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Tamura, Ryoji
Shimane University

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# ON THE NONPARAMETRIC TESTS BASED ON CERTAIN U-STATISTICS

By

**Ryoji TAMURA**

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§ 1. **Introduction** The object of this paper is to discuss the two sample tests for scale based on some of the certain *generalized U-statistics* which have been investigated by Hoeffiding [1], Lehmann [2], Sukhatme [3], [4], [5] and Fraser [6].

Let now $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent random samples from populations with comulative distribution functions $F(x-\xi)$ and $G(y-\eta)=F[(y-\eta)/\delta]$ respectively, $\xi$ and $\eta$ being the population medians of $X$ and $Y$, and $\delta$ a scale parameter. Moreover assume that $F(x)$ and $G(y)$ be absolutely continuous. The problem considered in this paper is to test the hypothesis that two populations have the same scale parameter against the alternative that the $Y$'s are more spread out than the $X$'s and vice versa, or in symbols

$$H: \delta=1, \quad A: \delta\neq 1 \quad (\text{or } A': \delta>1).$$

This is narrower than the alternative $F(x)\neq G(x)$. Let $\xi$ and $\eta$ be known, say $\xi=\eta=0$, without loss of generality, so that the distribution functions of $X$ and $Y$ differ only in the scale parameter. In this case several nonparametric tests have been proposed, particularly by Mood [7], Sukhatme [3], [5], and Tamura [8]. These tests are based on some of the statistics which are generally called as *the generalized U-statistics*. On the other hand, Sukhatme has proposed to use the modified statistics which are obtained by substituting their estimates into *the U-statistics* instead of $\xi$ and $\eta$, in case where the informations about $\xi$ and $\eta$ are unknown. We shall also propose the new statistic $\hat{Q}_N$ as the test criterion which is essentially *the modified generalized U-statistic*, and investigate some of properties of $\hat{Q}_N$.

§ 2. **The test based on the statistic $Q_N$ —*Q test*—.** When the location parameters $\xi$ and $\eta$ are known, *say* $\xi=\eta=0$, the auther [8] has proposed the statistic $Q_N$ as the test criterion

$$Q_N=\binom{m}{2}^{-1}\binom{n}{2}^{-1}\sum_{\alpha,\beta}\varphi(X_{\alpha_1}, X_{\alpha_2}, Y_{\beta_1}, Y_{\beta_2}), \tag{1}$$

*where*

$$\varphi(u_1,u_2,v_1,v_2)=\begin{cases}1 & \text{for } v_1<u_1u_2<v_2 \text{ or } v_2<u_1u_2<v_1\\ 0 & \text{otherwise}\end{cases},$$

*the notation* $v_1<u_1u_2<v_2$ *means that two* $u_1$, $u_2$ *lie between* $v_1$ *and* $v_2$ $(v_1<v_2)$, *and the summation runs over all subscripts* $\alpha$, $\beta$ *such that* $1\leqq\alpha_1<\alpha_2\leqq m$, $1\leqq\beta_1<\beta_2\leqq n$. Let $\rho_1$, $\rho_2$ be fixed non-negative numbers such that $m=N\rho_1$, $n=N\rho_2$ and $\rho_1+\rho_2=1$. Then it is easily shown that $Q_N$ is *the so-called generalized U-statistic* investigated by Lehmann and other authers. The mean value $\theta=EQ_N$ and the variance $\sigma^2=\operatorname{var}Q_N$ of $Q_N$ may be found after some computations as follows,

$$\theta=P(Y_1<X_1X_2<Y_2 \text{ or } Y_2<X_1X_2<Y_1)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F(y)-F(y')\}^2dG(y)dG(y') \tag{2}$$

$$\sigma^2=\binom{m}{2}^{-1}\binom{n}{2}^{-1}\{(a-\theta^2)mn^2+(b-\theta^2)mn^2+(4v+6\theta-5a-5b)mn+(t/2+2\theta^2/3-2a)m^2+(u/2+2\theta^2/3-2b)n^2+(2r-5t/2+10a+6b-8v-15\theta^2/2)m+(2s-5u/2+10b6a-8v-15\theta^2/2)n+(\theta+3t+3u+16v+9\theta^2-4r-4s-12a-12b)\}, \tag{3}$$

