma_{22}^{(2)}(1-\rho_2^2)} \frac{\hat{\sigma}_{11}^{(1)}\hat{\sigma}_{22}^{(1)}(1-r_1^2)}{\hat{\sigma}_{11}^{(2)}(1-r_2^2)}\lambda^2} \cdot d\hat{\sigma}_{11}^{(1)} d\hat{\sigma}_{22}^{(1)} d\hat{\sigma}_{11}^{(2)} dr_1 dr_2 d\lambda^2,$$

and the domain D_1 and D_2 in the sample spaces denote that

$$(3.3)\quad \begin{aligned} D_1 : 0<\lambda<\lambda_\alpha, & \begin{cases} 0<\frac{1}{N_1+N_2}\left(N_1\hat{\sigma}_{22}^{(1)}+N_2\frac{\hat{\sigma}_{11}^{(1)}\hat{\sigma}_{22}^{(1)}(1-r^2)\lambda^2}{\hat{\sigma}_{11}^{(2)}(1-r_2^2)}\right)<u_{22} \ \textit{for}\ \hat{\hat{\sigma}}_{22}^{(1)}, \\ 0<\frac{1}{N_1+N_2}(N_1\hat{\sigma}_{11}^{(1)}+N_2\hat{\sigma}_{11}^{(2)})<u_{11} \ \textit{for}\ \hat{\hat{\sigma}}_{11}^{(1)}, \\ -\infty<\frac{1}{N_1+N_2}\left\{N_1 r_1\sqrt{\hat{\sigma}_{11}^{(1)}\hat{\sigma}_{22}^{(1)}}+N_2 r_2\sqrt{\frac{\sigma_{11}^{(1)}\sigma_{22}^{(1)}(1-r_1^2)\lambda^2}{\sigma_{11}^{(2)}(1-r_1^2)}}\right.<u_{12} \ \textit{for}\ \hat{\hat{\sigma}}_{12}^{(1)}, \end{cases} \\ D_2 : \lambda_\alpha<\lambda<\infty, & \begin{cases} 0<\hat{\sigma}_{11}^{(1)}<u_{11},\ 0<\hat{\sigma}_{12}^{(1)}, \hat{\sigma}_{11}^{(2)}<\infty,\ -1<r_1, r_2<+1 \ \textit{for}\ \hat{\hat{\sigma}}_{11}^{(1)}, \\ 0<\hat{\sigma}_{22}^{(1)}<u_{22},\ 0<\hat{\sigma}_{11}^{(1)}, \hat{\sigma}_{11}^{(2)}<\infty,\ -1<r_1, r_2<+1 \ \textit{for}\ \hat{\hat{\sigma}}_{22}^{(1)}, \\ -\infty<r_1\sqrt{\hat{\sigma}_{11}^{(1)}\hat{\sigma}_{22}^{(1)}}<u_{12},\ 0<\hat{\sigma}_{11}^{(2)}<\infty,\ -1<r_2<1 \ \textit{for}\ \hat{\hat{\sigma}}_{12}^{(1)}. \end{cases} \end{aligned}$$

Proof: Let us put $Pr.\{\hat{\hat{\Sigma}}^{(1)}<u\}=Pr.\{(N_1\hat{\Sigma}^{(1)}+N_2\hat{\Sigma}^{(2)})/(N_1+N_2)<u, D_1\}+Pr.\{\hat{\Sigma}^{(1)}<u, D_2\}$ where D_1 and D_2 are the domains in sample space O_{N_1} and O_{N_2} defined by the above relations (a) and (b). Then the theorem is easily shown from the fact that the joint distribution function of $A^{(1)}$ and $A^{(2)}$ is denoted by the following Wishart distribution

$$(3.4)\qquad \prod_{i=1}^{2} \frac{|A^{(i)}|^{\frac{Ni-4}{2}} exp\left(-\frac{1}{2}tr\Sigma^{(i)^{-1}}A^{(i)}\right)}{2^{Ni-1}\pi^{1/2}|\Sigma^{(i)}|^{\frac{Ni-1}{2}}\prod_{j=1}^{2}\Gamma[(N_i-j)/2]}\, dA^{(i)},$$

and by using the transformations $r_i=\sqrt{\hat{\sigma}_{11}^{(i)}\hat{\sigma}_{22}^{(i)}}/\hat{\sigma}_{12}^{(i)}$ and afterwards $\lambda^2=\sigma_{11}^{(2)}\sigma_{22}^{(2)}(1-r_2^2)/\sigma_{11}^{(1)}\sigma_{22}^{(1)}(1-r_1^2)$.

