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LINEAR CONTROLLED STOCHASTIC PROCESSES WITH CONTINUOUS PARAMETER

By

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§ 1. **Summary and Introduction.** The theory of statistical controls was first discussed by T. Kitagawa [1], [2], as contained in successive processes. In a previous paper [3], the author studied this theory in detail by defining the linear controlled stochastic process (l. c. s. p.) for the discrete parameter case. A l. c. s. p. is the stochastic process which is transformed from an original stochastic process by the linear control. Some concrete types of discrete parameter l. c. s. p. and the fluctuation of the control system error were given in [3]. In the analogy to the discrete parameter l. c. s. p., we shall consider the continuous parameter l. c. s. p. in this paper. The definition and covariance function of continuous parameter l. c. s. p. are given in § 2 and § 3 respectively. In § 4 the mean square errors of continuous linear control system are considered. Some examples are given in § 5 to illustrate the structure and practical applications of continuous parameter l. c. s. p.. It is to be noted that throughout this paper we are concerned with the Loéve's second order random functions, and hence that in what follows the word "stochastic process" always implies such a second order random function.

§ 2. **Definition of continuous parameter l. c. s. p..** Let $\{x(t), -\infty<t<+\infty\}$ be a stochastic process and let us denote its mean and covariance functions by $m_x(t)$ and $r_x(t, s)$ respectively. In the correspondence to the vector $(\xi_1, \xi_2, \dots)$ and control matrix C defined in a previous paper [3], let us consider the two functions $\xi(t)$ and $C(u, v)$. We assume that the function $\xi(t)$ is continuous in $a\leq t<+\infty$ and the function $C(u, v)$ is bounded and measurable in $a\leq u, v<T$ for any T. In order to state the definition of continuous parameter l. c. s. p. we need the following

Lemma 2. 1. *If $r_x(s, t)$ is continuous in $a\leq s,\ t<+\infty$ then to each stochastic process $\{x(t), -\infty<t<+\infty\}$ there exists the stochastic process $\{y(t),\ a\leq t<+\infty\}$ satisfying the equation*

$$(2.1)\qquad y(t)=x(t)+\int_a^t du\int_a^u C(u, v)\{\xi(v)-y(v)\}\, dv,$$

and $y(t)$ *is given by*

$$y(t)=x(t)+\int_a^t K(t,u)\ \{\xi(u)-x(u)\}\ du, \tag{2.2}$$

with

$$K(s,t)=\sum_{k=1}^{\infty}(-1)^{k-1}g^{(k)}(s,t),$$

$$g^{(1)}(s,t)=g(s,t)=\int_t^s C(u,t)du, \tag{2.3}$$

$$g^{(n)}(s,t)=\int_t^s g^{(n-1)}(s,u)\ g(u,t)du, \qquad (n\geqq 2)$$

where the integral with respect to v *in* (2.1) *implies the limit in the quadratic mean.*

Proof. First we shall show the absolute convergence of $K(s,t)$ on each finite rectangle $(a,T)\times(a,T)$ for any assigned T. Let us put $\max_{a\leqq u,v\leqq T}|C(u,v)|=M$, then it is evident by the induction $|g^{(n)}(s,t)|\leq M^n|s-t|^{2n-1}$ $((2n-1)!)$, $(n=1,2,\dots)$, hence

$$|K(s,t)|\leq\sum_{k=1}^{\infty}|g^{(k)}(s,t)|<\sqrt{M}\ e^{\sqrt{M}|T-a|}, \tag{2.4}$$

which implies the absolute convergence of $K(s,t)$. The substitution of $y(t)$ defined by (2.2) into (2.1) shows directly the equation (2.1).

Now we define the continuous parameter l. c. s. p..