*where setting* $F_i=F(y_i)$, $G_i=G(y_i)$,

$$r=2\int_{-\infty}^{\infty}\int_{y_2}^{\infty}(F_1-F_2)^3\,dG_1dG_2$$

$$s=4\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}(F_1-F_2)^2G_2dG_2dG_1$$

$$t=2\int_{-\infty}^{\infty}\int_{-\infty}^{y_2}(F_2-F_1)^4dG_1dG_2$$

$$u=16\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}(F_1-F_2)^2\,G_2\,(1-G_1)\,dG_1\,dG_2$$

$$v=4\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}(F_1-F_2)^3G_2dG_2dG_1+4\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}(F_1-F_2)^2(F_2-F_3)dG_3dG_2dG_1$$

$$a=4\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}(F_1-F_2)^4G_2dG_2dG_1+6\int_{-\infty}^{\infty}\int_{-\infty}^{y_2}\int_{y_2}^{\infty}(F_1-F_2)^2(F_2-F_3)^2dG_1dG_3dG_2$$
$$+8\int_{-\infty}^{\infty}\int_{-\infty}^{y_2}\int_{y_2}^{\infty}(F_1-F_2)^3\,(F_2-F_3)dG_1dG_3dG_2$$

$$b=16\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}(F_1-F_2)^3G_2\,(1-G_1)\,dG_2dG_1+32\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}(F_1-F_2)\,(F_2-F_3)^2G_3dG_3$$
$$dG_2dG_1$$

$$+2\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{-\infty}^{y_3}(F_1-F_2)(F_2-F_3)(F_3-F_4)dG_4dG_3dG_2dG_1.$$

Under the null-hypothesis $F=G$, we can evaluate $\theta$ and $\sigma^2$ as follows

(4) $$\theta_0=E(Q_N|H)=1/6$$

(5) $$\sigma_0^2=\mathrm{var}\,(Q_N H)=(m+n)/45mn+o(1/N).$$

Futhermore, if we set

(6) $$\psi_1(x_1)=E\varphi(x_1, X_2, Y_1, Y_2)-\theta$$
$$\psi_2(y_1)=E\varphi(X_1, X_2, y_1, Y_2)-\theta,$$

where $x_1$, $y_1$ are arbitrary fixed numbers and the expectations is taken with respect to the random variable $X_1$, $X_2$, $Y_1$, $Y_2$, then the asymptotic distribution of $\sqrt{N}(Q_N-EQ_N)$ is the nondegenerate normal distribution since

(7) $$E\psi_1^2(X_1)+E\psi_2^2(Y_1)>0.$$

In a matter of fact,

$$\psi_2(y_1)=E\varphi(X_1, X_2, y_1, Y_2)-\theta$$
$$=\int_{-\infty}^{\infty}\{F(y)-F(y_1)\}^2dG(y)-\theta.$$

Setting $\int_{-\infty}^{\infty}\{F(y)-F(y_1)\}^2dG(y)=I(y_1)$, then $E\psi_2^2(Y_1)=0$ implies that

$$I(Y_1)=EI(Y_1).$$

However this identity cannot hold with probability one. This fact shows that the variance $\sigma^2$ of $Q_N$ is of order $N^{-1}$.

It will now be shown that the test based on $Q_N$ —*Q test*— is consistent, that is, the probability tends to one as $N$ tends to infinity that *the $Q_N$ test* will reject the hypothesis $H$ tested when the given alternative is true. In fact, the power function for the alternative $A$ is indicated as follows

(8) $$P(|Q_N-\theta_0|>\lambda_\alpha\sigma_0|A),$$

where $\lambda_\alpha$ is a constant that is defined by

$$(2\pi)^{-\frac{1}{2}}\int_{-\lambda\alpha}^{\lambda\alpha}\exp(-x^2/2)dx=1-\alpha.$$

Then the above probability (8) is transformed as follows

(9) $$1-\frac{1}{\sqrt{2\pi}\sigma}\int_{\theta_0-\lambda_\alpha\sigma_0}^{\theta_0+\lambda_\alpha\sigma_0}\exp\{-(y-\theta)^2/2\sigma^2\}dy=1-\frac{1}{\sqrt{2\pi}}\int_{t_1}^{t_2}\exp(-x^2/2)dx,$$

*where*

(10) $$t_1=(\theta_0-\lambda_\alpha\sigma_0-\theta)/\sigma,\ t_2=(\theta_0+\lambda_\alpha\sigma_0-\theta)/\sigma.$$

However $\sigma_0$ and $\sigma$ is of order $N^{-\frac{1}{2}}$ so that the above power tends to one as $N\to\infty$.