Theorem 3.2. *The mean value $E\{\hat{\hat{\Sigma}}^{(1)}\}$ and the mean square deviation M.S.D. $\{\hat{\hat{\Sigma}}^{(1)}\}$ of the estimate $\hat{\hat{\Sigma}}$* [1] *are given by*

$$(3.5)\quad E\{\hat{\hat{\sigma}}_{11}^{(1)}\}=\frac{n_1-1}{n_1}\sigma_{11}^{(1)}+\frac{n_2}{n_1+n_2}\frac{(1-\rho_2^2)^{(n_2-1)/2}}{\sqrt{\pi}\Gamma\left(\frac{n_2-1}{2}\right)}\sum_m\frac{(2\rho_2)^{\nu_2}}{\nu_2!}\Gamma\left(\frac{\nu_2+1}{2}\right)I_{\lambda_\alpha^2}$$

$$\left(\frac{n_2+\nu_2-1}{2}\right)\left\{\frac{2\sigma_{11}^{(2)}(1-\rho_2^2)}{n_2}\Gamma\left(\frac{n_2+\nu_2+1}{2}\right)-\frac{n_1-1}{n_1}\sigma_{11}^{(1)}\Gamma\left(\frac{n_2+\nu_2-1}{2}\right)\right\}, \nu_2\equiv 2m,$$

$$E\{\hat{\hat{\sigma}}_{22}^{(1)}\}=\frac{n_1-1}{n_1}\sigma_{22}^{(1)}-\frac{n_2}{n_1+n_2}\frac{n_1-1}{n_1}\sigma_{22}^{(1)}\frac{(1-\rho_2^2)^{\frac{n_2-1}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n_2-1}{2}\right)}\sum_m\frac{(2\rho_2)^{\nu_2}}{\nu_2!}\Gamma\left(\frac{\nu_2+1}{2}\right)$$

$$\Gamma\left(\frac{n_2+\nu_2-1}{2}\right) I_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2-1}{2}\right)+\frac{2}{n_1+n_2}\sigma_{22}^{(2)}(1-\rho_2^2)\frac{(1-\rho_2^2)^{(n_2-1)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n_2-1}{2}\right)}$$

$$\sum_m \frac{(2\rho_2)^{\nu_2}}{\nu_2!}\Gamma\left(\frac{\nu_2+1}{2}\right)\Gamma\left(\frac{n_2+\nu_2+1}{2}\right) I_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2+1}{2}\right), \quad \nu_2\equiv 2m,$$

$$E\{\hat{\sigma}_{12}^{(1)}\}=\frac{n_1-1}{n_1}\sqrt{\sigma_{11}^{(1)}\sigma_{22}^{(1)}}\rho_1-\frac{n_2}{n_1+n_2}\frac{n_1-1}{n_1}\sqrt{\sigma_{11}^{(1)}\sigma_{22}^{(1)}}\rho_1\frac{(1-\rho_2^2)^{(n_2-1)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n_2-1}{2}\right)}$$

$$\sum_m \frac{(2\rho_2)^{\nu_2}}{\nu_2!}\Gamma\left(\frac{\nu_2+1}{2}\right)\Gamma\left(\frac{n_2+\nu_2-1}{2}\right) I_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2-1}{2}\right)+\frac{2}{n_1+n_2}\sqrt{\sigma_{11}^{(2)}\sigma_{22}^{(2)}}$$

$$\rho_2\frac{(1-\rho_2^2)^{(n_2-1)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n_2-1}{2}\right)}\sum_m \frac{(2\rho_2)^{\nu_3}}{\nu_3!}\Gamma\left(\frac{\nu_3+2}{2}\right)\Gamma\left(\frac{n_2+\nu_3}{2}\right) I_{\lambda_\alpha^2}\left(\frac{n_2+\nu_3}{2}\right),$$

$$\nu_2\equiv 2m,\quad \nu_3\equiv 2m+1,$$

$M.S.D.\{\hat{\sigma}_{11}^{(1)}\}$

$$=\frac{1}{n_1}\left(2-\frac{1}{n_1}\right){\sigma_{11}^{(1)}}^2+\frac{1}{(n_1+n_2)^2}\frac{(1-\rho^2)^{(n_2-1)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n_2-1}{2}\right)}\sum_m \frac{(2\rho_2)^{\nu_2}}{\nu_2!}\Gamma\left(\frac{\nu_2+1}{2}\right)$$

$$\left\{\frac{n_1n_2-2n_1-n_2}{n_1^2}\sigma_{11}^{(1)2}\Gamma\left(\frac{n_2+\nu_2-1}{2}\right)-4(n_2+1)(1-\rho_2^2)\sigma_{11}^{(1)}\sigma_{11}^{(2)}\Gamma\left(\frac{n_2+\nu_2+1}{2}\right)\right.$$

$$\left.+4(1-\rho_2^2)^2{\sigma_{11}^{(2)}}^2\Gamma\left(\frac{n_2+\nu_2+3}{2}\right)\right\} I_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2-1}{2}\right), \quad \nu_2\equiv 2m,$$