Definition 2.1. *To each stochastic process* $\{x(t),-\infty<t<+\infty\}$ *a stochastic process* $\{y(t),a\leqq t<+\infty\}$ *satisfying the following equation* (2.5) *is called the continuous parameter l. c. s. p. obtained from* $\{x(t),-\infty<t<+\infty\}$.

$$y(t)=x(t)+\int_a^t du\int_a^u C(u,v)\{\xi(v)-y(v)\}\ dv. \tag{2.5}$$

The integral with respect to v implies the limit in the quadratic mean and a denotes the starting time point of control. The uniqness of this solution is shown in the following

Theorem 2.1. *Let* $\{y(t),a\leqq t<+\infty\}$ *be a continuous parameter l.c. s.p. obtained from* $\{x(t),-\infty<t<+\infty\}$. *If mean functions* $m_x(t)$, $m_y(t)$ *and covariance functions* $r_x(s,t)$, $r_y(s,t)$ *are continuous in* $a\leqq t<+\infty$ *and* $a\leqq s,\ t<+\infty$ *respectively, then* $\{y(t),\ a\leqq t<+\infty\}$ *is uniquely determined by the equation* (2.5).

Proof. Let $y_1(t)$ and $y_2(t)$ be any two solutions of (2.5) which have the continuous mean and covariance functions. Let us put $y_1(t)-y_2(t)=z(t)$.

Then in view of (2.5) we have

$$z(t) = \int_a^t du \int_a^u C(u, v) \{-z(v)\} dv, \tag{2.6}$$

so that

$$|E\{z(t)\}| \leqq \int_a^t |E\{z(v)\} g(t, v)| dv \leqq \frac{1}{n!} \{\max_{a \leqq v \leqq T} |E\{z(v)\} M| \cdot |T-a|^n\}, \tag{2.7}$$

where the last expression tends to zero as n tends to infinity. Since we may take n as large as we please, we can conclude $E\{z(t)\}=0$. From (2.6) the covariance function of $z(t)$, say $r_z(s, t)$, satisfy the following equation

$$r_z(s, t) = \int_a^s \int_a^t r_z(u, v) g(s, u) \bar{g}(t, v) \ du \ dv. \tag{2.8}$$

Since $r_{y_i}(t, s)$ $(i=1, 2)$ are continuous in both variables then $r_z(t, s)$ is contінous in both variables. Hence

$$|r_z(t, s)| \leq \max_{a \leqq u, v \leqq T} |r_z(u, v)| M^2 |T-a| \cdot |T-s|. \tag{2.9}$$

By the same way to mean function (2.7) we see $r_z(t, s)=0$ on $(a, T) \times (a, T)$ for any finite T. Thus uniqueness was proved under the given assumptions.

§ 3. **Covariance function of continuous parameter l. c. s. p..** Direct calculation gives us

Theorem 3.1. *The covariance function of continuous parameter l. c. s. p. $\{y(t), a \leqq t < +\infty\}$ is given by*

$$r_y(t, s) = r_x(t, s) - \int_a^t K(t, u) r_x(u, s) du - \int_a^s \bar{K}(s, v) \bar{r}_x(v, t) dv + \tag{3.1}$$

$$+ \int_a^t K(t, u) du \int_a^s \bar{K}(s, v) r_x(u, v) \ dv.$$

Theorem 3.2. *Let $\{x(t), -\infty < t < +\infty\}$ be weakly stationary stochastic process and let $K(s, t) = K(s-t)$ belong to $L_1(0, \infty)$. If we put $a=-\infty$ in the definition 2.1 then l. c. s. p. $\{y(t), -\infty < t < +\infty\}$ is weakly stationary and its covariance function is given by*

$$r_y(h) = \int_0^\infty e^{-i\lambda h} |1-k(\lambda)|^2 \ dF_x(\lambda), \tag{3.2}$$

where

$$h(\lambda) = \int_0^\infty e^{-i\lambda h} \ K(h) dh, \tag{3.3}$$

and $F_x(\lambda)$ *is spectral distribution function of* $\{x(t), -\infty<t<+\infty\}$.

Proof. This theorem is reduced from the expression

$$y(t)=x(t)+\int_0^\infty K(z)\{\xi(t-z)-x(t-z)\}dz. \tag{3.4}$$

Our particular interest will be often concerned with the case where $x(t)$ is weakly stationary and control time $t-a$ is sufficiently large. For this case we can caluculate the stationary part of $r_y(s,t)$ by the Theorem 3.2.

