On the homogeneous birth and death process with an absorbing barrier

Yamamoto, Sumiyasu
Nara Medical College

https://doi.org/10.5109/13000

出版情報：統計数理研究. 10 (1/2), pp.45-56, 1961-10. Research Association of Statistical Sciences

ON THE HOMOGENEOUS BIRTH AND DEATH PROCESS WITH AN ABSORBING BARRIER

By

Sumiyasu YAMAMOTO

(Nara Medical College)

Introduction

It sometimes or even more frequently happens in microscopic or molecular order biological experimentation that the specification of the response curve, which describes the relation between the intensity of the stimulus and the rate of the response of the subject under consideration, from hit-theoretical point of view when the response is quantal. In such a situation, the stimulation in progress should be considered as a process of hitting a target or a number of targets of the subject by some quantic and stochastic agents. Since there exists the process of recuperation in the subject from the effects of hit during the stimuiation is continued, the process which describes the fluctuation of the actual number of effective hits can legitimately be considered as a birth and death process. The occurrence of the response which is closely related to the process can be formulated as follows: There exists a fixed threshold number k inherent to the subject and once the process which describes the actual number of effective hits arrives at the threshold number k during the duration of the stimulus, the response occurs and not otherwise. The problem of specifying the response curve, therefore, turns out to that of solving the probability of absorption or first passage to the state k with respect to the birth and death process with an absorbing barrier.

In this paper, the general solution of the probability of absorption or the distribution of the first passage time will be given when the underlying stochastic process can be considered as the general homogeneous birth and death process. The asymptotic behavior of the probability of absorption will also be given when the intensity of the stimulus is adequately controlled.

1. The general homogeneous birth and death process with an absorbing barrier

Let $p_n(t)$ be the probability that at time t the process is in the state E_n $(n=0, 1, \cdots, k)$ and let the transition probabilities from the state E_n to the states E_{n+1} and E_{n-1} within infinitesimal time interval $(t, t+\Delta t)$ be $\lambda_n \Delta t$

$+o(\Delta t)$ and $\mu_n\Delta t+o(\Delta t)$ respectively, while those to other states except E_n, E_{n+1} and E_{n-1} be $o(\Delta t)$. The coefficient $\lambda_n(>0)$ is the probability of hit within unit duration of time when the process is in the state E_n and $\mu_n(>0)$ is that of a recuperation occurs out of n hits. Since E_0 is the ground state so that $\mu_0=0$ and E_k is an absorbing barrier so that $\lambda_k=\mu_k=0$.

The basic system of differential equations which characterize the process is

$$(1)\qquad \begin{aligned}
\frac{\partial p_0(t)}{\partial t} &= -\lambda_0 p_0(t)+\mu_1 p_1(t)\\
\frac{\partial p_n(t)}{\partial t} &= \lambda_{n-1}p_{n-1}(t)-(\lambda_n+\mu_n)p_n(t)+\mu_{n+1}p_{n+1}(t)\quad (n=1,2,\cdots,k-2)\\
\frac{\partial p_{k-1}(t)}{\partial t} &= \lambda_{k-2}p_{k-2}(t)-(\lambda_{k-1}+\mu_{k-1})p_{k-1}(t)\\
\frac{\partial p_k(t)}{\partial t} &= \lambda_{k-1}p_{k-1}(t)
\end{aligned}$$

or in matric notations

$$(1')\qquad \frac{\partial \boldsymbol{p}(t)}{\partial t}=R\boldsymbol{p}(t)$$

where $\boldsymbol{p}(t)=\begin{pmatrix} p_0(t)\\ p_1(t)\\ p_2(t)\\ \cdot\\ \cdot\\ p_{k-1}(t)\\ p_k(t)\end{pmatrix}$ and $R=\begin{pmatrix} -\lambda_0 & \mu_1 & & & & \\ \lambda_0 & -(\lambda_1+\mu_1) & \mu_2 & & \mathbf{0} & \\ & \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & \mathbf{0} & & \lambda_{k-2} & -(\lambda_{k-1}+\mu_{k-1}) & 0\\ & & & & \lambda_{k-1} & 0\end{pmatrix}$.