$M.S.D.\{\hat{\sigma}_{22}^{(1)}\}$

$$=\frac{1}{n_1}\left(2-\frac{1}{n_1}\right)\left({\sigma_{22}^{(1)}}^2+\frac{1}{(n_1+n_2)^2}\frac{(1-\rho_2^2)^{(n_2-1)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n_2-1}{2}\right)}\sum_m \frac{(2\rho_2)^{\nu_2}}{\nu_2!}\Gamma\left(\frac{\nu_2+1}{2}\right)\right.$$

$$\left\{\frac{n_1n_2-2n_1-n_2}{n_1^2}{\sigma_{22}^{(1)}}^2\Gamma_{\lambda_\alpha^2}\left(\frac{n_2+n_2-1}{2}\right)-4(n_2+1)(1-\rho_2^2)\sigma_{22}^{(1)}\sigma_{22}^{(2)}\Gamma_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2+1}{2}\right)\right.$$

$$\left.+4(1-\rho_2^2)^2{\sigma_{22}^{(1)}}^2\Gamma_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2+3}{2}\right)\right\}, \quad \nu_2=2m,$$

and

$M.S.D.\{\hat{\sigma}_{12}^{(1)}\}$

$$=\frac{n_1-1+n_2\rho_1^2}{n_1^2}\sigma_{11}^{(1)}\sigma_{22}^{(1)}+\frac{1}{(n_1+n_2)^2}\frac{(1-\rho_2^2)^{(n_2-1)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n_2-1}{2}\right)}\sum_m \frac{(2\rho_2)^{\nu_2}}{\nu_2!}$$

$$\left\{\frac{n_2(n_1-1)(n_1n_2\rho_1^2-2n_1-n_2)}{n_1^2}\sigma_{11}^{(1)}\sigma_{22}^{(1)}\Gamma\left(\frac{\nu_2+1}{2}\right)\Gamma_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2-1}{2}\right)\right.$$

$$-4(n_2+1)\rho_1(1-\rho^2)\sqrt{\sigma_{11}^{(1)}\sigma_{11}^{(2)}\sigma_{22}^{(1)}\sigma_{22}^{(2)}}\ \Gamma\left(\frac{\nu_3+2}{2}\right)\Gamma_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2}{2}\right)$$
$$+4(1-\rho_1^2)\sigma_{11}^{(2)}\sigma_{22}^{(2)}\Gamma\left(\frac{\nu_2+3}{2}\right)\Gamma_{\lambda_\alpha^2}\left(\frac{n_2+\nu_2+1}{2}\right)\Big\},\ \nu_2\equiv 2m,\ \nu_3\equiv 2m+1.$$

Proof: The mean valus $E\{\hat{\Sigma}^{(1)}\}$ is given by

$$(3.6)\quad \iint_{D_1}\frac{N_1\hat{\Sigma}^{(1)}+N_2\hat{\Sigma}^{(2)}}{N_1+N_2}f(\sigma_{11}^{(1)},\hat{\sigma}_{22}^{(1)},\hat{\sigma}_{11}^{(2)},r_1,r_2,\lambda)d\hat{\sigma}_{11}^{(1)}d\hat{\sigma}_{22}^{(1)}d\hat{\sigma}_{11}^{(2)}dr_1dr_2d\lambda$$
$$+\iint_{D_2}\hat{\Sigma}^{(1)}f(\hat{\sigma}_{11}^{(1)},\hat{\sigma}_{22}^{(1)},\hat{\sigma}_{11}^{(2)},r_1,r_2,\lambda)d\hat{\sigma}_{11}^{(1)}d\sigma_{22}^{(1)}d\hat{\sigma}_{11}^{(2)}dr_1dr_2d\lambda$$

and the mean square obtained from

$$(3.7)\quad M.S.D.\{\hat{\Sigma}^{(1)}\}=(E\{\hat{\sigma}_{ij}^{(1)2}\})-2(\hat{\sigma}_{ij}^{(1)}E\{\hat{\sigma}_{ij}^{(1)}\})+(\sigma_{ij}^{(1)2}),\ (i,j=1,2),$$

