Succesive Process of Statistical Inferences Applied to Linear Regression Analysis and Its Specialisations to Response Surface Analysis

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SUCCESSIVE PROCESS OF STATISTICAL INFERENCES APPLIED TO LINEAR REGRESSION ANALYSIS AND ITS SPECIALISATIONS TO RESPONSE SURFACE ANALYSIS

By

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§ 0. Introduction. The object of this paper is to discuss some aspects of sequential approaches associated with linear regression analysis under a formulation for which the estimation after preliminary test or tests of significance will play a fundamental rôle. The statistical procedures in which one or several preliminary tests of significance are involved before a final statistical decision such as test of significance or estimation will be applied have been discussed by various authors such as Bancroft [1], Kitagawa [1] $\sim$ [4], Bennett [1], Paull [1], Siotani [1], Bozivich, Bancroft and Hartley [1], and Huntsberger [1].

These procedures can naturally be applied to linear regression analysis. In § 1 we shall discuss biased estimation of linear regression coefficients under an incompletely specified model in the terminology used in Bezivich, Bancroft and Hartley [1], and then in § 2 estimations after preliminary test of significance are treated for the two and three stage cases. In § 3 and § 4 sequential designs of experiments in two and three stages are formulated respectively under specified statistical procedures in which test or tests of significance will decide whether we should perform further experiments in order to obtain more data for a better fit as linear regression or we should stop our experiments and give our estimation of linear regression on the basis of our data already obtained. In Theorems 1 and 2 the mean square error of our estimated regression function $\hat{\varphi}(x)$ to the true regression function $\varphi(x)$ is given for each assigned value of the independent argument $x=(x_1, x_2, \cdots, x_k)$, and then the overall distance function is defined over a given domain $\mathfrak{D}$ of the k-dimensional space by means of given weight function $w(x)$ and their expectation is considered.

Now the latter part of this paper is mainly concerned with the applications of general results obtained in the former to response surface analysis developed by Box and his collegues in a series of papers such as Box-Wilson [1], Box [1] $\sim$ [3], Box-Hunter [1] $\sim$]2]. Sequential approaches in some sense have been advocated in their approaches to response surface analysis.

Indeed Box-Hunter [2] gives five requirements (a)$\sim$(e) to experimental designs of order d which are desirable in their context. In particular two requirements (b) and (e) are intimately connected with our purpose of the present paper, because they say:

(b) It should allow a check to be made on the representational accuracy of the assumed polynomial.

(e) It should form a nucleus from which a statisfactory design of order $d+1$ can be built in case the assumed degree of polynomial proves inadequate.

However they did not give a formulation of statistical procedures to be applied to such sequential approaches. It is our view point that some formulation of statistical procedures can be done under certain situation and it will be of use for an experimenter by giving him the effects of his statistical procedures in term of the mean square errors and of the distance functions and also by giving him the possibility of getting an objective criterium to decide among possible sequences of designs of experiments.

§ 5 gives three examples of § 2 with respect to response surface analysis such as rotatable designs of the order two and three in the sense of Box-Hunter [2] in the two dimensions. § 6 gives one example of § 3 in connection with the rotatable design of the order two in response surface analysis, while § 7 treats with that of § 4 in connection with the rotatable design of the order three. These examples are concerned with the designs proposed (probably first) by Gardiner, Grandage and Hader [1] and discussed in Box-Hunter [2]. In § 8 an example of three stage sequential design associated with response surface analysis in the two dimensions is discussed. Numerical aspects which are quite important in discussing the merits and the demerits of statistical procedures formulated in this paper are not given here, but in § 9 some general remarks are given for the method of the evaluations of the probabilities and the mean values associated with the estimations after preliminary tests of significance.

In conclusion the author wishes to express his indebtness in preparing this paper to the talks with Prof. S. Wilks and Dr. G. Box which he could enjoy while he was in Princeton University from 1957 December to 1958 April by the Rockefeller Foundation grant.

§ 1. Biased estimation of linear regression coefficients under an incompletely specified model. Let us consider a linear regression model involving q unknown parameters $\beta_1, \beta_2, \cdots, \beta_{q-1}$ and β_q such that

$$(1.01) \qquad \varphi_q(x_1, x_2, \cdots, x_q) = \sum_{i=1}^{q} \beta_i x_i ,$$

where $x_1, x_2, \cdots, x_{q-1}$ and x_q are fixed variates.

Let D be a $q \times N$ design matrix such that

$$(1.02) \qquad D = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{q1} \\ x_{12} & x_{22} & \cdots & x_{q2} \\ \cdots & \cdots & \cdots & \cdots \\ x_{1\alpha} & x_{2\alpha} & \cdots & x_{q\alpha} \\ \cdots & \cdots & \cdots & \cdots \\ x_{1N} & z_{N2} & \cdots & x_{qN} \end{pmatrix}.$$

The observed value y_α at the point $x_\alpha = (x_{1\alpha}, x_{2\alpha}, \cdots, x_{q\alpha})$ is assumed to be

$$(1.03) \qquad y_\alpha = \sum_{i=1}^{q} \beta_i x_{i\alpha} + \varepsilon_\alpha ,$$

where $\{\varepsilon_\alpha\}$ $(\alpha = 1, 2, \cdots, N)$ are assumed to be distributed independently in the normal distribution $N(0, \sigma^2)$ with a common unknown variance σ^2.

We are now interested with the situation of an experimenter for whom the model (1.01) is not completely specified and who may assume under his own grounds a response function of the form

$$(1.04) \qquad \varphi_p(x_1, x_2, \cdots, x_q) = \sum_{i=1}^{p} \beta_i x_i$$

with a certain number p of unknown parameters $\beta_1, \beta_2, \cdots, \beta_{p-1}$ and β_p, where $p < q$.

Under this situation he may think it better to have the least square estimates of $\beta_1, \beta_2, \cdots, \beta_{p-1}$ and β_p under his assumption $\beta_{p+1} = \beta_{p+2} = \cdots = \beta_q = 0$. It is the purpose of this paragraph to discuss the biased estimations of the parameters and that of the response function to be derived under such situation.

The current procedure of the least square estimations will give him the normal equations

$$(1.05) \qquad \sum_{j=1}^{p} a_{ij} b_j = B_i , \qquad (i = 1, 2, \cdots, p)$$

where

$$(1.06) \qquad a_{ij} = \sum_{\alpha=1}^{N} x_{i\alpha} x_{j\alpha} , \qquad (i, j = 1, 2, \cdots, p)$$

$$(1.07) \qquad B_i = \sum_{\alpha=1}^{N} x_{i\alpha} y_\alpha . \qquad (i = 1, 2, \cdots, p) .$$

It is to be noted that $\{a_{ij}\}$ could be defined not only for $i, j = 1, 2, \cdots, p$ but also for $i, j = 1, 2, \cdots, q$.

Let the rank of the $p \times p$ matrix (a_{ij}) $(i, j = 1, 2, \cdots, p)$ be equal to p, and let its $p \times p$ inverse matrix be denoted by $(c_{ij}^{(1)}) = (a_{ij})^{-1}$.

Let us put

$$(1.08) \qquad e_\alpha \equiv y_\alpha - (b_1 x_{1\alpha} + b_2 x_{2\alpha} + \cdots + b_p x_{p\alpha}) \qquad (1 \leqq \alpha \leqq N)$$

(1.09) $\Delta_{ik} \equiv \sum_{j=1}^{p} c_{ij}^{(1)} a_{jk} \qquad (i=1, 2, \cdots, p;\ p+1 \leqq k \leqq q)$

(1.10) $A_{k\alpha} \equiv x_{k\alpha} - \sum_{i=1}^{p} \sum_{j=1}^{p} c_{ij}^{(1)} a_{jk} x_{i\alpha} \quad (p+1 \leqq k \leqq q;\ 1 \leqq \alpha \leqq N)$

(1.11) $H_\alpha \equiv \sum_{k=p+1}^{q} \beta_k A_{k\alpha} \qquad (1 \leqq \alpha \leqq N)$

(1.12) $L_i \equiv \sum_{\alpha=1}^{N} x_{i\alpha} H_\alpha \qquad (1 \leqq i \leqq q)$

(1.13) $\eta_i \equiv \sum_{\alpha=1}^{N} x_{i\alpha} \varepsilon_\alpha \qquad (1 \leqq i \leqq q)$.