To close this section, we calculate the asymptotic efficiency of this test *in the Mood's sense* for $G(x)=F(x/6)$. Then from (2) we can directly get

(11) $$\frac{\partial EQ_N}{\partial\sigma}\Big|_{\sigma=1}=4\int_{-\infty}^{\infty}xf(x)^2F(x)\,dx-2\int_{-\infty}^{\infty}xf(x)^2\,dx,$$

therefore the efficacy of *the Q test* is equal to

(12) $$180mn(m+n)^{-1}\{2\int_{-\infty}^{\infty}xf^2F\,dx-\int_{-\infty}^{\infty}xf^2\,dx\}^2.$$

Also, the efficacy of *the variance ratio F-test* is equal to

$$4mn(m+n)^{-1}(\beta_2-1).$$

Hence the asymptotic relative efficiency of our *Q test* with respect to the *variance ratio F-test* is given by

(13) $$e=45(\beta_2-1)\{2\int xf^2Fdx-\int xf^2dx\}^2,$$

*where*

$$\beta_2=\int_{-\infty}^{\infty}\{x-E(X)\}^4\,dF/[\int_{-\infty}^{\infty}\{x-E(X)\}^2dF]^2.$$

We can obtain after some computations the following table about the asymptotic relative efficiency with respect to *F-test* for some alternatives.

**Table**

| density function | asymptotic efficiency |
|---|---|
| 1. **normal** $f(x)=\frac{1}{\sqrt{2\pi}}\exp(-x^2/2),-\infty<x<\infty$ | $15/2\pi^2\fallingdotseq 0.76$ |
| 2. **uniform** $f(x)=\frac{1}{2}$, $-1\leqq x\leqq 1$ | 1 |
| 3. **exponential** $f(x)=\frac{1}{2}\exp(-\lvert x\rvert)$, $-\infty<x<\infty$ | 1.08 |

On the other hand, the asymptotic relative efficiency of Sukhatme's *T test* with respect to *F-test* is about 0.61 for the normal alternative and about 0.94 for the exponential, futhermore that of Sukhatme's *S test* is equal to 0.69 for the normal and 0.80 for the uniform. Hence it follows that our *Q test* is more efficient than not only Sukhatme's *T* and *S tests*, but *the variance ratio F-test* for some non-normal alternatives.

§ 3. **The test based on $\hat{Q}^N$.** The test in the above section presupposes the knowledge about the location of the two populations which are not always available. Then our concern is with the case that there are no know-

ledge about $\xi$ and $\eta$. In the latter case, we estimate $\xi$ and $\eta$ by the sample medians $\tilde{X}$ and $\tilde{Y}$, and use the deviations of the observations from the sample medians rather than the observations themselves. And we define the following new statistic $\hat{Q}_N$ which is known as *the modified generalized U-statistic* in general.

$$\hat{Q}_N=\binom{m}{2}^{-1}\binom{n}{2}^{-1}\sum_{\alpha}\sum_{\beta}\varphi\,(X_{\alpha_1}-\tilde{X}, X_{\alpha_2}-\tilde{X}, Y_{\beta_1}-\tilde{Y}, Y_{\beta_2}-\tilde{Y}), \tag{14}$$

*where the summation runs over all subscripts* $\alpha$, $\beta$ *such that* $1\leqq\alpha_1<\alpha_2\leqq m$, $1\leqq\beta_1<\beta_2\leqq n$ *and the function* $\varphi$ *means the same as* (1). In this section we shall now prove that $\hat{Q}_N$ has the same asymptotic normal distribution as the statistic $Q_N$, that is, in symbol

$$\lim_{N\to\infty} L_{\sqrt{N}}(\hat{Q}_N-EQ_N)=\lim_{N\to\infty} L_{\sqrt{N}}(Q_N-EQ_N)=N(0,\sigma_1^2) \tag{15}$$

under the some assumptions about $F$ and $G$, where $\lim_{N\to\infty} L\ T_N$ means the limiting distribution of $T$ and $\sigma_1^2$ is the asymptotic variance of $\sqrt{N}\,Q_N$. In order to prove the fact that $\hat{Q}_N$ has the same asymptotic distribution as the statistic $Q_N$, we must check up the regular conditions that have been given by Sukhatme. It is well known that $\sqrt{m}(\tilde{X}-\xi)$ and $\sqrt{n}(\tilde{Y}-\eta)$ are bounded in probability and each has a limiting distribution. Setting $A(t_1, t_2)$ as follows

$$A(t_1, t_2)=E\varphi(X_1-t_1, X_2-t_1, Y_1-t_2, Y_2-t_2|\xi=\eta=0), \tag{16}$$

where the exectation being taken with respect to all the $X$'s and $Y$'s, then it is necessary to show the following identities

$$\left.\frac{\partial A(t_1, t_2)}{\partial t_1}\right|_{t_1=t_2=0} = \left.\frac{\partial A(t_1, t_2)}{\partial t_2}\right|_{t_1=t_2=0} =0. \tag{17}$$