where $E\{\hat{\sigma}_{ij}^{(1)2}\}$ is given by

$$(3.8)\quad \iint_{D_1}\left(\frac{N_1\hat{\sigma}_{ij}^{(1)}+N_2\hat{\sigma}_{ij}^{(2)}}{N_1+N_2}\right)^2 f(\hat{\sigma}_{11}^{(1)},\hat{\sigma}_{22}^{(1)},\hat{\sigma}_{11}^{(2)},r_1,r_2,\lambda)d\hat{\sigma}_{11}^{(1)}d\hat{\sigma}_{22}^{(1)}d\hat{\sigma}_{11}^{(2)}dr_1dr_2d\lambda$$
$$+\iint_{D_2}\hat{\sigma}_{ij}^{(1)}f(\hat{\sigma}_{11}^{(1)},\hat{\sigma}_{22}^{(1)},\hat{\sigma}_{11}^{(2)},r_1,r_2,\lambda)d\hat{\sigma}_{11}^{(1)}d\hat{\sigma}_{22}^{(1)}d\hat{\sigma}_{11}^{(2)}dr_1dr_2d\lambda.$$

Integrating out $\hat{\sigma}_{11}^{(1)}$, $\hat{\sigma}_{22}^{(1)}$, $\hat{\sigma}_{11}^{(2)}$, r_1, r_2 and λ, we obtain the results (3.5) simplified by the following formulae:

$$\frac{(1-\rho^2)^{(N-1)/2}}{\sqrt{\pi}\Gamma\left(\frac{N-1}{2}\right)}\sum_{m=0}^{\infty}\frac{(2\rho)^\nu}{\nu!}\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{N+\nu-1}{2}\right)=1 \text{ for } \nu=2m,$$

$$\frac{(1-\rho^2)^{(N-1)/2}}{\sqrt{\pi}\Gamma\left(\frac{N-1}{2}\right)}\sum_{m=0}^{\infty}\frac{(2\rho)^\nu}{\nu!}\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{N+\nu}{2}\right)=\frac{N-1}{2}\frac{\rho}{1-\rho^2} \text{ for } \nu=2m+1,$$

$$(3.9)\quad \frac{(1-\rho^2)^{(N-1)/2}}{\sqrt{\pi}\Gamma\left(\frac{N-1}{2}\right)}\sum_{m}\frac{(2\rho)^\nu}{\nu!}\nu\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{N+\nu-1}{2}\right)=(N-1)\frac{\rho^2}{1-\rho^2} \text{ for } \nu=2m,$$

$$\frac{(1-\rho^2)^{(N-1)/2}}{\sqrt{\pi}\Gamma\left(\frac{N-1}{2}\right)}\sum_{m}\frac{(2\rho)^\nu}{\nu!}\nu\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{N+\nu}{2}\right)=\frac{(N-1)\rho}{2(1-\rho^2)}\frac{1+n\rho^2}{1-\rho^2} \text{ for } \nu=2m+1,$$

$$\frac{(1-\rho^2)^{(N-1)/2}}{\sqrt{\pi}\Gamma\left(\frac{N-1}{2}\right)}\sum_{m}\frac{(2\rho)^\nu}{\nu!}\nu^2\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{N+\nu-1}{2}\right)=\frac{(N-1)\rho^2\{(n-1)\rho^2+2\}}{(1-\rho^2)^2}$$
$$\text{for } \nu=2m.$$

Corollary 3.1. *Let us consider in particular an inference procedure defined by $\lambda_\alpha=0$, that is to say, this corresponds to the case when we never use poling of data. We have then the following results,*

$$(3.10)\quad E\{\hat{\sigma}_{11}^{(1)}\}=(N_1-1)\sigma_{11}^{(1)}/N_1,\qquad M.S.D.\{\hat{\sigma}_{11}^{(1)}\}=(2-1/N_1)\sigma_{11}^{(1)^2}/N_1,$$
$$E\{\hat{\sigma}_{22}^{(1)}\}=(N_1-1)\sigma_{22}^{(1)}/N_1,\qquad M.S.D.\{\hat{\sigma}_{22}^{(1)}\}=(2-1/N_1)\sigma_{22}^{(1)^2}/N_1,$$
$$E\{\hat{\sigma}_{12}^{(1)}\}=(N_1-1)\sqrt{\sigma_{11}^{(1)}\sigma_{22}^{(1)}}\rho_1/N_1,\ M.S.D.\{\hat{\sigma}_{12}^{(1)}\}=(N_1-1+N_1\rho_1^2)\sigma_{11}^{(1)}\sigma_{22}^{(1)}/N_1^2.$$