Then the fundamental aspects of the biased estimations under the present situation can be enunciated in

Lemma 1. (1°) *We have*

(1.14) $$b_i = \beta_i + \sum_{k=p+1}^{q} \beta_k \Delta_{ik} + \sum_{\alpha=1}^{N} \varepsilon_\alpha \Big(\sum_{j=1}^{p} c_{ij}^{(1)} x_{j\alpha}\Big) \qquad (1 \leqq i \leqq p)$$

(1.15) $$\sum_{\alpha=1}^{N} e_\alpha^2 = \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)^2 - \sum_{i=1}^{p} \sum_{j=1}^{p} c_{ij}^{(1)} (\eta_i + L_i)(\eta_j + L_j) + \sum_{i=1}^{p} \sum_{j=1}^{p} c_{ij}^{(1)} L_i L_j .$$

(2°) $(\sum_{\alpha=1}^{N} e_\alpha^2 - \sum_{i=1}^{p} \sum_{j=1}^{p} c_{ij}^{(1)} L_i L_j) \sigma^{-2}$ *is distributed according to the non-central chi-square distribution with the* $(N-p)$ *degrees of freedem and the non-centricity parameter*

(1.16) $$\lambda = \sum_{\alpha=1}^{N} H_\alpha^2 \Big/ 2\sigma^2 .$$

Specially when $\beta_{p+1} = \beta_{p+2} = \cdots = \beta_q = 0$, *then* $\sum_{\alpha=1}^{N} e_\alpha^2 / \sigma^2$ *is distributed according to the central chi-square distribution with the* $(N-p)$ *degrees of freedom.*

Proof. *Ad* (1°): (1.14) follows from (1.05) in view of (1.09). Indeed we have

(1.17) $$\begin{aligned} b_i &= \sum_{j=1}^{p} c_{ij}^{(1)} B_j = \sum_{j=1}^{p} c_{ij}^{(1)} \Big(\sum_{\alpha=1}^{N} x_{j\alpha} \Big(\sum_{k=1}^{q} \beta_k x_{k\alpha} + \varepsilon_\alpha\Big)\Big) \\ &= \sum_{j=1}^{p} c_{ij}^{(1)} \Big(\sum_{k=1}^{q} \beta_k a_{jk} + \sum_{\alpha=1}^{N} x_{j\alpha} \varepsilon_\alpha\Big) \\ &= \beta_i + \sum_{k=p+1}^{q} \beta_k \Delta_{ik} + \sum_{j=1}^{p} c_{ij}^{(1)} \sum_{\alpha=1}^{N} x_{j\alpha} \varepsilon_\alpha . \end{aligned}$$

Now we have, in view of (1.10),

$$(1.18)\qquad \begin{aligned} e_\alpha &= y_\alpha - \sum_{i=1}^{p} b_i x_{i\alpha} \\ &= \sum_{i=1}^{q} \beta_i x_{i\alpha} + \varepsilon_\alpha - \sum_{i=1}^{p} x_{i\alpha}\Big(\beta_i + \sum_{k=p+1}^{q} \beta_k \Delta_{ik}\Big) \\ &\qquad - \sum_{i=1}^{p} x_{i\alpha} \sum_{j=1}^{p} c_{ij}^{(1)} \sum_{\beta=1}^{N} x_{j\beta}\varepsilon_\beta \\ &= \sum_{k=p+1}^{q} \beta_k A_{k\alpha} + \sum_{\beta=1}^{N}\Big(\delta_{\beta\alpha} - \sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}^{(1)} x_{i\alpha} x_{j\beta}\Big)\varepsilon_\beta , \end{aligned}$$

which, in view of (1.11), (1.12) and (1.13), gives us (1.15), as we were to prove.

Ad (2°): The remaining part of our Lemma 1 can be proved by the transformation of the quadratic form

$$(1.19)\qquad Q \equiv \sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}^{(1)} \zeta_i \zeta_j = \sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}^{(1)} (\eta_i + L_i)(\eta_j + L_j) , \qquad (\zeta_i \equiv \eta_i + L_i) ,$$

into the sum of squares. Indeed the classical method due to Jacobi's transformation yields us, in our particular case, that

$$(1.20)\qquad Q = \frac{Z_1^2}{D_{1,1}} + \frac{Z_2^2}{D_{1,1}D_{2,2}} + \cdots + \frac{Z_p^2}{D_{p-1,p-1}D_{p,p}}$$

with

$$(1.21)\qquad Z_i = D_{i,i}\zeta_i + D_{i,i+1}\zeta_{i+1} + \cdots + D_{i,p}\zeta_p$$

$$(1.22)\qquad \mathrm{Cov.}(Z_i, Z_j) = 0 \qquad (i \neq j)$$

$$(1.23)\qquad \mathrm{Var.}(Z_i) = \sigma^2 D_{i,i} D_{i-1,i-1} ,$$

where

$$(1.24)\qquad D_{i,h} = \begin{vmatrix} c_{11}^{(1)} & c_{12}^{(1)} & \cdots & c_{1,i-1}^{(1)} & c_{1h}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} & \cdots & c_{2,i-1}^{(1)} & c_{2h}^{(1)} \\ \vdots & & & \vdots & \vdots \\ c_{i-1,1}^{(1)} & c_{i-1,2}^{(1)} & \cdots & c_{i-1,i-1}^{(1)} & c_{i-1,h}^{(1)} \\ c_{i1}^{(1)} & c_{i2}^{(1)} & \cdots & c_{i,i-1}^{(1)} & c_{i,h}^{(1)} \end{vmatrix} ,$$

for $1 \leq i \leq h \leq p$.

Consequently we have, in view of (1.20), (1.12) and (1.13),

$$(1.25)\qquad \begin{aligned} Z_i (D_{i-1,i-1} D_{i,i})^{-1/2} &= \sum_{\alpha=1}^{N} (D_{i-1,i-1} D_{i,i})^{-1/2} \sum_{j=i}^{p} D_{i,j} x_{j\alpha} (H_\alpha + \varepsilon_\alpha) \\ &\equiv \sum_{\alpha=1}^{N} d_{i\alpha} (H_\alpha + \varepsilon_\alpha) , \end{aligned}$$

where

$$(1.26)\qquad d_{i\alpha} \equiv (D_{i-1,i-1} D_{i,i})^{-1/2} \sum_{j=i}^{p} D_{i,j} x_{j\alpha}$$

and

$$\sum_{\alpha=1}^{N} d_{i\alpha}^2 = 1, \quad \sum_{\alpha=1}^{N} d_{i\alpha} d_{j\alpha} = 0 \qquad (i \neq j) \tag{1.27}$$

in virtue of (1.24) and (1.26).

We can apply the same transformation to the last term in the right-hand side of (1.15). We have finally

$$\sum_{\alpha=1}^{N} e_\alpha^2 = \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)^2 - \sum_{i=1}^{p} \Big(\sum_{\alpha=1}^{N} d_{i\alpha} (H_\alpha + \varepsilon_\alpha) \Big)^2 + \sum_{i=1}^{p} \Big(\sum_{\alpha=1}^{N} d_{i\alpha} H_\alpha \Big)^2 , \tag{1.28}$$

which completes the proof of Lemma 1.

§ **2. Estimations after preliminary test of significance.** Under the situation of the incompletely specified model enunciated in § 1, the experimenter may appeal to the estimation procedure after preliminary test of significance. It is the purpose of this paragraph to formulate some of his possible procedures on two fundamental cases.

In addition to the assumption made in § 1, let u_ν^2 be an unbiassed estimate of the variance σ^2 such that $\nu u_\nu^2/\sigma^2$ is distributed in the *chi*-square distribution with the ν degrees of freedom, and let u_ν^2 and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N$ be mutually independent. This assumption can be satisfied in various important cases. For instance, (i) u_ν^2 may be obtained by another independent experiment, or (ii) for each assigned $x_\alpha = (x_{1\alpha}, x_{2\alpha}, \cdots, x_{q\alpha})$, there may be m replications of indpendent experiments giving us $y_\alpha^{(1)}, y_\alpha^{(2)}, \cdots, y_\alpha^{(m)}$ $(\alpha = 1, 2, \cdots, N)$ and hence

$$y_\alpha = (y_\alpha^{(1)} + y_\alpha^{(2)} + \cdots + y_\alpha^{(m)})/m$$

$$u_\nu^2 = \sum_{\alpha=1}^{N} \sum_{h=1}^{m} (y_\alpha^{(h)} - y_\alpha)^2/(m-1)N, \qquad \nu = (m-1)N.$$

[1] **Two stage case.** Let us define our statistical procedure in the following way.