§ 4. **Mean square error of continuous control sysytem.** Under some situations our main error of control system can be expressed by the expectation of square sum of manipulated values which is called mean square error of our control system. Since manipulated value is given by $\int_a^t du \int_a^u C(u,v)\times$ $\times\{\xi(v)-y(v)\}dv$, the mean square error $E\{S(t)\}$ is defined by

$$E\{S(t)\}=E\{\int_a^t \left|\int_a^s du\int_a^u C(u,v)(\xi(v)-y(v))\,dv\right|^2 ds\}. \tag{4.1}$$

Theorem 4.1. *We have*

$$E\{S(t)\}=\int_a^t |h(a,s)|^2\,ds+\int_a^t ds\int_a^s \bar{g}(s,u)\,du\int_a^s g(s,v)r_y(v,u)\,dv, \tag{4.2}$$

where

$$h(a,s)=\int_a^s\{\xi(v)-m_x(v)\}K(s,v)dv.$$

In particular when $\{y(t), a\leq t<+\infty\}$ *is weakly stationary, that is,*

$$r_y(v,u)=\int_{-\infty}^{+\infty} e^{i(v-u)\lambda}\,dF_y(\lambda), \tag{4.3}$$

then

$$E\{S(t)\}=\int_a^t \{|h(a,s)|^2+\int_{-\infty}^{+\infty}|G(s,a,\lambda)|^2 dF_y(\lambda)\}\,ds, \tag{4.4}$$

where we have put

$$G(s,a,\lambda)=\int_a^s g(s,v)e^{i\lambda v}dv. \tag{4.5}$$

Proof. In expression (4.1) if $\xi(v)-y(v)$ is replaced by $\{\xi(v)-m_y(v)\}$ $-\{y(v)-m_y(v)\}$ then we have

$$E\left[\int_a^t ds\int_a^s g(s,v)\,dv\int_a^s \bar{g}(s,u)\{(\xi(v)-m_y(v))-(y(v)-m_y(v))\}\times\right. \tag{4.6}$$

$$\left.\times\{(\bar{\xi}(u)-\bar{m}_y(u))-(\bar{y}(u)-\bar{m}_y(u))\}\,du\right].$$

Taking the expectation under integral sign we obtain

$$(4.7)\qquad E\{S(t)\} = \int_a^t \left| \int_a^s (\xi(v) - m_y(v)) g(s,v) dv \right|^2 ds + $$
$$+ \int_a^t ds \int_a^s \overline{g}(s,u)\, du \int_a^s g(s,v) r_y(v,u)\, dv.$$

The integral with respect to v of the first term on the right hand side is written

$$(4.8)\qquad \int_a^s \{\xi(v) - m_y(v)\} g(s,v)\, dv =$$
$$= \int_a^s g(s,v) \left\{ \int_a^v K(v,u)(m_x(u) - \xi(u)) du - (m_x(v) - \xi(v)) \right\} dv.$$

On the other hand, from (2.3) we obtain

$$(4.9)\qquad K(s,u) = g(s,u) - \int_a^s K(v,u) g(s,v)\, dv,$$

hence (4.8) reduces to

$$(4.10)\qquad \int_a^s \{\xi(v) - m_y(v)\} g(s,v) dv = \int_a^s K(s,u) \{\xi(u) - m_x(u)\} du$$
$$= h(a,s),$$

which proves (4.2). (4.4) is easily obtained by the substitution of (4.3) into (4.2).

In Theorem 4.1 $h(a,s)$ is the functional of the diffefence between $\xi(v)$ and $m_y(v)$ hence the first term of right hand side of (4.2) is the total weighting sum of the difference $\xi(v) - m_y(v)$ over the interval (a,t). Therefore if our control system is constructed so as to satisfy the condition $\xi(v) = m_y(v)$, $v \geqq a$, then the first term of right hand side of (4.2) vanishes. Consequently $E\{S(t)\}$ is considered as the measure of our control system error. Since the condition of unbiasedness $\xi(v) = m_y(v)$, $v \geqq a$, is equivalent to

$$(4.11)\qquad m_x(u) - \xi(u) = \int_a^u K(u,v) \{m_x(v) - \xi(v)\} dv, \qquad u \geqq a$$

to each $m_x(u) - \xi(u)$ we must decide the $K(u,v)$ satisfying the integral equation (4.11) and minimizing $E\{S(t)\}$ for any fixed t.