Let us examine the latent roots of the matrix R. The characteristic polynomial $\varphi_{k+1}(x)$ of R is

$$(2)\qquad \varphi_{k+1}(x)=|xE_{k+1}-R|=xf_k(x)\,,$$

where E_{k+1} is the unit matrix of degree $k+1$ and $f_k(x)$ is the characteristic polynomial of the matrix A_k:

$$(3)\qquad A_k=\begin{pmatrix} -\lambda_0 & \mu_1 & & & & \\ \lambda_0 & -(\lambda_1+\mu_1) & \mu_2 & & \mathbf{0} & \\ & \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & \mathbf{0} & & \lambda_{k-3} & -(\lambda_{k-2}+\mu_{k-2}) & \mu_{k-1}\\ & & & & \lambda_{k-2} & -(\lambda_{k-1}+\mu_{k-1})\end{pmatrix}$$

It is easy to see that $x=0\equiv x_k$ is a latent root of the matrix R and the remaining k latent roots are the roots of the equation $f_k(x)=0$.

As to the latent roots of the matrix (3), the following theorem holds.

Theorem I. *All latent roots x_i of the matrix A_k are simple, real and satisfying the inequality*

$$(4)\qquad -2\max_{0\leq j\leq k-1}(\lambda_j+\mu_j)\leq x_i<0\quad (i=0,1,\cdots,k-1;\ \mu_0\equiv 0).$$

In order to prove this theorem[1], we shall first prove the following three lemmas.

Lemma 1.[2] *All latent roots of a real matrix $A_n(n\times n)=(\!(a_{ij})\!)$ $(i, j=1, 2, \cdots, n)$ satisfying the following two conditions are real and simple:*

(*i*) $a_{ij}=0$ *when* $|i-j|\geq 2$,

(*ii*) $a_{jj-1}a_{j-1j}>0$ *for all* $j=2,3,\cdots,n$.

Moreover, these latent roots are separated by $n-1$ latent roots of the contracted matrix A_{n-1} $(\overline{n-1}\times\overline{n-1})=(\!(a_{ij})\!)$ $(i,j=1,2,\cdots,n-1)$.

Proof: Let A_h be the contracted square matrix consist of the first h rows and h columns of the matrix A_n and $f_h(x)=|xE_h-A_h|$ be its characteristic polynomial. The expansion of the determinant shows that the following relations hold:

$$(5)\qquad f_h(x)=(x-a_{hh})f_{h-1}(x)-a_{hh-1}a_{h-1h}f_{h-2}(x),\ (h=2,3,\cdots,n).$$

Using the condition (ii) and the above relations we may verify that the sequence of polynomials

$$(6)\qquad f_n(x),\ f_{n-1}(x),\cdots,f_1(x)\ \text{and}\ f_0(x)\equiv 1$$

forms Strum's series in the extended sense (Takagi (1937)). Since the highest order term of the polynomials $f_h(x)$ is x^h, the number of changes of sign in the series at ∞, $V(\infty)$, is zero and those at $-\infty$, $V(-\infty)$, is n. Hence the roots of $f_n(x)=0$, the latent roots of A_n, are all real and simple. The roots of $f_{n-1}(x)=0$, furthermore, separate each other the roots of $f_n(x)=0$.

Lemma 2. (Bartlett (1955), pp 52–3) *For any latent root x of a real matrix A $(n\times n)=(\!(a_{ij})\!)$ satisfying the conditions $a_{ij}\geq 0 (i\neq j)$ and $\sum_k a_{kj}=0$*

---

1) Professor T. Kitagawa of Kyushu University has kindly pointed out to the author the proofs of the simplicity of the latent roots of the matrix A_n and a modified one which have given by Ledermann and Reuter (1954) after the preparation of this paper. The proof given here, however, is more simple and straightforward one compared with that cited above.