(1°) Test the hypothesis $H_0(p; q)$: $\beta_{p+1} = \beta_{p+2} = \cdots = \beta_q = 0$ at an assigned level of significance α_1 by appealing to the F-test where

$$F = \sum_{\alpha=1}^{N} \Big(y_\alpha - \sum_{j=1}^{p} b_j^{(1)} x_{j\alpha} \Big)^2 \Big/ (N-p) u_\nu^2 . \tag{2.01}$$

$(2°)_1$ If the F-test is non-significant (A_1), that is, $F \leq F_\nu^{N-p}(\alpha_1)$, which is the α_1-point of the F-distribution with the pair of the degrees of freedom $(N-p, \nu)$, we give the estimates

$$\begin{cases} b_i = b_i^{(1)} & (i = 1, 2, \cdots, p) \\ b_i = 0 & (i = p+1, \cdots, q) . \end{cases} \tag{2.02}$$

$(2^\circ)_2$ If the F-test is significant (R_1), that is $F > F_\nu^{N-p}(\alpha_1)$, we give the estimates

$$b_i = b_i^{(2)} \qquad (i = 1, 2, \cdots, q) , \tag{2.03}$$

where $\{b_i^{(2)}\}$ are the solutions of the normal equations

$$\sum_{i=1}^{q} a_{ij} b_j^{(2)} = B_i \qquad (i = 1, 2, \cdots, q) , \tag{2.04}$$

with

$$a_{ij} = \sum_{\alpha=1}^{N} x_{i\alpha} x_{j\alpha} \qquad (i, j = 1, 2, \cdots, q) \tag{2.05}$$

$$B_i = \sum_{\alpha=1}^{N} x_{i\alpha} y_\alpha \qquad (i = 1, 2, \cdots, q) . \tag{2.06}$$

By means of our estimates $\{b_i\}$ $(i = 1, 2, \cdots, q)$ let us define the estimate of the response function by

$$\hat{\varphi}(x) = b_1 x_1 + b_2 x_2 + \cdots + b_q x_q \tag{2.07}$$

for each assigned point $x = (x_1, x_2, \cdots, x_q)$, which is equal to

$$\hat{\varphi}(x) = b_1^{(1)} x_1 + b_2^{(1)} x_2 + \cdots + b_p^{(1)} x_p \tag{2.08}$$

under the circumstance (A_1) and to

$$\hat{\varphi}(x) = b_1^{(2)} x_1 + b_2^{(2)} x_2 + \cdots + b_q^{(2)} x_q \tag{2.09}$$

under the circumstance (R_1) respectively.

It is evident that $\hat{\varphi}(x)$ does not generally give an unbiassed estimate of $\varphi_q(x)$ for each assigned x. Consequently for each assigned x we shall be interested with the mean square error defined by

$$E\left\{\left|\hat{\varphi}(x) - \varphi_q(x)\right|^2\right\} = \sum_{i=1}^{q} \sum_{j=1}^{q} E\left\{(b_i - \beta_i)(b_i - \beta_j)\right\} x_i x_j . \tag{2.10}$$

Furthermore it may be interesting to consider the norm such as

$$E\{Q\} \equiv E\left\{\int_{\mathfrak{D}} \left|\hat{\varphi}(x) - \varphi_q(x)\right|^2 w(x)\, dx_1 \cdots dx_q\right\} , \tag{2.11}$$

where $w(x) \equiv w(x_1, x_2, \cdots, x_q)$ is an assigned weight-function and $\mathfrak{D}$ is the domain in our consideration in the q-dimensional euclidian space.

[2] **Three stage case.** Let us consider the situation when the true response function is given by

$$\varphi_r(x) = \sum_{i=1}^{r} \beta_i x_i . \tag{2.12}$$

Let us suppose an experimenter for whom the model is not completely specified and who may assume other response functions such as $\varphi_p(x)$ or

$\varphi_q(x)$, where p and q $(1 \leq p < q < r)$ are chosen by him on his own grounds. Let us define our statistical procedure in the following way.

(1°) Test the hypothesis $H_0(p;r)$: $\beta_{p+1} = \beta_{p+2} = \cdots = \beta_r = 0$ at an assigned level of significance α_1 by appealing to the F-test where F is defined as in (2.01).

$(2°)_1$ If the F-test is non-significant (A_1), proceed similarly as in $(2°)_1$ in the two stage case.

$(2°)_2$ If the F-test is significant (R_1), then let us define $\{b_i^{(2)}\}$ $(i = 1, 2, \cdots, q)$ as in $(2°)_2$ in the two stage case.

Test the hypothesis $H_0(q;r)$: $\beta_{q+1} = \beta_{q+2} = \cdots = \beta_r = 0$ at an assigned level of significance α_2 by appealing to the F-test where F is defined by

$$F = \sum_{\alpha=1}^{N}\left(y_\alpha - \sum_{j=1}^{q} b_j^{(2)} x_{j\alpha}\right)^2 \Big/ (N-q)\, u_\nu^2 . \tag{2.13}$$

$(3°)_1$ If the F-test is non-significant (A_2), that is, $F \leq F_\nu^{N-q}(\alpha_2)$, where α_2 is an assigned level of significance, then we give the estimates

$$\begin{cases} b_i = b_i^{(2)} & (i = 1, 2, \cdots, q) \\ b_i = 0 & (i = q+1, \cdots, r) . \end{cases} \tag{2.14}$$

$(3°)_2$ If the F-test is significant (R_2), that is, $F > F_\nu^{N-p}(\alpha_2)$, then we give the estimates

$$b_i = b_i^{(3)} \qquad (i = 1, 2, \cdots, r) , \tag{2.15}$$

where $\{b_i^{(3)}\}$ are the solutions of the normal equations

$$\sum_{i=1}^{r} a_{ij} b_j^{(3)} = B_i \qquad (i = 1, 2, \cdots, r) \tag{2.16}$$

with

$$a_{ij} = \sum_{\alpha=1}^{N} x_{i\alpha} x_{j\alpha} \qquad (i, j = 1, 2, \cdots, r) \tag{2.17}$$

$$B_i = \sum_{\alpha=1}^{N} x_{i\alpha} y_\alpha \qquad (i = 1, 2, \cdots, r) . \tag{2.18}$$

Under this procedure, let us define an estimate of the response function by

$$\hat{\varphi}(x) = b_1 x_1 + b_2 x_2 + \cdots + b_r x_r , \tag{2.19}$$

which is equal to

$$\begin{cases} \hat{\varphi}(x) = b_1^{(1)} x_1 + \cdots + b_p^{(1)} x_p & (\text{under } (A_1)) \\ \hat{\varphi}(x) = b_1^{(2)} x_1 + \cdots + b_q^{(2)} x_q & (\text{under } (R_1 A_2)) \\ \hat{\varphi}(x) = b_1^{(3)} x_1 + \cdots + b_r^{(3)} x_r & (\text{under } (R_1 R_2)) \end{cases} \tag{2.20}$$

respectively. We shall be interested with

$$E\left\{\left|\hat{\varphi}(x) - \varphi_r(x)\right|^2\right\} \tag{2.21}$$

for each assigned $x = (x_1, x_2, \cdots, x_r)$, and also with

$$E\{Q\} \equiv E\left\{\int_{\mathfrak{D}} \cdots \int \left|\hat{\varphi}(x) - \varphi_r(x)\right|^2 w(x)\, dx_1 \cdots dx_r\right\} \tag{2.22}$$

with a certain weight function $w(x) = w(x_1, x_2, \cdots, x_r)$.

We are not giving here the evaluations of the quantities (2.10), (2.11), (2.21) and (2.22), because they can be readily seen from the corresponding evaluations to be given in the following two paragraphs in connection with the sequential designs in which two or three stage designs of experiments can be performed in order to have better fitting for response function by appealing to estimations after preliminary test (or tests) of significance.

§ 3. **Two stage sequential design.** Let us consider the two stage sequential design defined in the following way under the incompletely specified linear regression model in the sense enunciated in § 1.

