A Note on a Nonparametric Two-Sample Test

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A NOTE ON A NONPARAMETRIC TWO-SAMPLE TEST

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1. Introduction

Let X and Y be the two random variables which follow the distribution functions of continuous type $F(x)$ and $G(y)$, respectively. The test of the hypothesis $H_0: F(x)=G(x)$ is treated here. For this purpose a slightly modified test of David-Okamoto's test [2], [6] will be applied.

Let $x_i, i=1, 2, \dots, N$, and $y_j, j=1, 2, \dots, M$, be N and M observations of X and Y . These x_i and y_j are mixed and arranged in order of magnitude. Denote these by

$$(1) \quad z_1 < z_2 < \dots < z_L$$

where $L=N+M$. We shall consider here only the case when $L=n\alpha$ where α is a fixed integer. Divide (1) into n classes from the left so that each class contains α members. That is

$$(2) \quad z_1, z_2, \dots, z_\alpha; z_{\alpha+1}, \dots, z_{2\alpha}; \dots; \dots, z_{n\alpha}.$$

The number of x_i involved in each class is denoted by $i_k, k=1, 2, \dots, n$. Consider a function of i

$$(3) \quad \begin{aligned} h(i) &= 0, & \text{when } i \neq 0, \\ &= 1, & \text{when } i = 0. \end{aligned}$$

Then

$$v = \sum_{k=1}^n h(i_k)$$

is the number of classes which contain no x 's. And $u=n-v$ is the number of classes which contain some x 's.

A large value of v suggests that the hypothesis should be rejected.

Therefore we use the v as a test function and call this test v -test.

In many occasions, a test of goodness of fit can be extended to a two sample test, Darling [1]. For example, Kolmogorov's test of fit by $D_n = \sup |F(x) - F_n(x)|$ has a corresponding two sample test by $D_{n,m} = \sup |F_n(x) - G_m(x)|$ while ω^2 test of fit by $\omega_n^2 = n \int_{-\infty}^{\infty} (F(x) - F_n(x))^2 dF(x)$ has a correspon-

ding two sample test by $\omega_{n,m}^2 = \frac{mn}{m+n} \int_{-\infty}^{\infty} (F_n(x) - G_m(x))^2 d\frac{nF_n + mG_m}{n+m}$.

David [2] proposed a test of fit by the following procedure. Let x_i , $i=1, 2, \dots, N$, be N independent observations of X . Since $F(x)$ is continuous, there are real numbers $\{a_i\}$, $i=0, 1, 2, \dots, n-1$ such that $F(a_{i+1}) - F(a_i) = 1/n$ where $a_0 = -\infty$ and $a_n = +\infty$. Let v_0 be the number of intervals (a_{i-1}, a_i) which contain no x 's. If v_0 is too large, the hypothesis that the distribution of X is $F(x)$ will be rejected. The detailed character of this test was treated by Okamoto [6], Weiss [7], Kitabatake [5], and the application of the test to life testing was treated by Ishii [4]. It is clear that the v -test above mentioned is a two sample test corresponding to the test of fit by David-Okamoto.

2. Distribution of ν

Let us consider the behavior of x 's in series (2). The probability that Nx 's are contained in at most k classes is

$$\binom{k\alpha}{N} / \binom{L}{N}.$$

This expression has a meaning only when $k\alpha \geq N$. When $k\alpha < N$, we let the probability 0.

Denote by p_i the probability that i classes in the above k classes contain all of the Nx 's and that each contains at least one of the x 's. Then we have

$$\binom{k\alpha}{N} / \binom{L}{N} = \sum_{i=1}^k \binom{k}{i} p_i.$$

Therefore for every positive integer ν

$$\begin{aligned} & \sum_{k=1}^{\nu} (-1)^{\nu-k} \binom{\nu}{k} \binom{k\alpha}{N} / \binom{L}{N} \\ &= \sum_{k=1}^{\nu} (-1)^{\nu-k} \binom{\nu}{k} \sum_{i=1}^k \binom{k}{i} p_i \\ &= \sum_{i=1}^{\nu} p_i \binom{\nu}{i} \sum_{k=i}^{\nu} (-1)^{\nu-k} \binom{\nu-i}{k-i} = p_i. \end{aligned}$$

Thus

$$P\{u=\nu\} = \binom{n}{\nu} p_{\nu} = \binom{n}{\nu} \sum_{k=1}^{\nu} (-1)^{\nu-k} \binom{\nu}{k} \binom{k\alpha}{N} / \binom{L}{N}.$$