2) cf. Yamamoto, *Sûgaku*, 11, (1959), pp 14–5. (in Japanese)

for all i and j, there exists a certain diagonal element a_{mm} for which the following inequality holds:

$$(7)\qquad |a_{mm}-x| \leq |a_{mm}|$$

where $|c|$ *denotes the absolute value of any quantity c.*

Lemma 3. *We have the following identity*:

$$(8)\qquad A_n=\begin{vmatrix} a_1+b_1 & -b_2 & & & & \\ -a_1 & a_2+b_2 & -b_3 & & \mathbf{0} & \\ & -a_2 & a_3+b_3 & -b_4 & & \\ & & \cdot & \cdot & \cdot & \\ & \mathbf{0} & & -a_{n-2} & a_{n-1}+b_{n-1} & -b_n \\ & & & & -a_{n-1} & a_n+b_n \end{vmatrix}$$

$$=a_1a_2\cdots\cdots a_n+b_1a_2\cdots\cdots a_n+b_1b_2a_3\cdots\cdots a_n+\cdots$$
$$+b_1b_2\cdots\cdots b_{n-1}a_n+b_1b_2\cdots\cdots b_{n-1}b_n.$$

Proof: Since $A_n=(a_n+b_n)A_{n-1}-a_{n-1}b_nA_{n-2}$, and $A_1=a_1+b_1$, $A_2=a_1a_2+b_1a_2+b_1b_2$, it is easy to prove the identity by induction.

Proof of the Theorem I: Since $\lambda_0, \lambda_1, \cdots, \lambda_{k-2}$ and $\mu_1, \mu_2, \cdots, \mu_{k-1}$ are positive so that $\lambda_h\mu_{h+1}>0$ $(h=0, 1, \cdots, k-2)$, it can be seen by Lemma 1 that all latent roots of the matrix (3) are real and simple. Moreover, from Lemma 3 we have $f_k(0)=\lambda_0\lambda_1\cdots\lambda_{k-1}>0$ and $f_0(0)\equiv 1>0$, and a slight modification of the proof of Lemma 1 that $V(0)=0$ instead of $V(\infty)=0$ shows that all latent roots are negative. The results will lead us to the required inequality as an immediate consequence of the Lemma 2.

It follows from this theorem that all roots of the characteristic polynomial $\varphi_{k+1}(x)$ are real as well as simple because k roots of $\varphi_{k+1}(x)=0$ are also simple negative roots of $f_k(x)=0$ and an extra root is $x_k=0$. We shall denote these latent roots of R in the ascending order of magnitudes as follows:

$$(9)\qquad x_0<x_1<x_2<\cdots<x_{k-1}<x_k=0.$$

Let s_j and t_j' be normalized column and row latent vectors of the matrix R corresponding respectively to the latent roots x_j, i.e.,

$$(10)\qquad Rs_j=x_js_j,\quad t_j'R=x_jt_j' \text{ and } t_i's_j=\delta_{ij},$$

then we have the spectral resolution of R

$$(11)\qquad R=\sum_{j=0}^{k} x_js_jt_j'$$

say, where $x_k=0$ but is included to complete the set of latent vectors, so that $\sum_j s_jt_j'=E_{k+1}$. The solution of (1) or (1′) may, therefore, be obtained as follows:

From (1′) and (11) we have

(12) $$\frac{\partial \boldsymbol{t}_j'\boldsymbol{p}(t)}{\partial t}=\boldsymbol{t}_j'R\boldsymbol{p}(t)=x_j\boldsymbol{t}_j'\boldsymbol{p}(t).$$

Hence it follows that

(13) $$\boldsymbol{t}_j'\boldsymbol{p}(t)=\boldsymbol{t}_j'\boldsymbol{p}(0)e^{x_jt} \qquad (j=0, 1, \cdots, k).$$

Multiplying $\boldsymbol{s}_j$ on both sides of (13) from the left and summing from $j=0$ to $j=k$, the required solution can be expressed as follows:

(14) $$\boldsymbol{p}(t)=\sum_{j=0}^{k} e^{x_jt}\,\boldsymbol{s}_j\,\boldsymbol{t}_j'\ \boldsymbol{p}(0).$$

The latent roots of the matrix R can in general not be expressed explicitly by the algebraic functions of the constants λ's and μ's except the cases of $k=1$ and 2, so that the solution (14) is unfortunately quite a formal one. (The case of $k=1$ is only a trivial one where no counter process can occur. We shall, therefore, consider exclusively the case where $k\geqq 2$ in the followings.)