(1°) (i) Make the first experiment involving N_1 observations $\{y_\alpha\}$ ($\alpha = 1, 2, \cdots, N_1$) according to the $p \times N_1$ design matrix $D^{(1)}$ defined by

$$D^{(1)} = \begin{pmatrix} x_{11}^{(1)} & \cdots & x_{p1}^{(1)} \\ x_{12}^{(1)} & \cdots & x_{p2}^{(1)} \\ \cdots & \cdots & \cdots \\ x_{1\alpha}^{(1)} & \cdots & x_{p\alpha}^{(1)} \\ \cdots & \cdots & \cdots \\ x_{1N_1}^{(1)} & \cdots & x_{pN_1}^{(1)} \end{pmatrix}. \tag{3.01}$$

(ii) Let an estimate $b_i^{(1)}$ of β_i be given by

$$b_i^{(1)} = \sum_{j=1}^{p} c_{ij}^{(1)} B_j^{(1)} \qquad (i = 1, 2, \cdots, p), \tag{3.02}$$

where $(c_{ij}^{(1)})$ is the inverse matrix of the $p \times p$ matrix $(a_{ij}^{(1)})$ $(i, j = 1, 2, \cdots, p)$ with

$$a_{ij}^{(1)} = \sum_{\alpha=1}^{N_1} x_{i\alpha}^{(1)} x_{j\alpha}^{(1)} \qquad (i, j = 1, 2, \cdots, p) \tag{3.031}$$

and

$$B_j^{(1)} = \sum_{\alpha=1}^{N_1} x_{j\alpha}^{(1)} y_\alpha \qquad (j = 1, 2, \cdots, p). \tag{3.032}$$

(iii) Let u_ν^2 be an unbiassed estimate of the variance σ^2 such that $\nu u_\nu^2/\sigma^2$ is distributed in the *chi*-square distribution with the ν degrees of freedom, and let u_ν^2 and $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{N_1-1}\ \varepsilon_{N_1})$ be mutually independent.

(iv) Test the hypothesis $H_0(q; p)$: $\beta_{p+1} = \beta_{p+2} = \cdots = \beta_q = 0$ at an assigned level of significance α_1 by appealing to the F-test where

(3.04) $$F=\sum_{\alpha=1}^{N_1}\left(y_\alpha-\sum_{j=1}^{p} b_j^{(1)}x_{j\alpha}\right)^2 (N_1-p)u_\nu^2 .$$

$(2^\circ)_1$ If the F-test is non-significant (A_1), that is, $F\leqq F_\nu^{N_1-p}(\alpha_1)$, then we stop the experimentation, and we give the estimates

(3.05) $$\begin{cases} b_i=b_i^{(1)} & (i=1,2,\cdots,p) \\ b_i=0 & (i=p+1,p+2,\cdots,q) . \end{cases}$$

$(2^\circ)_2$ If the F-test is significant (R_1), that is, $F>F_\nu^{N_1-p}(\alpha_1)$, we proceed in the following way.

(i) Make the second experiment involving (N_2-N_1) further observations $\{y_\theta\}$ $(\theta=N_1+1, N_1+2, \cdots, N_2)$ according to the $q\times N_2$ design matrix $D^{(2)}$ defined by

(3.06) $$D^{(2)}=\begin{pmatrix} x_{11}^{(2)} & x_{21}^{(2)} & \cdots & x_{p1}^{(2)} & x_{p+11}^{(2)} & \cdots & x_{q1}^{(2)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{1N_1}^{(2)} & x_{2N_1}^{(2)} & \cdots & x_{pN_1}^{(2)} & x_{p+1N_1}^{(2)} & \cdots & x_{qN_1}^{(2)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{1N_2}^{(2)} & x_{2N_2}^{(2)} & \cdots & x_{pN_2}^{(2)} & x_{p+1N_2}^{(2)} & \cdots & x_{qN_2}^{(2)} \end{pmatrix},$$

where we assume that

(3.07) $$x_{i\alpha}^{(2)}=x_{i\alpha}^{(1)} \qquad (i=1,2,\cdots,p;\ \alpha=1,2,\cdots,N_1) ,$$

and that $x_{k\alpha}^{(2)}$ $(k=p+1,\cdots,q;\ \alpha=1,2,\cdots,N_1)$ are assigned through the first experiment.

(ii) Let an estimate of β_i be given by

(3.08) $$b_i^{(2)}=\sum_{j=1}^{q} c_{ij}^{(2)}B_j^{(2)} \qquad (i=1,2,\cdots,q) ,$$

where $(c_{ij}^{(2)})$ is the inverse matrix of the $q\times q$ matrix $(a_{ij}^{(2)})$ $(i,j=1,2,\cdots,q)$ and

(3.091) $$a_{jk}^{(12)}=\sum_{\alpha=1}^{N_1} x_{j\alpha}^{(1)}x_{k\alpha}^{(2)}=\sum_{\alpha=1}^{N_1} x_{j\alpha}^{(2)}x_{k\alpha}^{(2)} \qquad (j=1,2,\cdots,p;\ k=p+1,\cdots,q) .$$

(3.092) $$a_{ij}^{(2)}=\sum_{\theta=1}^{N_2} x_{i\theta}^{(2)}x_{j\theta}^{(2)} \qquad (i,j=1,2,\cdots,q) .$$

(3.10) $$B_j^{(2)}=\sum_{\theta=1}^{N_2} x_{j\theta}^{(2)}y_\theta \qquad (j=1,2,\cdots,q) .$$

Let us now define

(3.11) $$b_i=b_i^{(2)} \qquad (i=1,2,\cdots,q) .$$

Under this procedure, let us define an estimate of the response function for each assigned $x=(x_1,x_2,\cdots,x_q)$ by

$$\hat{\varphi}(x) \equiv b_1 x_1 + b_2 x_2 + \cdots + b_q x_q, \tag{3.12}$$

similarly as in (2.07).

Let us put

$$\Delta_{ik}^{(1)} \equiv \sum_{j=1}^{p} c_{ij}^{(1)} a_{jk}^{(12)} \qquad (i=1,2,\cdots,p;\ k=p+1,\cdots,q) \tag{3.13}$$

$$\Delta_i^{(1)}(\beta;\ p,q) \equiv \sum_{k=p+1}^{q} \beta_k \Delta_{ik}^{(1)} \equiv \Delta_i^{(1)} \qquad (i=1,2,\cdots,p) \tag{3.14}$$

$$G_{i\alpha}^{(1)} \equiv \sum_{j=1}^{p} c_{ij}^{(1)} x_{j\alpha}^{(1)} \qquad (i=1,2,\cdots,p;\ \alpha=1,2,\cdots,N_1) \tag{3.15}$$

$$\zeta_i^{(1)} \equiv \sum_{\alpha=1}^{N_1} G_{i\alpha}^{(1)} \varepsilon_\alpha \qquad (i=1,2,\cdots,p) \tag{3.16}$$

$$G_{i\theta}^{(2)} \equiv \sum_{j=1}^{q} c_{ij}^{(2)} x_{j\theta}^{(2)} \qquad (i=1,2,\cdots,q;\ \theta=1,2,\cdots,N_2) \tag{3.17}$$

$$\zeta_i^{(2)} \equiv \sum_{\theta=1}^{N_2} G_{i\theta}^{(2)} \varepsilon_\theta \qquad (i=1,2,\cdots,q)\,. \tag{3.18}$$

Lemma 2. *We have*

(1°) $E_{A_1}\{(b_i-\beta_i)(b_j-\beta_j)\}$

$$= \begin{cases} \Delta_i^{(1)}(\beta;\ p,q)\,\Delta_j^{(1)}(\beta;\ p,q) + c_{ij}^{(1)}\sigma^2 & (1 \leqq i,j \leqq p) \\ -\Delta_i^{(1)}(\beta;\ p,q)\,\beta_j & (1 \leqq i \leqq p < j \leqq q) \\ \beta_i\beta_j & (p+1 \leqq i,j \leqq q)\,. \end{cases} \tag{3.19}$$

(2°) $E_{R_1}\{(b_i-\beta_i)(b_j-\beta_j)\} = E_{R_1}\{\zeta_i^{(2)}\zeta_j^{(2)}\}$

$$= \sum_{\alpha=1}^{N_1}\sum_{\beta=1}^{N_1} G_{i\alpha}^{(2)} G_{j\beta}^{(2)} E_{R_1}\{\varepsilon_\alpha\varepsilon_\beta\} + \sigma^2 \sum_{\gamma=N_1+1}^{N_2} G_{i\gamma}^{(2)} G_{j\gamma}^{(2)} \tag{3.20}$$

for $1 \leqq i,j \leqq q$.