Then

$$(4) \quad P\{v=\nu\} = \binom{n}{\nu} \sum_{k=1}^{n-\nu} (-1)^{n-\nu-k} \binom{n-\nu}{k} \binom{k\alpha}{N} / \binom{L}{N} \quad \nu=0, 1, \dots, n-1.$$

The s -th factorial moment of v is

$$(5) \quad E(v^{(s)}) = \frac{((n-s)\alpha)^{(s)} n!}{(n\alpha)^{(s)} (n-s)!},$$

where

$$x^{(s)} = x(x-1)\dots(x-s+1).$$

Putting $s=1, 2$

$$(6) \quad E(v) = n \frac{(L-\alpha)^{(N)}}{L^{(N)}}$$

$$E(v(v-1)) = n(n-1) \frac{(L-2\alpha)^{(N)}}{L^{(N)}}.$$

In the case when N is fixed and M tends to infinity, the probability (4) tends to $p(v=\nu)$ in Okamoto [6].

In the case when both N and M tend to infinity, we can assume many conditions on the increasing order of N, M, α and n . In the following we shall treat the case that α and n tend to infinity with order $\sqrt{L}$ and $M=n(\alpha-r), N=nr$ where r is fixed. In this case the factorial moment (5) becomes

$$E(v^{(s)}) \sim \left(1 - \frac{s}{n}\right)^{nr} \frac{n!}{(n-s)!}.$$

On the asymptotic behavior of v the following theorem is developed. The proof is similar to that of theorem 1 in Okamoto [6] and theorem 1 in Ishii [4]. Therefore, the proof is omitted.

Theorem 1.

v/n is asymptotically normally distributed with mean e^{-r} and variance $e^{-2r}(e^r - 1 - r)/n$, where $r = N/n = \text{const.}$

3. Application to life test.

Suppose that life testing of X and Y start with Nx items and My items and stops when $L_1 = n_1\alpha$ deaths in total occur, where L_1 is a certain preassigned number. Let $z_i, i=1, 2, \dots, L_1$, be the ordered observations of X and Y . Denote by N_1 and M_1 the number of X and Y in $L_1 z$'s.

$$L_1 = n_1\alpha = N_1 + M_1$$

$$L = n\alpha = N + M$$

$$L_1 \leq L.$$

Divide these z_i into n_1 classes in the same way as in section 1

$$(7) \quad z_1, z_2, \dots, z_\alpha; z_{\alpha+1}, \dots, z_{2\alpha}; \dots, z_{n_1\alpha}.$$

Let v_1 be the number of classes that contain no x 's in the above n_1 classes. We shall use the v_1 as a test function of testing the hypothesis H_0 .

N_1 follows the hypergeometric distribution

$$p\{N_1\} = \frac{\binom{N}{N_1} \binom{M}{M_1}}{\binom{L}{L_1}}.$$

The probability of v_1 under the condition that N_1 is fixed is

$$(8) \quad P\{v_1 = \nu, N_1\} = \binom{n_1}{\nu} \sum_{k=1}^{n_1-\nu} (-1)^{n_1-\nu-k} \binom{n_1-\nu}{k} \binom{k}{N_1}.$$

Then

$$(9) \quad P\{v_1 = \nu, N_1\} = P\{v_1 = \nu, N_1\} \cdot P\{N_1\}$$

$$(10) \quad P\{v_1 = \nu\} = \sum_{N_1} P\{v_1 = \nu, N_1\}.$$

The s -th factorial moment of v_1 is

$$E(v_1^{(s)}) = \frac{((n-s)\alpha)^{(N)} n_1!}{(n\alpha)^{(N)} (n_1-s)!}.$$

Putting $s=1, 2$

$$E(v_1) = n_1 \frac{(L-\alpha)^{(N)}}{L^{(N)}}$$

$$E(v_1(v_1-1)) = n_1(n_1-1) \frac{(L-2\alpha)^{(N)}}{L^{(N)}}.$$

In the case when N and n_1/n are fixed and M tends to infinity, the probability (9) tends to $P(v_1=\nu)$ in Ishii [4]. In the case when both N and M tend to infinity, we shall add a condition that $n_1/n=t$ is fixed to the assumptions in section 2. Under these assumptions we have the following theorem on the asymptotic behavior of v_1 . The proof is also similar to that of theorem 1 in Ishii [4]. Therefore it is omitted.

Theorem 2.

v_1/n_1 is asymptotically normally distributed with mean e^{-r} and variance $e^{-2r}(e^r-1-tr)/n$, where $r=N/n$ and $t=n_1/n$ are constants.

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