The latent vectors, however, can be expressed explicitly, though complicated, by the corresponding latent root and the constants λ's and μ's. Thus, the latent vectors $\boldsymbol{s}_k$ and $\boldsymbol{t}_k'$ corresponding to the root $x_k=0$ are easily seen that $\boldsymbol{s}_k'=(0, 0, \cdots, 0, 1)$ and $\boldsymbol{t}_k'=(1, 1, \cdots, 1, 1)$. The column latent vector corresponding to each of the negative latent roots x_j may be obtained by solving the set of equations

(15) $$(x_jE_{k+1}-R)\boldsymbol{s}_j=0.$$

Assuming the $(k+1)$-th component of $\boldsymbol{s}_j$ be one; $s_{jk}=1$, we have

$$s_{jk-1}=x_j/\lambda_{k-1},$$

(16) $$s_{jh}=\frac{x_j}{\lambda_h\lambda_{h+1}\cdots\lambda_{k-1}}\begin{vmatrix} x_j+\lambda_{h+1}+\mu_{h+1} & -\mu_{h+2} & & & \mathbf{0} \\ -\lambda_{h+1} & x_j+\lambda_{h+2}+\mu_{h+2} & -\mu_{h+3} & & \\ & \cdot & \cdot & \cdot & \\ & \mathbf{0} & \cdot & \cdot & \cdot \\ & & & -\lambda_{k-2} & x_j+\lambda_{k-1}+\mu_{k-1} \end{vmatrix}$$

$$(h=0, 1, \cdots, k-2).$$

By solving the set of equations

(17) $$\boldsymbol{t}_j'(x_jE_{k+1}-R)=0$$

under the assumption that the first component of $\boldsymbol{t}_j'$; $t_{j0}=c_j$, we have also

(18) $$t_{jh}=\frac{c_j}{\lambda_0\lambda_1\cdots\lambda_{h-1}}\begin{vmatrix} x_j+\lambda_0 & -\mu_1 & & & \mathbf{0} \\ -\lambda_0 & x_j+\lambda_1+\mu_1 & -\mu_2 & & \\ & \cdot & \cdot & \cdot & \\ & \mathbf{0} & \cdot & \cdot & -\mu_{h-1} \\ & & & -\lambda_{h-2} & x_j+\lambda_{h-1}+\mu_{h-1} \end{vmatrix}$$

$$(h=1, 2, \cdots, k-1).$$

It is easy to see that $t_{jk}=0$ for all $j=0, 1, \cdots, k-1$.

The coefficients c_j's may be determined by the normalizing conditions; thus

$$c_j = \frac{\lambda_0\lambda_1\cdots\lambda_{k-1}}{\sum_{h=0}^{k-1} x_j S_{jh} T_{jh}} \tag{19}$$

where
$$S_{jh} = \begin{vmatrix} x_j+\lambda_{h+1}+\mu_{h+1} & -\mu_{h+2} & & & \mathbf{0} \\ -\lambda_{h+1} & x_j+\lambda_{h+2}+\mu_{h+2} & -\mu_{h+3} & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ \mathbf{0} & & & -\lambda_{k-2} & x_j+\lambda_{k-1}+\mu_{k-1} \end{vmatrix}$$
$(h=0, 1, \cdots, k-2)$

$$S_{jk-1} = 1,$$

and
$$T_{jh} = \begin{vmatrix} x_j+\lambda_0 & -\mu_1 & & & \mathbf{0} \\ -\lambda_0 & x_j+\lambda_1+\mu_1 & -\mu_2 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ \mathbf{0} & & & -\lambda_{h-2} & x_j+\lambda_{h-1}+\mu_{h-1} \end{vmatrix}$$
$(h=1, 2, \cdots, k-1)$

$$T_{j0} = 1.$$

In the case where the process starts from the ground state E_0, i.e., $\boldsymbol{p}'(0)=(1, 0, \cdots, 0)$, the solution of (1) may be simplified as follows:

$$\boldsymbol{p}(t) = \boldsymbol{s}_k + \sum_{j=0}^{k-1} c_j \boldsymbol{s}_j e^{x_j t}. \tag{20}$$

Hence we have the following theorem:

Theorem II. *When the underlying process which describes the stimulus-response relation is specified by* (1), *the probability of absorption or occurrence of the response within the duration of the stimulation,* $p_k(T)$, *is expressed as follows*:

$$p_k(T) = 1 + \sum_{j=0}^{k-1} c_j e^{x_j T}, \tag{21}$$

where x_j*'s are the negative latent roots of the matrix* R *and* c_j*'s are the functions of* λ_j*'s and* μ_j*'s as expressed in* (19).