Proof. Let us note first that (i) under the condition (A_1)

$$(b_i-\beta_i)(b_j-\beta_j) = \begin{cases} (b_i^{(1)}-\beta_i)(b_j^{(1)}-\beta_j) & (i,j=1,2,\cdots,p) \\ -(b_i^{(1)}-\beta_i)\,\beta_j & (i=1,2,\cdots,p;\ j=p+1,\cdots,q) \\ \beta_i\beta_j & (i,j=p+1,\cdots,q) \end{cases} \tag{3.21}$$

and (ii) that under the condition (R_1)

$$(b_i-\beta_i)(b_j-\beta_j) = (b_i^{(2)}-\beta_i)(b_j^{(2)}-\beta_j) \tag{3.22}$$

for $1 \leqq i,j \leqq p$.

Let us now notice that $(b_1^{(1)}, b_2^{(1)}, \cdots, b_p^{(1)})$ is independent of the statistic

$$\sum_{\alpha=1}^{N_1}\Big(y_\alpha - \sum_{j=1}^{p} b_j^{(1)} x_{j\alpha}\Big)^2, \tag{3.23}$$

because it is orthogonal to the linear subspace spanned by the set of p linear forms

$$\sum_{\gamma=1}^{N_1} x_{i\gamma}^{(1)} \varepsilon_\gamma \qquad (i=1, 2, \cdots, p)\ . \tag{3.24}$$

Ad (1°): In view of (3.21) we have, for $1 \leqq i, j \leqq p$,

$$\begin{aligned} E_{A_1}\{(b_i^{(1)}-\beta_i)(b_j^{(1)}-\beta_j)\} &= E\{(b_i^{(1)}-\beta_i)(b_j^{(1)}-\beta_j)\} \\ &= \Delta_i^{(1)}(\beta; p, q)\,\Delta_j^{(1)}(\beta; p, q) + c_{ij}^{(1)}\sigma^2 \end{aligned} \tag{3.251}$$

and, for $1 \leqq i \leqq p < k \leqq q$,

$$E_{A_1}\{(b_i^{(1)}-\beta_i)\beta_j\} = E\{(b_i^{(1)}-\beta_i)\}\beta_j = \Delta_i^{(1)}(\beta; p, q)\beta_j\,, \tag{3.252}$$

which gives us (3.19). The case $p+1 \leqq i, j \leqq q$ is trivial.

Ad (2°): We have

$$b_i^{(2)} - \beta_i = \sum_{\theta=1}^{N_2}\left(\sum_{j=1}^{q} c_{ij}^{(2)} x_{j\theta}^{(2)}\right)\varepsilon_\theta = \zeta_i^{(2)} \tag{3.261}$$

$$\zeta_i^{(2)}\zeta_j^{(2)} = \left(\sum_{\alpha=1}^{N_1} G_{i\alpha}^{(2)}\varepsilon_\alpha + \sum_{\gamma=N_1+1}^{N_2} G_{i\gamma}^{(2)}\varepsilon_\gamma\right)\left(\sum_{\beta=1}^{N_1} G_{j\beta}^{(2)}\varepsilon_\beta + \sum_{\theta=N_1+1}^{N_2} G_{j\theta}^{(2)}\varepsilon_\theta\right). \tag{3.262}$$

But we have, for $1 \leqq \alpha,\ \beta \leqq N_1,\ N_1+1 \leqq r,\ \theta \leqq N_2$,

$$E_{R_1}\{\varepsilon_\alpha\varepsilon_\theta\} = E_{R_1}\{\varepsilon_\alpha\}E\{\varepsilon_\theta\} = 0 \tag{3.27}$$

$$E_{R_1}\{\varepsilon_\gamma\varepsilon_\theta\} = E_{R_1}\{\varepsilon_\gamma\}E\{\varepsilon_\theta\} = 0 \qquad (r \neq \theta) \tag{3.28}$$

$$E_{R_1}\{\varepsilon_\gamma^2\} = E\{\varepsilon_\gamma^2\} = \sigma^2 \tag{3.29}$$

because ε_γ and $\varepsilon_\theta\,(N_1+1 \leqq r,\ \theta \leqq N_2)$ are independent of the statistic F which is defined by the function of $\{\varepsilon_\alpha\}$ $(1 \leqq \alpha \leqq N_1)$.

Consequently we have

$$\begin{aligned} &E_{R_1}\{(b_i^{(2)}-\beta_i)(b_j^{(2)}-\beta_j)\} \\ &= \sum_{\alpha=1}^{N_1}\sum_{\beta=1}^{N_1} G_{i\alpha}^{(2)}G_{i\beta}^{(2)}E_{R_1}\{\varepsilon_\alpha\varepsilon_\beta\} + \sum_{\gamma=N_1+1}^{N_2} G_{j\gamma}^{(2)}G_{j\gamma}^{(2)}\sigma^2\,, \end{aligned} \tag{3.30}$$

as we were to prove.

Theorem 1. *Under the two stage sequential design enunciated in this paragraph, we have:*

(1°) *The mean square error of the estimate* $\hat{\varphi}(x)$ *at each assigned point* $x=(x_1, x_2, \cdots, x_q)$

$$\begin{aligned} &E\{|\hat{\varphi}(x)-\varphi(x)|^2\} \\ &= \Pr.\{A_1\}\Big[\sum_{i=1}^{p}\sum_{j=1}^{p}(\Delta_i^{(1)}(\beta;p,q)\,\Delta_j^{(1)}(\beta;p,q) + c_{ij}^{(1)}\sigma^2)\,x_i x_j \\ &\quad - 2\sum_{i=1}^{p}\sum_{j=p+1}^{q}\Delta_i^{(1)}(\beta;p,q)\,\beta_j x_i x_j + \sum_{i=p+1}^{q}\sum_{j=p+1}^{q}\beta_i\beta_j x_i x_j\Big] \\ &\quad + \Pr.\{R_1\}\Big[\sum_{i=1}^{q}\sum_{j=1}^{q}\Big\{\sum_{\alpha=1}^{N_1}\sum_{\beta=1}^{N_1} G_{i\alpha}^{(2)}G_{j\beta}^{(2)}E_{R_1}\{\varepsilon_\alpha\varepsilon_\beta\} + \sum_{\gamma=N_1+1}^{N_2} G_{i\gamma}^{(2)}G_{j\gamma}^{(2)}\sigma^2\Big\}\,x_i x_j\Big] \end{aligned} \tag{3.31}$$

(2°) *The norm*

$$E\{Q\} = \int\int_{\mathfrak{D}} E\left\{|\hat{\varphi}(x) - \varphi(x)|^2\right\} w(x)\, dx_1 \cdots dx_q . \tag{3.32}$$

§4. **Three stage sequential design.** Let us consider the situation when the true response function is given by

$$\varphi_r(x) = \sum_{i=1}^{r} \beta_i x_i . \tag{4.01}$$

Let us suppose an experimenter for whom the model is not completely specified and who may assume other response functions such as $\varphi_p(x)$ defined in (1.04) or $\varphi_q(x)$ defined in (1.02), where p and q $(1 \leq p < q < r)$ are chosen by him. Under this situation he may appeal to the three stage sequential design defined in the following way.

(1°) Let us proceed the steps (1°) (i) ~ (iv) similarly as those corresponding to (1°) (1) ~ (iv) of § 3.

$(2°)_1$ Let us proceed similarly as in $(2°)_1$ in § 3, except that

$$b_i = \begin{cases} b_i^{(1)} & (i = 1, 2, \cdots, p) \\ 0 & (i = p+1, \cdots, r) . \end{cases} \tag{4.02}$$

$(2°)_2$ Let us proceed the steps $(2°)_2$ (i) ~ (ii) similarly as those corresponding to in $(2°)_2$ (i) ~ (ii) of § 3.

(iii) Let u_ν^2 and $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{N_2-1}, \varepsilon_{N_2})$ be mutually independent.

(iv) Test the hypothesis $H_0(p, r) : \beta_{q+1} = \beta_{q+2} = \cdots = \beta_r = 0$ at an assigned level of significance α_2 by appealing to the F-test where

$$F = \sum_{\alpha=1}^{N_2} \Big(y_\alpha - \sum_{j=1}^{q} b_j^{(2)} x_{j\alpha} \Big)^2 \Big/ (N_2 - q) u_\nu^2 . \tag{4.03}$$

$(3°)_1$ If the F-test is non-significant (A_2), that is, $F \leq F_\nu^{N-q}(\alpha_2)$, which is the α_2-point of the F-distribution with the pair of the degrees of freedom $(N_2 - q, \nu)$, then we stop the experimentation, and we give the estimates

$$b_i = \begin{cases} b_i^{(2)} & (i = 1, 2, \cdots, q) \\ 0 & (i = q+1, \cdots, r) . \end{cases} \tag{4.04}$$

$(3°)_2$ If the F-test is significant (R_2), that is, $F > F_\nu^{N_2-q}(\alpha_2)$, we proceed in the following way.

