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AN INEQUALITY FOR THE WEIGHTED SUM OF χ^2 VARIATES

By

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The distribution of a weighted sum of χ^2 variates is investigated by several authors, H. Robbins and E. J. G. Pitman [1] and G. E. P. Box [2], for instance. We now present a new, simple inequality for the tail probability.

Theorem. Let X_i ($i=1, \dots, k$) follow independently a χ^2 distribution with n_i degrees of freedom and let $a_1, \dots, a_k$ be positive constants, then for every constant c ,

$$Pr(\sum_{i=1}^k a_i X_i < c) \leq Pr(aX < c),$$

where

$$a = (\prod_{i=1}^k a_i^{n_i})^{1/n}, \quad n = \sum_{i=1}^k n_i,$$

and X is a χ^2 variate with n degrees of freedom.

Proof. By decomposing X_i ($i=1, \dots, k$) into n_i independent χ^2 variates with one degree of freedom it is seen that it suffices to prove the theorem when all n_i 's are one. Then $n=k$ and $a = (\prod_{i=1}^n a_i)^{1/n}$. We have for positive c

$$\begin{aligned} (1) \quad P &= Pr(\sum_{i=1}^n a_i X_i < c) \\ &= \frac{1}{(\sqrt{2\pi})^n} \int \dots \int_E e^{-\sum_{i=1}^n x_i^2/2} \prod_{i=1}^n dx_i, \end{aligned}$$

where E denotes the ellipsoid in R^n defined by the inequality $\sum_{i=1}^n a_i x_i^2 < c$.

Let S be the sphere in R_n defined by $a \sum_{i=1}^n x_i^2 < c$. Since the volumes of two regions E and S are equal and since the integrand of (1) is a decreasing function of the distance from the origin, it holds that

$$P \leq \frac{1}{(\sqrt{2\pi})^n} \int \dots \int_S e^{-\sum_{i=1}^n x_i^2/2} \prod_{i=1}^n dx_i = Pr(aX < c).$$

The theorem means that $\sum_{i=1}^k a_i X_i$ is stochastically larger than or equal to aX in the sense of Mann-Whitney [3]. Since the property "stochastically

larger than or equal to" is preserved under the addition of (or multiplication by) any independent (or non-negative, independent) random variable, we have readily

Corollary. Besides the conditions in the theorem, let Y be a χ^2 variate with n' degrees of freedom independent of $X_1, \dots, X_k$. Then for every constant c ,

$$Pr\left(\frac{1}{n} \sum_{i=1}^k a_i X_i \bigg/ \frac{Y}{n'} < c\right) \leq Pr(aF < c),$$

where F follows an F distribution with n and n' degrees of freedom.

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