It may easily be seen from (20) that all states except E_k are transient provided all λ's and μ's are constants, so that $\lim_{T\to\infty} p_k(T)=1$.

2. Asymptotic behavior of the probability of absorption

In a situation such that the experimenter has to contemplate the counter process of the subject for the stimulus, i e., the recuperation process from the effect of hit during the stimulation is continued, the theoretical response curve turns out to be a very complicated one as described in section 1. Such a situation is always met especially in biological experimentation.

As far as the determination of the threshold number k is concerned, the experimenter would sometimes administer the duration of stimulus as short as possible in order to make the effect of counter process as small as possible, because the administration of intensity as well as duration of stimulus would always be in his own disposal. If we can clear away the counter process or death process, theoretical curve may well be simplified as follows:

(22) $$p_k(T)=1-$$

$$(-1)^{k-1}\sum_{j=0}^{k-1}\frac{\lambda_0\lambda_1\cdots\lambda_{j-1}\lambda_{j+1}\cdots\lambda_{k-1}}{(\lambda_j-\lambda_0)(\lambda_j-\lambda_1)\cdots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\cdots(\lambda_j-\lambda_{k-1})}e^{-\lambda_j T}$$

provided all λ's are not equal. The case in which some λ's are equal may be inferred from (22).

Alternatively, asymptotic behavior of the process under the situation that the duration of the stimulus is sufficiently extended would be of interest, in as much as reasonable assumptions for the mechanism of counter process could be established. We shall, therefore, discuss here the asymptotic behavior of the probability of absorption or occurrence of a response given by the formula (21) for large T under the following assumptions:

Assumption (a): All λ's are the same order of magnitudes, say $\lambda_j=O(\lambda)$[3] $(j=0, 1, \cdots k-1)$.

Assumption (b): All μ's are the same order of magnitudes, say $\mu_j=O(\mu)$ $(j=1, 2, \cdots k-1)$.

The assumption (a) is quite reasonable practically because all λ's are proportional to the intensity of stimulus which may be fixed throughout the stimulation process is continued. The assumption (b) may be acceptable in almost all practical cases because all μ's depend only on the mechanism of recuperation of the subject. Proper selection of the time scale render us to make the following additional assumption:

Assumption (c): μ is a finite order of magnitude, i.e., $\mu_j=O(1)$ $(j=1, 2, \cdots, k-1)$,

because μ_j is the reciprocal of mean recuperation time of the process.

Under the above three assumptions, the following theorem holds for the probability of absorption.

Theorem III. *Under the assumptions (a), (b) and (c), the probability of absorption can be expressed as follows:*

---

3) In this paper, the symbol '$a=O(b)$' implies that 'a is of the same order b', i.e., a/b as well as b/a remain bounded. The symbol '$a=o(b)$' implies that 'a is of a smaller order than b', i.e., a/b tends to zero.

$$(23) \qquad p_k(T) = 1 - e^{-\frac{\lambda_0\lambda_1\cdots\lambda_{k-1}}{\mu_1\mu_2\cdots\mu_{k-1}}T(1+o(\lambda))}(1+O(\lambda))$$

provided $\lambda = o(1)$.

Proof: The condition $\lambda = o(1)$ would be indispensable for the purpose of controlling the probability between zero and one. When $\lambda = O(1)$, it is evident that the probability of absorption tends to one because all x_j's are negative constants.

Let us evaluate the largest non-zero latent root x_{k-1} of the matrix R in (1′) or A_k in (3). This is the largest zero of the characteristic polynomial

$$(24) \qquad f_k(x) = \{-A_k, k\} + \{-A_k, k-1\}x + \cdots + \{-A_k, 1\}x^{k-1} + x^k,$$

where $\{-A_k, h\}$ denote the sum of all principal $h \times h$ minors of $-A_k$.