(i) Make the third experiment involving $N_3 - N_2$ further observations $\{y_\psi\}$ $(\psi = N_2 + 1, N_2 + 2, \cdots, N_3)$ according to the design matrix $D^{(3)}$ defined by

$$D^{(3)} = \begin{pmatrix} x_{11}^{(3)} & x_{21}^{(3)} & \cdots & x_{p1}^{(3)} & \cdots & x_{q1}^{(3)} & \cdots & x_{r1}^{(3)} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_{1N_1}^{(3)} & x_{2N_1}^{(3)} & \cdots & x_{pN_1}^{(3)} & \cdots & x_{qN_1}^{(3)} & \cdots & x_{rN_1}^{(3)} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_{1N_2}^{(3)} & x_{2N_2}^{(3)} & \cdots & x_{pN_2}^{(3)} & \cdots & x_{qN_2}^{(3)} & \cdots & x_{rN_2}^{(3)} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_{1N_3}^{(3)} & x_{2N_3}^{(3)} & \cdots & x_{pN_3}^{(3)} & \cdots & x_{qN_3}^{(3)} & \cdots & x_{rN_3}^{(3)} \end{pmatrix}, \tag{4.05}$$

where we assume that

$$x_{i\alpha}^{(3)} = x_{i\alpha}^{(2)} = x_{i\alpha}^{(1)} \qquad (i=1, 2, \cdots, p;\ \alpha = 1, 2, \cdots, N_1) \tag{4.06}$$

$$x_{k\theta}^{(3)} = x_{k\theta}^{(2)} \qquad (i=p+1, \cdots, q;\ \theta = 1, 2, \cdots, N_2) \tag{4.07}$$

and that $x_{k\theta}^{(2)}$ $(k=q+1, \cdots, r;\ \theta=1, 2, \cdots, N_2)$ are obtained without any additional experiments.

(ii) Let the estimate of β_i $(i=1, 2, \cdots, r)$ be given by

$$b_i^{(3)} = \sum_{j=1}^{r} c_{ij}^{(3)} B_j^{(3)} \qquad (i=1, 2, \cdots, r)\ , \tag{4.08}$$

where $(c_{ij}^{(3)})$ is the inverse matrix of the $r \times r$ matrix $(a_{ij}^{(3)})$ $(i, j=1, 2, \cdots, r)$ and

$$a_{ij}^{(3)} = \sum_{\psi=1}^{N_3} x_{i\psi}^{(3)} x_{j\psi}^{(3)}\ , \tag{4.091}$$

$$a_{jk}^{(23)} = \sum_{\theta=1}^{N_2} x_{j\theta}^{(2)} x_{k\theta}^{(2)} = \sum_{\theta=1}^{N_2} x_{j\theta}^{(3)} x_{k\theta}^{(3)} \qquad \begin{pmatrix} j=1, 2, \cdots, q; \\ k=q+1, \cdots, r \end{pmatrix}, \tag{4.092}$$

$$B_j^{(3)} = \sum_{\psi=1}^{N_3} x_{j\psi}^{(3)} y_\psi\ . \tag{4.093}$$

Let us define

$$b_i = b_i^{(3)} \qquad (i=1, 2, \cdots, r)\ . \tag{4.10}$$

Under this procedure, let us define the estimated response function by $\hat{\varphi}(x)$ as in (2.22).

In addition to the quantities similarly defined as in § 3, let us define

$$\Delta_{ik}^{(2)} \equiv \sum_{j=1}^{q} c_{ij}^{(2)} a_{jk}^{(23)} \qquad (i=1, 2, \cdots, q;\ k=q+1, \cdots, r) \tag{4.11}$$

$$\Delta_i^{(2)}(\beta;\ q, r) \equiv \sum_{k=q+1}^{r} \beta_k \Delta_{ik}^{(2)} \qquad (i=1, 2, \cdots, q) \tag{4.12}$$

$$G_{i\theta}^{(2)} \equiv \sum_{j=1}^{q} c_{ij}^{(2)} x_{j\theta}^{(2)} \qquad (i=1, 2, \cdots, q;\ \theta=1, 2, \cdots, N_2) \tag{4.13}$$

$$\zeta_i^{(2)} \equiv \sum_{\theta=1}^{N_2} G_{i\theta}^{(2)} \varepsilon_\theta \qquad (i=1, 2, \cdots, q) \tag{4.14}$$

$$G_{i\psi}^{(3)} \equiv \sum_{j=1}^{r} c_{ij}^{(3)} x_{j\psi}^{(3)} \qquad (i=1, 2, \cdots, r;\ \psi=1, 2, \cdots, N_3)\,, \tag{4.15}$$

where the $r \times r$ matrix $(c_{ij}^{(3)})$ is the inverse matrix of the $r \times r$ matrix $(a_{ij}^{(3)})$.

Lemma 3. *We have*

(1°) $E_{A_1}\{(b_i-\beta_i)(b_j-\beta_j)\}$

$$= \begin{cases} \Delta_i^{(1)}(\beta;\, p, r)\, \Delta_j^{(1)}(\beta;\, p, r) + c_{ij}^{(1)}\sigma^2 & (\text{for } i, j = 1, 2, \cdots, p) \\ -\Delta_i^{(1)}(\beta;\, p, r)\, \beta_j & (1 \leqq i \leqq p;\ p+1 \leqq j \leqq r) \\ \beta_i \beta_j & (i, j = p+1, \cdots, r)\,. \end{cases} \tag{4.16}$$

(2°) $E_{R_1 A_2}\{(b_i-\beta_i)(b_j-\beta_j)\}$

$$= \begin{cases} \Delta_i^{(2)}(\beta;\, q, r)\, \Delta_j^{(2)}(\beta;\, q, r) + \Delta_i^{(2)}(\beta;\, q, r)\, E_{R_1}\{\zeta_j^{(2)}\} & \\ + \Delta_j^{(2)}(\beta;\, q, r)\, E_{R_1}\{\zeta_i^{(2)}\} + E_{R_1}\{\zeta_i^{(2)}\zeta_j^{(2)}\} & (1 \leqq i, j \leqq q) \\ -(\Delta_i^{(2)}(\beta;\, q, r) + E_{R_1}\{\zeta_i^{(2)}\})\, \beta_j & (1 \leqq i \leqq q+1 \leqq j \leqq r) \\ \beta_i \beta_j & (q+1 \leqq i, j \leqq r)\,. \end{cases} \tag{4.17}$$

(3°) $E_{R_1 R_2}\{(b_i-\beta_i)(b_j-\beta_j)\}$

$$= E_{R_1 R_2}\{\zeta_i^{(3)}\zeta_j^{(3)}\} \tag{4.18}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\theta=1}^{N_2} \sum_{\varphi=1}^{N_2} G_{i\theta}^{(3)} G_{\varphi}^{(3)} E_{R_1 R_2}\{\varepsilon_\theta \varepsilon_\varphi\}$$

for $i, j = 1, 2, \cdots, r$.

Proof: This is quite similar to that of Lemma 2.

Theorem 2. *Under the three stage sequential design enunciated in this paragraph, we have:*

(1°) *the mean square error of the estimate* $\hat{\varphi}(x)$ *at each assigned point* $x = (x_1, x_2, \cdots, x_r)$

$$E\{|\hat{\varphi}(x) - \varphi(x)|^2\}$$

$$\begin{aligned} &= \Pr.\{A_1\}\Big[\sum_{i=1}^{p} \sum_{j=1}^{p} (\Delta_i^{(1)}(\beta;\, p, r)\, \Delta_j^{(1)}(\beta;\, p, r) + c_{ij}^{(1)}\sigma^2)\, x_i x_j \\ &\qquad - 2 \sum_{i=1}^{p} \sum_{j=p+1}^{r} \Delta_i^{(1)}(\beta;\, p, r)\, \beta_j x_i x_j + \sum_{i=p+1}^{r} \sum_{j=p+1}^{r} \beta_i \beta_j x_i x_j\Big] \\ &\quad + \Pr.\{R_1 A_2\}\Big[\sum_{i=1}^{q} \sum_{j=1}^{q} (\Delta_i^{(2)}(\beta;\, q, r)\, \Delta_j^{(2)}(\beta;\, p, r) \\ &\qquad + \Delta_i^{(2)}(\beta;\, q, r)\, E_{R_1}\{\zeta_j^{(2)}\} \\ &\qquad + \Delta_j^{(2)}(\beta;\, q, r)\, E_{R_1}\{\zeta_i^{(2)}\} + E_{R_1}\{\zeta_i^{(2)}\zeta_j^{(2)}\}) \end{aligned} \tag{4.19}$$

$$-2\sum_{i=1}^{q}\sum_{j=q+1}^{r}(\varDelta^{(2)}(\beta; q, r)+E_{R_1}\{\zeta_i^{(2)}\})\beta_j$$

$$+\sum_{i=q+1}^{r}\sum_{j=q+1}^{r}\beta_i\beta_j x_i x_j\Big]$$

$$+\Pr.\{R_1R_2\}\sum_{i=1}^{r}\sum_{j=1}^{r}E_{R_1R_2}\{\zeta_i^{(3)}\zeta_j^{(3)}\}x_i x_j .$$