§ 5. **Examples of continuous parameter l. c. s. p.'s.** In this § we give some types of $C(u,v)$ and the corresponding continuous parameter l. c. s. p.'s

$\{y(t),\ 0\leqq t<+\infty\}$ obtained from the weakly stationary stochastic process $\{x(t),\ -\infty<t<+\infty\}$. We denote the stationary part of $r_y(s,t)$ by $\tilde{r}_y(s,t)$.

[A]. *General exponential type.* We take $C(u,t)=d_u[e^{\beta(u-t)}+\delta(u,t)]$, where $\delta(u,t)=0$ for $u<t$, $\delta(u,t)=1$ for $u\geqq t$, and β is any real or complex number. Then $g(s,t)=e^{\beta(s-t)}$ and $K(s,t)=e^{(\beta-1)(s-t)}$ so that

$$y(t)=x(t)+\int_0^t e^{(\beta-1)(t-u)}\{\xi(u)-x(u)\}du. \tag{5.1}$$

From Theorem 3.2 $\tilde{r}_y(h)$ becomes

$$\tilde{r}_y(h)=\int_{-\infty}^{+\infty} e^{i\lambda h}\frac{\beta^2+\lambda^2}{(\beta-1)^2+\lambda^2}\,dF_x(\lambda) \tag{5.2}$$

where we have assumed that β is real and $\beta\leqq 1$. Accordingly the mean square error corresponding to $\tilde{r}_y(h)$ is given by

$$E\{S(t)\}=\int_0^t\Big\{\Big|\int_0^s(\xi(v)-m_x(v))\,e^{(\beta-1)(s-v)}\,dv\Big|^2+ \tag{5.3}$$

$$+\int_{-\infty}^{+\infty}\Big|\frac{e^{i\lambda s}-e^{\beta s}}{\beta-i\lambda}\Big|^2\,dF_x(\lambda)\Big\}\,ds.$$

[B]. *Power type.* We take $C(u,t)=\alpha(u-t)^k$ where $\alpha>0$ and k is non-negative integer. Then $g(s,t)=\frac{\alpha}{\alpha+1}(s-t)^{k+1}$ and $K(s,t)=\sum_{k=1}^{\infty}(-1)^{n-1}\times$ $\times\frac{(\alpha\cdot k!)^n}{\{n(k+2)-1\}!}(s-t)^{n(k+2)-1}$. For example we take $k=0$, then $K(s,t)=$ $=\sqrt{\alpha}\sin\sqrt{\alpha}\,(s-t)$ and

$$y(t)=x(t)+\int_0^t\sqrt{\alpha}\sin\sqrt{\alpha}\,(t-u)\{\xi(u)-x(u)\}du. \tag{5.4}$$

Using Theorem 3.1

$$\tilde{r}_y(h)=\int_{-\infty}^{+\infty}e^{i\lambda h}\frac{\lambda^4-\alpha^2+4\alpha}{(\alpha-\lambda^2)^2}\,dF_x(\lambda). \tag{5.5}$$

In this case the mean square error corresponding to $\tilde{r}_y(h)$ is given by

$$E\{S(t)\}=\int_0^t\Big\{\Big|\int_0^s(\xi(v)-m_x(v))\sqrt{\alpha}\sin\sqrt{\alpha}\,(s-v)\,dv\Big|^2+ \tag{5.6}$$

$$+\int_{-\infty}^{+\infty}\Big|\frac{1-i\lambda s-e^{i\lambda s}}{\alpha}\Big|^2\frac{\lambda^4-\alpha^2+4\alpha}{(\alpha-\lambda^2)^2}\,dF_x(\lambda)\Big\}\,ds.$$

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References

[1] T. KITAGAWA, *Successive processes of statistical controls* (1), Mem. Fac. Sci., Kyushu Univ. Ser. A, Vol. **7** (1952), pp. 13–28.
[2] T. KITAGAWA, *Successive processes of statistical controls* (2), Mem. Fac. Sci., Kyushu Univ. Ser. A, Vol. **13**, No. 1 (1959), pp. 1–16.
[3] S. KANÔ, *Linear controlled stochastic processes,* Science Reports of Kagoshima Univ., No. 8, (1959), pp. 1–14.
[4] MICHEL LOÈVE, *Probability theory,* New York (1955).