Let M_{ij} be a principal $\overline{j-i-1} \times \overline{j-i-1}$ minor of $-A_k$ which lies between i-th row (column) and j-th row (column) of the matrix $-A_k$, i.e.,

$$(25) \qquad M_{ij} = \begin{vmatrix} \lambda_i + \mu_i & -\mu_{i+1} & & & \mathbf{0} \\ -\lambda_i & \lambda_{i+1} + \mu_{i+1} & -\mu_{i+2} & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & \mathbf{0} & & -\lambda_{j-3} & \lambda_{j-2} + \mu_{j-2} \end{vmatrix}$$

$(0 \leq i < j \leq k+1)$,

where M_{0j} and M_{ik+1} denote the principal minors situated in the upper and lower corner of $-A_k$ respectively. We shall assume here that $\mu_0 = 0$ and that $M_{ii+1} = 1$ for all $i = 0, 1, \cdots, k$. Then, the coefficients of the characteristic polynomial (24) may be expressed by using (25) and Lemma 3 as follows:

$$\{-A_k, k\} = M_{0k+1} = \lambda_0\lambda_1 \cdots \lambda_{k-1}$$

$$\{-A_k, k-1\} = \sum_{i=1}^{k} M_{0i}M_{ik+1} = \mu_1\mu_2 \cdots \mu_{k-1} + \mu_1\mu_2 \cdots \mu_{k-2}\lambda_{k-1}$$
$$+ \cdots + \lambda_1\lambda_2 \cdots \lambda_{k-1}$$
$$+ \sum_{i=2}^{k} \lambda_0 \cdots \lambda_{i-2}(\mu_i \cdots \mu_{k-1} + \mu_i \cdots \mu_{k-2}\lambda_{k-1} + \cdots + \lambda_i \cdots \lambda_{k-1})$$

$$(26) \qquad \{-A_k, k-h\} = \sum_{1 \leq i_1 < i_2 < \cdots < i_h \leq k} M_{0i_1}M_{i_1i_2} \cdots M_{i_hk+1}$$
$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_h \leq k} \prod_{r=0}^{h} \mu_{i_r} \cdots \mu_{i_{r+1}-2} + \cdots + \lambda_{i_r} \cdots \lambda_{i_{r+1}-2}$$
$$\{-A_k, 1\} = \sum_{i=0}^{k-1} (\lambda_i + \mu_i),$$

where $i_0 = 0$ and $i_{h+1} = k+1$.

From the assumptions (a), (b) and (c) we can evaluate the order of these coefficients as follows:

$$\begin{aligned}&\{-A_k, k\}=\lambda_0\lambda_1 \cdots \lambda_{k-1}=O(\lambda^k)\\ (27)\qquad &\{-A_k, k-1\}=\mu_1\mu_2 \cdots \mu_{k-1}(1+O(\lambda))=O(1)\\ &\{-A_k, k-h\}=O(1), \qquad (h=2, 3, \cdots, k=1).\end{aligned}$$

Since $f_k(0)=\{-A_k, k\}>0, f_k'(0)=\{-A_k, k-1\}>0$ and $f_k(0)=2\{-A_k, k-2\}>0$, the first approximate value $x_{k-1}^{(1)}$ of x_{k-1} may be obtaimed by using Newton's method as follows:

$$(28)\qquad x_{k-1}^{(1)}=-\frac{f_k(0)}{f_k'(0)}=-\frac{\{-A_k, k\}}{\{-A_k, k-1\}}=-\frac{\lambda_0\lambda_1 \cdots \lambda_{k-1}}{\mu_1\mu_2 \cdots \mu_{k-1}}(1+O(\lambda))$$

or $\qquad =O(\lambda^k)$.