(2°) *The norm*

$$E\{Q\}=\int\!\!\int_{\mathfrak{D}} E\{|\hat{\varphi}(x)-\varphi(x)|^2\}\,w(x)\,dx_1\cdots dx_r . \tag{4.20}$$

§ 5. **Estimations after preliminary test of significance in the models associated with the rotatable designs in the two dimensions.** We are giving here three examples of § 2 with reference to response surface analysis.

[1] **The rotatable design of the order two in the two dimensions.** Let us consider a model of response surface of the second degree in the two dimensions defined by

$$\varphi_6(x)=\beta_0+\beta_1x_1+\beta_2x_2+\beta_{11}x_1^2+\beta_{22}x_2^2+\beta_{12}x_1x_2 . \tag{5.01}$$

Let us consider an experimenter for whom the model is not completely specified and who may assume a response surface of the first degree, i.e.,

$$\varphi_3(x)=\beta_0+\beta_1x_1+\beta_2x_2 . \tag{5.02}$$

Under this situation let us assume that he has N observations

$$y_\alpha=\beta_0+\beta_1x_{1\alpha}+\beta_2x_{2\alpha}+\beta_{11}x_{1\alpha}^2+\beta_{22}x_{2\alpha}^2+\beta_{12}x_{1\alpha}x_{2\alpha}+\varepsilon_\alpha , \tag{5.03}$$

where $\{\varepsilon_\alpha\}$ are subject to the same conditions defined in § 1.

Let us consider the transformations of the fixed variables and the unknown parameters $(\beta_0, \beta_1, \cdots, \beta_{12})$ into $\{x'_{i\alpha}\}$ $(i=1, 2, \cdots, 6)$ and $(\beta'_1, \beta'_2, \cdots, \beta'_6)$ respectively by

$$\begin{aligned} x_{0\alpha}&=x'_{1\alpha}, & x_{1\alpha}&=x'_{2\alpha}, & x_{2\alpha}&=x'_{3\alpha} \\ x_{1\alpha}^2&=x'_{4\alpha}, & x_{2\alpha}^2&=x'_{5\alpha}, & x_{1\alpha}x_{2\alpha}&=x'_{6\alpha}, \end{aligned} \tag{5.04}$$

and

$$\begin{aligned} \beta_0&=\beta'_1, & \beta_1&=\beta'_2, & \beta_2&=\beta'_3 \\ \beta_{11}&=\beta'_4, & \beta_{22}&=\beta'_5, & \beta_{12}&=\beta'_6 . \end{aligned} \tag{5.05}$$

The transformations reduce our formulation into a special case of that defined in § 1. We are now concerned with the design matrix D associated with a rotatable design of the order two in the sense of Box-Hunter [1] for which

$$(5.06)\qquad (a_{ij}) = \begin{pmatrix} N & 0 & 0 & aN & aN & 0 \\ 0 & aN & 0 & 0 & 0 & 0 \\ 0 & 0 & aN & 0 & 0 & 0 \\ aN & 0 & 0 & 3\lambda_1 a^2 N & \lambda_1 a^2 N & 0 \\ aN & 0 & 0 & \lambda_1 a^2 N & 3\lambda_1 a^2 N & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 a^2 N \end{pmatrix}.$$

The choices of (5.01) and (5.02) correspond to $q=6$ and $p=3$ respectively. In the consequence we can give various constants enunciated in Lemma 1 as follows.

$$(5.07)\qquad (c_{ij}^{(1)}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} = \begin{pmatrix} N & 0 & 0 \\ 0 & aN & 0 \\ 0 & 0 & aN \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{N} & 0 & 0 \\ 0 & \frac{1}{aN} & 0 \\ 0 & 0 & \frac{1}{aN} \end{pmatrix}.$$

$$(5.081)\qquad A_{4\alpha} = x'_{4\alpha} - \sum_{i=1}^{3}\sum_{j=1}^{3} c_{jj}^{(1)} a_{j4} x'_{j\alpha} = x_{1\alpha}^2 - a$$

$$(5.082)\qquad A_{5\alpha} = x'_{5\alpha} - \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}^{(1)} a_{j5} x'_{j\alpha} = x_{2\alpha}^2 - a$$

$$(5.083)\qquad A_{\alpha 6} = x'_{6\alpha} - \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}^{(1)} a_{j6} x'_{j\alpha} = x_{1\alpha} x_{2\alpha}$$

$$(5.09)\qquad H_\alpha = \sum_{k=4}^{6} \beta'_k A_{k\alpha} = \beta_{11}(x_{1\alpha}^2 - a) + \beta_{22}(x_{2\alpha}^2 - a) + \beta_{12} x_{1\alpha} x_{2\alpha}$$

$$(5.101)\qquad L_1 = \sum_{\alpha=1}^{N} H_\alpha x'_{1\alpha} = \sum_{\alpha=1}^{N} H_\alpha = 0$$

$$(5.102)\qquad L_2 = \sum_{\alpha=1}^{N} H_\alpha x'_{2\alpha} = \sum_{\alpha=1}^{N} H_\alpha x_{1\alpha} = 0$$

$$(5.103)\qquad L_3 = \sum_{\alpha=1}^{N} H_\alpha x'_{3\alpha} = \sum_{\alpha=1}^{N} H_\alpha x_{2\alpha} = 0$$

$$(5.111)\qquad \eta_1 = \sum_{\alpha=1}^{N} \varepsilon_\alpha x'_{1\alpha} = \sum_{\alpha=1}^{N} \varepsilon_\alpha$$

$$(5.112)\qquad \eta_2 = \sum_{\alpha=1}^{N} \varepsilon_\alpha x'_{2\alpha} = \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{1\alpha}$$

$$(5.113)\qquad \eta_3 = \sum_{\alpha=1}^{N} \varepsilon_\alpha x'_{3\alpha} = \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{2\alpha}$$

$$(5.12)\qquad \sum_{\alpha=1}^{N} e_\alpha^2 = \sum_{\alpha=1}^{N}(H_\alpha + \varepsilon_\alpha)^2 - \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}^{(1)} \eta_i \eta_j$$
$$= \sum_{\alpha=1}^{N}(H_\alpha + \varepsilon_\alpha)^2 - \frac{1}{N}\Big(\sum_{\alpha=1}^{N} \varepsilon_\alpha\Big)^2 - \frac{1}{aN}\Big(\sum_{\alpha=1}^{N} \varepsilon_\alpha x_{1\alpha}\Big)^2 - \frac{1}{aN}\Big(\sum_{\alpha=1}^{N} \varepsilon_\alpha x_{2\alpha}\Big)^2$$

(5.13) $$\sum_{\alpha=1}^{N} H_\alpha^2 = a^2 N\{(3\lambda_1 - 1)(\beta_{11}^2 + \beta_{22}^2) + \lambda_1 \beta_{12}^2 + 2(\lambda_1 - 1)\beta_{11}\beta_{22}\},$$

where $\lambda_1 = N/2n_1$.