Otherwise if we apply an inference procedure of always pooling defined by $\lambda_\alpha=\infty$, *the results are given by*

$$(3.11)\quad E\{\hat{\sigma}_{11}^{(1)}\}=\{(N_1-1)\sigma_{11}^{(1)}+(N_2-1)\sigma_{11}^{(2)}\}/(N_1+N_2),\ M.S.D.\{\hat{\sigma}_{11}^{(1)}\}$$
$$=\frac{(N_1^2-1)\sigma_{11}^{(1)^2}+2(N_1-1)(N_2-1)\sigma_{11}^{(1)}\sigma_{11}^{(2)}+(N_2^2-1)\sigma_{11}^{(2)^2}}{(N_1+N_2)^2}$$
$$+\frac{2\sigma_{11}^{(1)}\{(N_1-1)\sigma_{11}^{(1)}+(N_2-1)\sigma_{11}^{(2)}\}}{N_1+N_2}+\sigma_{11}^{(1)^2},$$

$$E\{\hat{\sigma}_{22}^{(1)}\}=\{(N_1-1)\sigma_{22}^{(1)}+(N_2-1)\sigma_{22}^{(2)}\}/(N_1+N_2),\ M.S.D.\{\hat{\sigma}_{22}^{(1)}\}$$
$$=\frac{(N_1^2-1)\sigma_{22}^{(1)^2}+2(N_1-1)(N_2-1)\sigma_{22}^{(1)}\sigma_{22}^{(2)}+(N_2^2-1)\sigma_{22}^{(2)^2}}{(N_1+N_2)^2}$$
$$+\frac{2\sigma_{22}^{(1)}\{(N_1-1)\sigma_{22}^{(1)}+(N_2-1)\sigma_{11}^{(2)}\}}{N_1+N_2}+\sigma_{22}^{(2)^2},$$
$$E\{\sigma_{12}^{(1)}\}=\{(N_1-1)\sqrt{\sigma_{11}^{(1)}\sigma_{22}^{(1)}}\rho_1+(N_2-1)\sqrt{\sigma_{11}^{(2)}\sigma_{22}^{(2)}}\rho_2\}/(N_1+N_2),$$

$$M.S.D.\{\hat{\sigma}_{12}^{(1)}\}=\frac{1}{(N_1+N_2)^2}\{(N_1-1)(1+N_1\rho_1^2)\sigma_{11}^{(1)}\sigma_{22}^{(1)}+2(N_1-1)(N_2-1)\rho_1\rho_2$$
$$\sqrt{\sigma_{11}^{(1)}\sigma_{22}^{(1)}\sigma_{22}^{(2)}\sigma_{22}^{(2)}}+(N_2-1)(1+N_2\rho_2^2)\sigma_{11}^{(2)}\sigma_{22}^{(2)}\}$$
$$-\frac{2\sqrt{\sigma_{11}^{(1)}\sigma_{22}^{(1)}}\rho_1}{N_1+N_2}\left\{(N_1-1)\sqrt{\sigma_{11}^{(1)}\sigma_{22}^{(1)}}\rho_1+(N_2-1)\sqrt{\sigma_{11}^{(2)}\sigma_{22}^{(2)}}\rho_2\right\}+\sigma_{11}^{(1)}\sigma_{22}^{(1)}\rho_1^2.$$

Corollary 3.2. *When we define that* $\hat{\Sigma}^{(i)}=A^{(i)}/(N_i-1)$ *in our procedure,* $\hat{\Sigma}^{(1)}$ *gives us the unbiassed estimate in case that* $\rho_1=\rho_2=0$, $N_1=N_2$ *and* $\Sigma^{(1)}=\Sigma^{(2)}$ *regardless of a value of* λ_α *and the mean square deviation of* $\hat{\Sigma}^{(i)}$ *is greater than that defined by always pooling procedure and is less than that defined by never-pooling procedure, independent of the assumption* $N_1=N_2$.

Corollary 3.3. *In case when we assume that* $\rho_i=0$ *and* $\Sigma^{(1)}=\Sigma^{(2)}$ *the present inference procedure shows us the smaller mean square error of the estimate* $\hat{\sigma}_{ii}^{(1)^2}$ *regardless of* N_1 *and* N_2 *than that defined by the pooling procedure of Bancroft* (1) *in univariate case,* $(i=1,2)$.

§ 4. Pooling of sample generalized variances.