The correction term for the second approximate value $x_{k-1}^{(2)}$ obtained from $x_{k-1}^{(1)}$ is

$$(29)\qquad -\frac{f_k(x_{k-1}^{(1)})}{f_k'(x_{k-1}^{(1)})}=-\frac{\{-A_k, k-2\}x_{k-1}^{(1)2}+\cdots+x_{k-1}^{(1)k}}{\{-A_k, k-1\}+2\{-A_k, k-2\}x_{k-1}^{(1)}+\cdots+kx_{k-1}^{(1)k-1}}$$
$$=O(\lambda^{2k})$$

so that the order is smaller than λ^{k+1}. The required largest root x_{k-1} can therefore be expressed as

$$(30)\qquad x_{k-1}=-\frac{\lambda_0\lambda_1 \cdots \lambda_{k-1}}{\mu_1\mu_2 \cdots \mu_{k-1}}(1+O(\lambda)).$$

Since Lemma 1 shows that the largest latent root of the contracted matrix A_{k-1} lies between x_{k-2} and x_{k-1} and the order of which is λ^{k-1} as mentioned above, the following order relations hold for k different roots:

$$(31)\qquad O(1)\leq x_0<x_1<\cdots<x_{k-2}\leq O(\lambda^{k-1})<x_{k-1}=O(\lambda^k)<0.$$

Thus for any negative latent root x_j, the order relation $x_j=O(\lambda^p)$ $(0\leq p\leq k)$ may hold.

Let us next evaluate the order of the coefficients c_j's by evaluating the order of the components of latent vectors s_j and t_j' corresponding to x_j $(j=0, 1, \cdots, k-1)$.

Since $x_j=O(\lambda^p)$ we have from (16) the followings:

(i) When $1\leq p\leq k$,

$$\begin{aligned}(32)\qquad s_{jk} &= 1\\ s_{jk-1} &= x_j/\lambda_{k-1}=O(\lambda^{p-1})\\ s_{jh} &= \frac{x_j}{\lambda_h\lambda_{h+1} \cdots \lambda_{k-1}}(\{M_{h+1k+1}, k-h\}+\{M_{h+1k+1}, k-h-1\}x_j\\ &\qquad +\cdots+x_j^{k-h})\\ &= \frac{\mu_{h+1} \cdots \mu_{k-1}}{\lambda_h\lambda_{h+1} \cdots \lambda_{k-1}}x_j\,(1+O(\lambda))=O(\lambda^{-k+h+p}) \quad (h=0, 1, \cdots, k-2)\end{aligned}$$

$(h=0, 1, \cdots, k-2)$.

(ii) When $0 \leq p \leq 1$, we have also

(32′) $$s_{jk} = 1$$
$$s_{jk-1} = O(\lambda^{p-1})$$
$$s_{jh} = O(\lambda^{-k+h+p}) \quad (h=0, 1, \cdots, k-2).$$

We have also from (18) the followings:

(33) $$t_{.0} = c_j$$
$$t_{jh} = \frac{c_j}{\lambda_0 \cdots \lambda_{h-1}} (\{M_{0h+1}, h\} + \{M_{0h+1}, h-1\} x_j + \cdots + x_j^h)$$
$$= c_j(1+O(\lambda^{-h})x_j) \quad (h=1, 2, \cdots, k-1)$$
$$t_{jk} = 0$$

The normalizing constants c_j's are therefore determined as follows:

(i) For $x_{k-1} = -\frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_{k-1}} (1+O(\lambda))$ we have

(34) $$c_{k-1} = -(1+O(\lambda)).$$

(ii) For $x_j = O(\lambda^p)$ $(0 \leq p \leq k-1)$ we have

(34′) $$c_j = O(\lambda^{k-p}) = o(1).$$

Hence (21) can be expressed by (23) as stated in Theorem III.

When λ is so controlled as $\lambda = O(T^{-1/k})$ or the magnitude of the stimulus is $O(T^{k-1/k})$ for large T, (23) may be reduced as follows:

(23′) $$p_k(T) = 1 - e^{-\frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_{k-1}} T} (1+O(T^{-1/k})).$$

When the experimenter fails to control the intensity λ, the probability tends to one provided the order of λ is larger than $T^{-1/k}$ and tends to zero provided the order of λ is smaller than $T^{-1/k}$.

