An Inequality for the Weighted Sum of $ x^2 $ Variates

Okamoto, Masashi
Osaka University

https://doi.org/10.5109/12996

出版情報：統計数理研究. 9 (2/3), pp.69-70, 1960-10. Research Association of Statistical Sciences
バージョン：
権利関係：

AN INEQUALITY FOR THE WEIGHTED SUM OF χ^2 VARIATES

By

Masashi OKAMOTO

(*Received* December 15, 1959)

The distribution of a weighted sum of χ^2 variates is investigated by several authors, H. Robbins and E. J. G. Pitman [1] and G. E. P. Box [2], for instance. We now present a new, simple inequality for the tail probability.

Theorem. Let X_i $(i=1, \cdots, k)$ follow independently a χ^2 distribution with n_i degrees of freedom and let $a_1, \cdots, a_k$ be positive constants, then for every constant c,

$$Pr(\sum_{i=1}^{k} a_i X_i < c) \leq Pr(aX < c) ,$$

where

$$a = (\prod_{i=1}^{k} a_i^{n_i})^{1/n}, \quad n = \sum_{i=1}^{k} n_i,$$

and X is a χ^2 variate with n degrees of freedom.

Proof. By decomposing X_i $(i=1, \cdots, k)$ into n_i independent χ^2 variates with one degree of freedom it is seen that it suffices to prove the theorem when all n_i's are one. Then $n=k$ and $a=(\prod_{i=1}^{n} a_i)^{1/n}$. We have for positive c

$$(1) \qquad P = Pr(\sum_{i=1}^{n} a_i X_i < c) = \frac{1}{(\sqrt{2\pi})^n} \int \cdots \int_E e^{-\sum_{i=1}^{n} x_i^2/2} \prod_{i=1}^{n} dx_i,$$

where E denotes the ellipsoid in R^n defined by the inequality $\sum_{i=1}^{n} a_i x_i^2 < c$.

Let S be the sphere in R_n defined by $a\sum_{i=1}^{n} x_i^2 < c$. Since the volumes of two regions E and S are equal and since the integrand of (1) is a decreasing function of the distance from the origin, it holds that

$$P \leq \frac{1}{(\sqrt{2\pi})^n} \int \cdots \int_S e^{-\sum_{i=1}^{n} x_i^2/2} \prod_{i=1}^{n} dx_i = Pr\ (aX < c).$$

The theorem means that $\sum_{i=1}^{k} a_i X_i$ is stochastically larger than or equal to aX in the sense of Mann-Whitney [3]. Since the property "stochastically larger than or equal to" is preserved under the addition of (or multiplication by) any independent (or non-negative, independent) random variable, we have readily

Corollary. Besides the conditions in the theorem, let Y be a χ^2 variate with n' degrees of freedom independent of $X_1, \cdots, X_k$. Then for every constant c,

$$Pr\left(\frac{1}{n}\sum_{i=1}^{k} a_i X_i \Big/ \frac{Y}{n'} < c\right) \leq Pr(aF < c),$$

where F follows an F distribution with n and n' degrees of freedom.

OSAKA UNIVERSITY.

References

[1] H. ROBBINS and E. J. G. PITMAN, *Application of the method of mixtures to quadratic forms in normal variates*, Ann. Math. Stat., **20** (1949), 552-560.

[2] G. E. P. BOX, *Some thoerems on quadratic forms applied in the study of analysis of variance problems, I. Effect of inequality of variance in the one-way classification*, Ann. Math. Stat., **25** (1954), 290-302.

]3] H. B. MANN and D. R. WHITNEY, *On a test of whether one of two random variables is stochastically larger than the other*, Ann. Math. Stat., **18** (1947), 50-60.