Let O_{N_1}: $(\boldsymbol{x}_1^{(1)},\boldsymbol{x}_2^{(1)},\cdots,\boldsymbol{x}_{N_1}^{(1)})$ be a random sample of N_1 vector observations from a bivariate normal population $N[\boldsymbol{\mu}^{(1)},\Sigma^{(1)}]$, and let O_{N_2}: $(\boldsymbol{x}_1^{(2)},\boldsymbol{x}_2^{(2)},$

$\cdots, \boldsymbol{x}_{N_2}^{(2)})$ be another random sample of N_2 from some bivariate normal population $N[\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}^{(2)}]$. The population generalized variances $|\boldsymbol{\Sigma}^{(1)}|$ and $|\boldsymbol{\Sigma}^{(2)}|$ may or may not be distinct. Here let us suppose that we may pool the square roots of the two sample generalized variances $|\boldsymbol{S}^{(1)}|^{1/2}$ and $|\boldsymbol{S}^{(2)}|^{1/2}$ and form an estimate of the square root of the same population generalized variance in case when testing of the hypothesis that $|\boldsymbol{\Sigma}^{(1)}|=|\boldsymbol{\Sigma}^{(2)}|$ for the alternative $|\boldsymbol{\Sigma}^{(1)}|>|\boldsymbol{\Sigma}^{(2)}|$ leads us to the decision that the hypothesis $|\boldsymbol{\Sigma}^{(1)}|=|\boldsymbol{\Sigma}^{(2)}|$ cannot be rejected.

Our rule of inference procedure is as follows:

(i) Let $\boldsymbol{S}^{(i)}$ be sample variance-covariance matrix defined by

$$\boldsymbol{S}^{(i)}=\frac{1}{N_i-1}\sum_{j=1}^{N_i}(\boldsymbol{x}_j^{(i)}-\bar{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)}-\bar{\boldsymbol{x}}^{(i)})',$$

where $\bar{\boldsymbol{x}}^{(i)}=\sum_{j=1}^{N_i}\boldsymbol{x}_j^{(i)}/N_i$ and let $\boldsymbol{A}^{(i)}$ be defined by $A^{(i)}=(N_i-1)S^{(i)}$.

(ii) Now let us introduce the estimate $|\hat{\boldsymbol{\Sigma}}^{(1)}|^{1/2}$ of $|\boldsymbol{\Sigma}^{(1)}|^{1/2}$ defined in the following way:

(a) $|\hat{\boldsymbol{\Sigma}}^{(1)}|^{1/2}=(|\boldsymbol{A}^{(1)}|^{1/2}+|\boldsymbol{A}^{(2)}|^{1/2})/(N_1+N_2-4)$, if $\lambda=|\boldsymbol{A}^{(1)}|^{1/2}/|\boldsymbol{A}^{(2)}|^{1/2}\leq\lambda_\alpha$,

(b) $|\hat{\boldsymbol{\Sigma}}^{(1)}|^{1/2}=|\boldsymbol{A}^{(1)}|^{1/2}/(N_1-2)$, if $\lambda=|\boldsymbol{A}^{(1)}|^{1/2}/|\boldsymbol{A}^{(2)}|^{1/2}>\lambda_\alpha$,

where the non-negative switching constant prescribed λ_α denotes a value of α-percent point of F-distribution with the pair $(2N_1-4,\ 2N_2-4)$ of degrees of freedom.

Theorem 4.1. *The probability distribution function of the statistic* $|\hat{\boldsymbol{\Sigma}}^{(1)}|^{1/2}$ *is given by*

$$(4.1)\qquad Pr.\{|\hat{\boldsymbol{\Sigma}}^{(1)}|^{1/2}<x\}=\int_0^{\Delta} b_{r_1,r_2}(t)\, I\left(\frac{(r_1+r_2)x}{|\boldsymbol{\Sigma}^{(1)}|^{1/2}t+|\boldsymbol{\Sigma}^{(2)}|^{1/2}(1-t)},r_1+r_2\right)dt$$
$$+\int_{\Delta}^{1} b_{r_1,r_2}(t)\, I\left(\frac{r_1x}{|\boldsymbol{\Sigma}^{(1)}|^{1/2}t},\ r_1+r_2\right)dt,$$

where we put

$$b_{r_1,r_2}(t)=\frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)}t^{r_1-1}(1-t)^{r_2-1},\ I(u,r)=\frac{1}{\Gamma(r)}\int_0^u v^{r-1}e^{-v}dv,$$

$$(4.2)\qquad \Delta=\frac{|\boldsymbol{\Sigma}^{(2)}|^{1/2}\lambda_\alpha}{|\boldsymbol{\Sigma}^{(1)}|^{1/2}+|\boldsymbol{\Sigma}^{(2)}|^{1/2}\lambda_\alpha},\ r_i=(N_i-2)/2\ for\ i=1,2.$$

Proof: The joint elementary probability of $|\boldsymbol{A}^{(1)}|^{1/2}$ and $|\boldsymbol{A}^{(2)}|^{1/2}$ is given by

$$(4.3)\qquad \prod_{i=1}^{2}\frac{(H_i^2)^{N_i-3}exp[-H_i^2/2]}{2^{N_i-2}\Gamma(N_i-2)}\ dH_i^2,\ (0<H_i^2<\infty),$$

where $H_i^2 = 2|A^{(i)}|^{1/2}/|\Sigma^{(i)}|^{1/2}$.