Since $A(t_1, t_2)$ is equal to the next probability from (16)

$$P(Y_1-t_2<X_1-t_1\ X_2-t_1<Y_2-t_2 \text{ or } Y_2-t_2<X_1-t_1\ X_2-t_1<Y_1-t_2),$$

we can transform $A(t_1, t_2)$ to the following integral representation

$$A(t_1, t_2) =\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F(y'+t_1-t_2)-F(y+t_1-t_2)\}^2 dG(y)dG(y'). \tag{18}$$

Hence it follows by differentiating (18) that

$$\left.\frac{\partial A(t_1, t_2)}{\partial t_1}\right|_{t_1=t_2=0} =4\{\int_{-\infty}^{\infty}F(x)f(x)dG(x)-\int_{-\infty}^{\infty}F(x)dG(x)\int_{-\infty}^{\infty}f(x)dG(x)\}.$$

Assume that $f(x)$ and $g(x)$ are symmetric about the origin, then we can prove after some computations that the value of the first partial derivative

to $t_1$ vanishes in $t_1=t_2=0$. Similarly we can get the identity

$$\frac{\partial A(t_1,t_2)}{\partial t_2}\Big|_{t_1=t_2=0}=0.$$

Next set

(19) $W(x_1,x_2,y_1,y_2,t_1,t_2)=\varphi(x_1-t_1,x_2-t_1,y_1-t_2,y_2-t_2)-A(t_1,t_2)$,

then it must be proved that the following inequality follows

(20) $E\,W(X_1,X_2,Y_1,Y_2,t_1,0)-W(X_1,X_2,Y_1,Y_2,0,0)\leqq M_1t_1$
$E\,|W(X_1,X_2,Y_1,Y_2,0,t_2)-W(X_1,X_2,Y_1,Y_2,0,0)\leqq M_2t_2$,

*where $M_1$ and $M_2$ are certain constants.* In fact, for $t_1>0$, we get

$$E\,W(X_1,X_2,Y_1,Y_2,t_1,0)-W(X_1,X_2,Y_1,Y_2,0,0)|$$
$$\leqq E\,|\varphi(X_1-t_1,X_2-t_1,Y_1,Y_2)-\varphi(X_1,X_2,Y_1,Y_2)+A(t_1,0)-A(0,0).$$

Hence it follows from (18)

(21) $$A(t_1,0)-A(0,0)\leqq 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{F(y'+t_1)-F(y')|+F(y+t_1)-F(y)\}dG(y')dG(y)$$
$$\leqq 4at_1,$$

since $F(y+t)-F(y)|\leqq at$ if the distribution function $F(x)$ has a derivative $f(x)$ bounded in absolute value by $a$. On the other hand, the expected value $E\,|\varphi(X_1-t_1,X_2-t_1,Y_1,Y_2)-\varphi(X_1,X_2,Y_1,Y_2)|$ is equal to the probability

(22) $P\{\varphi(X_1-t_1,X_2-t_1,Y_1,Y_2)=1$ and $\varphi(X_1,X_2,Y_1,Y_2)=0\}$
$+P\{\varphi(X_1-t_1,X_2-t_1,Y_1,Y_2)=0$ and $\varphi(X_1,X_2,Y_1,Y_2)=1\}$.

This probability is transformed to the following integral form

$$4\Big[\iint_{x<x'}\{G(x')-G(x'-t_1)\}G(x-t_1)dF(x)dF(x')$$
$$+\iint_{x<x'}\{G(x)-G(x-t_1)\}\{1-G(x')\}dF(x)\,dF(x)\Big]$$

and this value is bounded by

$$8\int_{-\infty}^{\infty}\{G(x)-G(x-t_1)\}dF(x)\leqq 8bt_1,$$

provided the distribution function $G(x)$ has a derivative $g(x)$ bounded in absolute value by $b$. It can be therefore shown that

(23) $E|W(X_1,X_2,Y_1,Y_2,t_1,0)-W(X_1,X_2,Y_1,Y_2,0,0)\leqq(4a+8b)t_1$.

Similarly it can be shown that

(24) $E\,W(X_1, X_2, Y_1, Y_2, 0, t_2) - W(X_1, X_2, Y_1, Y_2, 0, 0) \leqq (8a+4b)t_2.$

Thus the statistic $\hat{Q}_N$ has the same asymptotic normal distribution as the statistic $Q_N$ from the above results, so that it has been shown that the test based on the statistic $\hat{Q}_N$ is asymptotically distribution free and has the same asymptotic properties as the test on $Q_N$. This consequence is stated in the following theorem.

**Theorem** If the $X$'s and $Y$'s are distributed symmetrically about the respective medians and have bounded density functions, the test of the hypothesis $H$ based on the statistic $\hat{Q}_N$ is asymptotically distribution free.

SHIMANE UNIVERSITY.

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