When the intensity of the stimulus is maintained constant, since λ_j is proportional to the intensity I, i.e., $\lambda_j = \alpha_j I$, and is interpreted as the reciprocal of mean arrival time—or mean duration up to a new hit occurs—when the process is in the state E_j,

$$\bar{\lambda} = (\lambda_0 \lambda_1 \cdots \lambda_{k-1})^{1/k} = I(\alpha_0 \alpha_1 \cdots \alpha_{k-1})^{1/k} = I\bar{\alpha}$$

can be regarded as the reciprocal of the geometric mean of these mean arrival times and is proportional to the intensity of the stimulus. Since μ_j can also be interpreted as the reciprocal of mean holding time—mean duration up to a death occurs—τ_j of the state E_j, if we denote the geometric mean of μ_j by $\bar{\mu}$, $\bar{\mu} = (\mu_1 \mu_2 \cdots \mu_{k-1})^{1/(k-1)}$, it can be interpreted as the reciprocal of a general mean holding time $\bar{\tau}=(\tau_1\tau_2\cdots\tau_{k-1})^{1/(k-1)}$.

When we introduce these parameters such as $\bar{\lambda}=I\bar{\alpha}$ and $\bar{\mu}=\bar{\tau}^{-1}$, the asymptotic probability of absorption can be expressed as

$$(35)\qquad p_k(T)\doteqdot 1-e^{-\frac{(\bar{\alpha}IT)^k}{(T/\bar{\tau})^{k-1}}}\ (T\gg 1).$$

Now we can introduce from this relation the asymptotic relation between the megnitude of stimulus IT and its duration T both in log unit which yields a fixed probability of absorption, say p. Thus from (35) we have

$$(36)\qquad \frac{(\bar{\alpha}IT)^k}{(T/\bar{\tau})^{k-1}}\doteqdot -\log\ (1-p)\qquad (T\gg 1)$$

or

$$(37)\qquad \log IT\doteqdot\frac{k-1}{k}\log T-\log\bar{\alpha}-\frac{k-1}{k}\log\bar{\tau}+\log\ (-\log\ (1-p)).$$

The results show that the asymptotic relation of the magnitude of stimulus in log unit to the duration of stimulus in log unit when T is large is linear, the direction coefficient being $(k-1)/k$.

Hence we have a new problem of the linear regression, the direction parameter being discrete. The idea of utilizing this asymptotic behavior of the probability of absorption may be found in the paper due to Bouman and Van der Velden (1947). They had treated a somewhat different case where the holding time is always a constant,[4] and intended to estimate the threshld number of quanta for seeing. The data given by them, however, were quite insufficient in that T was not sufficiently large to permit such a asymptotic approximation. Theoretical assumptions employed for the mechanism of death, moreover, are inadequate. But the discussion related to these physiological disputes are beyond the scope of this paper.

Acknowledgements: The author wishes to express his best thanks to Professor Junjiro Ogawa of Nihon University for valuable discussions and encouragements. He also thanks to Professor Toshio Kitagawa of Kyushu University for valuable suggestions.

4) Their treatments are not rigorous except the case of $k=2$. Yamamoto et al (1959) obtained a more improved results.

References

(1) Bartlett, M. S. (1955). *An introduction to stochastic processes.* London: Cambridge Univ. Press.

(2) Bouman, M. A. and van der Velden, H. A. (1947). The two-quanta explanation of the dependence of the threshold values and visual acuity on the visual angle and the time of observation. *J. Optic. Soc. America*, **37**, 908-19.

(3) Ledermann, W. and Reuter, G. E. H. (1954). Spectral theory for the differential equations of simple birth and death processes. *Phil. Trans. A*, **246**, 321-69.

(4) Takagi, T. (1937). *Daisûgaku Kôgi.* Tokyo: Iwanami.

(5) Weal, R. A. (1955). The absolute threshold of vision. *Physiol. Rev.*, **35**, 233-46.

(6) Yamamoto, S. (1959). Aru kata no gyôretsu no koyûchi ni tsuite. *Sûgaku*, **11**, 14-5. (in Japanese)

(7) Yamamoto, S. *et al.* (1959). Shikaku no ikichi-kôryôshi-sû ni tsuite. *Sûrikagaku-Kenkyû Dai-5-han Hôkoku*, No. **4**, 15-72. (in Japanese)