Here adopting the transformations of $H_1^2 = R^2 \cos^2\theta$ and $H_2^2 = R^2 \sin^2\theta$ and afterwards $t = \cos^2\theta$ and $V = R^2/2$, (4.3) becomes

$$(4.4)\quad \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)} t^{r_1-1}(1-t)^{r_2-1}dt \frac{V^{r_1+r_2-1}}{\Gamma(r_1+r_2)} \exp[V]dV,\ (0<t<1,\ 0<V<\infty).$$

Further decomposing the probability distribution function into two parts defined by

$$(4.5)\quad Pr.\{|\hat{\Sigma}^{(1)}|^{1/2}<x\} = Pr.\{|\hat{\Sigma}^{(1)}|^{1/2}<x, \lambda\le\lambda_\alpha\} + Pr.\{|\hat{\Sigma}^{(1)}|^{1/2}<x, \lambda>\lambda_\alpha\}$$
$$= Pr.\{[|\Sigma^{(1)}|^{1/2}t + |\Sigma^{(2)}|^{1/2}(1-t)]V/(r_1+r_2)<x,\ 0<t\le|\Sigma^{(2)}|^{1/2}\lambda_\alpha/(|\Sigma^{(1)}|^{1/2}+|\Sigma^{(2)}|^{1/2}\lambda_\alpha)\}$$
$$+ Pr.\{|\Sigma^{(1)}|^{1/2}tV/r_1<x,\ |\Sigma^{(2)}|^{1/2}\lambda_\alpha/(|\Sigma^{(1)}|^{1/2}+|\Sigma^{(2)}|^{1/2}\lambda_\alpha)<t<1\},$$

we may obtain (4.1) to be proved.

Theorem 4.2. *The mean value $E\{|\hat{\Sigma}^{(1)}|^{1/2}\}$ of the estimate $|\hat{\Sigma}^{(1)}|^{1/2}$ is given by*

$$(4.6)\quad E\{|\hat{\Sigma}^{(1)}|^{1/2}\} = \left[1+\frac{N_2-2}{N_1+N_2-4}\left\{I_{x_0}(N_1-2, N_2-1)\frac{|\Sigma^{(2)}|^{1/2}}{|\Sigma^{(1)}|^{1/2}} - I_{x_0}(N_1-1, N_2-2)\right\}\right] \cdot |\Sigma^{(1)}|^{1/2},$$

where we put that $x_0 = \frac{|\Sigma^{(2)}|^{1/2}}{|\Sigma^{(1)}|^{1/2}}\lambda_\alpha \Big/ \left(1+\frac{|\Sigma^{(2)}|^{1/2}}{|\Sigma^{(1)}|^{1/2}}\lambda_\alpha\right)$ *and* $I_{x_0}(a,b) = \int_0^{x_0} x^{a-1}(1-x)^{b-1} dx/B(a,b)$.

Proof: Since $2|A^{(i)}|^{1/2}/|\Sigma^{(i)}|^{1/2}$ distribute mutually independently as χ^2-distribution with $2N_2-4$ degrees of freedom, $(i=1,2)$, the joint distribution of $|A^{(1)}|^{1/2}$ and $|A^{(2)}|^{1/2}$ is given by

$$(4.7)\qquad C\cdot \prod_{i=1}^{2} |A^{(i)}|^{(N_i-3)/2} \exp\{-|A^{(i)}|^{1/2}/|\Sigma^{(i)}|^{1/2}\} d|A^{(i)}|^{1/2},$$

where $C = \prod_{i=1}^{2}\{\Gamma(N_i-2)|\Sigma^{(i)}|^{(N_i-2)/2}\}^{-1}$. Then integrating out Q_1 after making use the transformations, $Q_1 = |A^{(1)}|^{1/2}+|A^{(2)}|^{1/2}$ and $Q_2 = |A^{(1)}|^{1/2}/|A^{(2)}|^{1/2}$, the expected value E_1 of $Q_1/(N_1+N_2-4)$ under $Q_2\le\lambda_2$ is given by

$$(4.8)\quad E_1 = \frac{1}{(N_1+N_2-4)Pr.\{Q_2\le\lambda_\alpha\}}\{(N_1-2)I_{x_0}(N_1-1, N_2-2)|\Sigma^{(1)}|^{1/2}$$
$$+ (N_2-2)I_{x_0}(N_1-2, N_2-1)|\Sigma^{(2)}|^{1/2}\},$$

where $x_0 = |\Sigma^{(2)}|^{1/2}\lambda_\alpha/\{|\Sigma^{(1)}|^{1/2}+\lambda_\alpha|\Sigma^{(2)}|^{1/2}\}$.

While the expected value E_2 of $|A^{(1)}|^{1/2}/(N_1-2)$ under $Q_2>\lambda_\alpha$ is similarly as follows:

$$(4.9)\qquad E_2 = \{1-I_{x_0}(N_1-1, N_2-2)\}|\Sigma^{(1)}|^{1/2}/Pr.\{Q_2>\lambda_\alpha\}.$$

Since $Pr.\{Q_2\le\lambda_\alpha\}$ and $Pr.\{Q_2>\lambda_\alpha\}$ are easily given by

(4.10) $\{(N_1-2)I_{x_0}(N_1-1,N_2-2)|\Sigma^{(1)}|^{1/2}+(N_2-2)I_{x_0}(N_1-2,N_2-1)|\Sigma^{(2)}|^{1/2}\}/(N_1+N_2-4)$ and $\{1-I_{x_0}(N_1-1,N_2-2)\}\cdot|\Sigma^{(2)}|^{1/2}$ respectively, combining (4.8) and (4.9) we obtain (4.6) to be proved.

Theorem 4.3. *The mean square deviation* $M.S.D.\{\widehat{|\Sigma^{(1)}|^{1/2}}\}$ *of the estimate* $\widehat{|\Sigma^{(1)}|^{1/2}}$ *is given by*

(4.11) $M.S.D.\{\widehat{|\Sigma^{(1)}|^{1/2}}\}$

$$
\begin{aligned}
&=\frac{1}{(N_1+N_2-4)^2}\{(N_1-1)(N_1-2)I_{x_0}(N_1,N_2-2)\cdot|\Sigma^{(1)}| \\
&\quad+2(N_1-2)(N_2-2)|\Sigma^{(1)}|^{1/2}|\Sigma^{(2)}|^{1/2}I_{x_0}(N_1-1,N_2-1) \\
&\quad+(N_2-1)(N_2-2)I_{x_0}(N_1-2,N_2)|\Sigma^{(2)}|\}+\frac{N_1-1}{N_1-2}\{1-I_{x_0}(N_1,N_2-2)\}\cdot|\Sigma^{(1)}| \\
&\quad-\left[1+\frac{2(N_2-2)}{N_1+N_2-4}\left\{I_{x_0}(N_1-2,N_2-1)\frac{|\Sigma^{(2)}|^{1/2}}{|\Sigma^{(1)}|^{1/2}}-I_{x_0}(N_1-1,N_2-2)\right\}\right]|\Sigma^{(1)}|.
\end{aligned}
$$

Proof : Using the same method as theorem 2.2., we may prove this theorem.

Corollary 4.1. *Specially, an inference procedure defined by the "never popling", that is,* $\lambda_\alpha=0$, *shows us that* $E\{\widehat{|\Sigma^{(1)}|^{1/2}}\}=|\Sigma^{(1)}|^{1/2}$, $M.S.D.\{\widehat{|\Sigma^{(1)}|^{1/2}}\}=|\Sigma^{(1)}|/(N_1-2)$. *An inference procedure defined by the "always pooling", that is,* $\lambda_\alpha=\infty$, *shows us that*

$$
E\{\widehat{|\Sigma^{(1)}|^{1/2}}\}=\frac{N_1-2}{N_1+N_2-4}|\Sigma^{(1)}|^{1/2}+\frac{N_2-2}{N_1+N_2-4}|\Sigma^{(2)}|^{1/2},
$$

$$
\begin{aligned}
M.S.D.\{\widehat{|\Sigma^{(1)}|^{1/2}}\}&=\frac{1}{(N_1+N_2-4)^2}[(N_1-2)^2+(N_2-2)^2\}|\Sigma^{(1)}| \\
&\quad-2(N_2-2)^2|\Sigma^{(1)}|^{1/2}|\Sigma^{(2)}|^{1/2}+(N_2-1)(N_2-2)|\Sigma^{(2)}|].
\end{aligned}
$$

Corollary 4.2. *In case when the dispersion matrices of two bivariate normal populations are assumed that* $\rho^{(i)}=0$ *and* $\sigma_{11}^{(i)}=\sigma_{22}^{(i)}=\sigma$, *the present inference procedure shows us the smaller variance of the estimate* $\hat{\sigma}^2$ *than that defined by the pooling rule of Bancroft* [1] *in univariate case.*

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