[2] **The rotatable design of the order three in the two demensions (1).** Let us consider a model of response surface of the third degree in the two dimensions defined by

(5.14) $$\varphi_{10}(x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \beta_{111} x_1^3 + \beta_{112} x_1^2 x_2 + \beta_{122} x_1 x_2^2 + \beta_{222} x_2^3 .$$

Let us consider an experimenter for whom the model is not completely specified and who may assume a response surface of the first degree, i.e. $\varphi_3(x)$ enunciated in (5.02). Let us consider the additional transformations to those difined in (5.04) and (5.05) such that

(5.15) $$x_{1\alpha}^3 = x'_{7\alpha}, \quad x_{1\alpha}^2 x_{2\alpha} = x'_{8\alpha}, \quad x_{1\alpha} x_{2\alpha}^2 = x'_{9\alpha}, \quad x_{2\alpha}^3 = x'_{10\alpha},$$

(5.16) $$\beta_{111} = \beta'_7, \quad \beta_{112} = \beta'_8, \quad \beta_{122} = \beta'_9, \quad \beta_{222} = \beta'_{10} .$$

We are now concerned with the design matrix D associated with a rotatable design of the order three in the sense of Box-Hunter [1] for which

(5.17) $$(a_{ij}) = \begin{pmatrix} N & 0 & 0 & aN & aN & 0 & 0 & 0 & 0 & 0 \\ 0 & aN & 0 & 0 & 0 & 0 & 3\lambda_1 a^2 N & 0 & \lambda_1 a^2 N & 0 \\ 0 & 0 & aN & 0 & 0 & 0 & 0 & \lambda_1 a^2 N & 0 & 3\lambda_1 a^2 N \\ aN & 0 & 0 & 3\lambda_1 a^2 N & \lambda_1 a^2 N & 0 & 0 & 0 & 0 & 0 \\ aN & 0 & 0 & \lambda_1 a^2 N & 3\lambda_1 a^2 N & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 a^2 N & 0 & 0 & 0 & 0 \\ 0 & 3\lambda_1 a^2 N & 0 & 0 & 0 & 0 & 15\lambda_2 a^3 N & 0 & 3\lambda_2 a^3 N & 0 \\ 0 & 0 & \lambda_1 a^2 N & 0 & 0 & 0 & 0 & 3\lambda_2 a^3 N & 0 & 3\lambda_2 a^3 N \\ 0 & \lambda_1 a^2 N & 0 & 0 & 0 & 0 & 3\lambda_2 a^3 N & 0 & 3\lambda_2 a^3 N & 0 \\ 0 & 0 & 3\lambda_1 a^2 N & 0 & 0 & 0 & 0 & 3\lambda_2 a^3 N & 0 & 15\lambda_2 a^3 N \end{pmatrix}.$$

We have

(5.181) $$(c_{ij}^{(1)}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{N} & 0 & 0 \\ 0 & \frac{1}{aN} & 0 \\ 0 & 0 & \frac{1}{aN} \end{pmatrix}.$$

(5.182) $$A_{k\alpha} = x'_{k\alpha} - \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij}^{(1)} a_{jk} x'_{i\alpha} = x'_{k\alpha} - \left(\frac{a_{1k}}{N} x'_{1\alpha} + \frac{a_{2k}}{aN} x'_{2\alpha} + \frac{a_{3k}}{aN} x'_{3\alpha}\right).$$

and hence

$$(5.191)\qquad A_{4\alpha} = x_{1\alpha}^2 - a$$

$$(5.192)\qquad A_{5\alpha} = x_{2\alpha}^2 - a$$

$$(5.193)\qquad A_{6\alpha} = x_{1\alpha} x_{2\alpha}$$

$$(5.194)\qquad A_{7\alpha} = x_{1\alpha}^3 - 3\lambda_1 a x_{1\alpha}$$

$$(5.195)\qquad A_{8\alpha} = x_{1\alpha}^2 x_{2\alpha} - \lambda_1 a x_{2\alpha}$$

$$(5.196)\qquad A_{9\alpha} = x_{1\alpha} x_{2\alpha}^2 - \lambda_1 a x_{1\alpha}$$

$$(5.197)\qquad A_{10\alpha} = x_{2\alpha}^3 - 3\lambda_1 a x_{2\alpha}$$

$$(5.20)\qquad \begin{aligned} H_\alpha &= \beta_{11}(x_{1\alpha}^2 - a) + \beta_{22}(x_{2\alpha}^2 - a) + \beta_{12} x_{1\alpha} x_{2\alpha} \\ &\quad + \beta_{111}(x_{1\alpha}^3 - 3\lambda_1 a x_{1\alpha}) + \beta_{112}(x_{1\alpha}^2 x_{2\alpha} - \lambda_1 a x_{2\alpha}) \\ &\quad + \beta_{122}(x_{1\alpha} x_{2\alpha}^2 - \lambda_1 a x_{1\alpha}) + \beta_{222}(x_{2\alpha}^3 - 3\lambda_1 a x_{2\alpha}) \end{aligned}$$

and

$$(5.211)\qquad L_1 = \sum_{\alpha=1}^N H_\alpha x'_{1\alpha} = \sum_{\alpha=1}^N H_\alpha = 0$$

$$(5.212)\qquad L_2 = \sum_{\alpha=1}^N H_\alpha x'_{2\alpha} = \sum_{\alpha=1}^N H_\alpha x_{1\alpha} = 0$$

$$(5.213)\qquad L_3 = \sum_{\alpha=1}^N H_2 x'_{3\alpha} = \sum_{\alpha=1}^N H_\alpha x_{2\alpha} = 0 .$$

Hence the relation similar to (5.12) holds

$$(5.22)\qquad \begin{aligned} \sum_{\alpha=1}^N e_\alpha^2 &= \sum_{\alpha=1}^N (H_\alpha + \varepsilon_\alpha)^2 - \frac{1}{N}\Big(\sum_{\alpha=1}^N \varepsilon_\alpha\Big)^2 - \frac{1}{aN}\Big(\sum_{\alpha=1}^N \varepsilon_\alpha x_{1\alpha}\Big)^2 \\ &\quad - \frac{1}{aN}\Big(\sum_{\alpha=1}^N \varepsilon_\alpha x_{2\alpha}\Big)^2 , \end{aligned}$$

and

$$(5.23)\qquad \begin{aligned} \sum_{\alpha=1}^N H_\alpha^2 &= a^2 N \{ (3\lambda_1 - 1)(\beta_{11}^2 + \beta_{22}^2) + 2(\lambda_1 - 1)\beta_{11}\beta_{22} + \lambda_1 \beta_{12}^2 \} \\ &\quad + a^3 N \{ 3(5\lambda_2 - 3\lambda_1^2)(\beta_{111}^2 + \beta_{222}^2) \\ &\qquad + \ (3\lambda_2 - \lambda_1^2)(\beta_{112}^2 + \beta_{122}^2) \\ &\qquad + 6(\lambda_2 - \lambda_1^2)(\beta_{111} + \beta_{122} + \beta_{222}\beta_{112}) \} . \end{aligned}$$

[3] The rotatable design of the order three in the two dimensions (2). Under the hypothesis of (5.14)$\sim$(5.17), let us consider an experimenter for whom the model is not completely specified and who may assume a response surface of the second degree, i.e., $\varphi_6(x)$ enunciated in (5.01).

We have in this case

$$(5.24)\qquad (c_{ij}^{(1)}) = \begin{pmatrix} N & 0 & 0 & aN & aN & 0 \\ 0 & aN & 0 & 0 & 0 & 0 \\ 0 & 0 & aN & 0 & 0 & 0 \\ aN & 0 & 0 & 3\lambda_1 a^2 N & \lambda_1 a^2 N & 0 \\ aN & 0 & 0 & \lambda_1 a^2 N & 3\lambda_1 a^2 N & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 a^2 N \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{2\lambda_1}{(2\lambda_1-1)N} & 0 & 0 & \frac{-1}{2(2\lambda_1-1)aN} & \frac{-1}{2(2\lambda_1-1)aN} & 0 \\ 0 & \frac{1}{aN} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{aN} & 0 & 0 & 0 \\ \frac{-1}{2(2\lambda_1-1)aN} & 0 & 0 & \frac{3\lambda_1-1}{4\lambda_1(2\lambda_1-1)a^2N} & \frac{1-\lambda_1}{4\lambda_1(2\lambda_1-1)a^2N} & 0 \\ \frac{-1}{2(2\lambda_1-1)aN} & 0 & 0 & \frac{1-\lambda_1}{4\lambda_1(2\lambda_1-1)a^2N} & \frac{3\lambda_1-1}{4\lambda_1(2\lambda_1-1)a^2N} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_1 a^2 N} \end{pmatrix}.$$

Consequently we have

$$(5.251)\qquad A_{7\alpha} = x'_{7\alpha} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(1)} a_{j7} x'_{j\alpha} = x_{1\alpha}^3 - 3\lambda_1 a x_{1\alpha}$$

$$(5.252)\qquad A_{8\alpha} = x'_{8\alpha} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(1)} a_{j8} x'_{j\alpha} = x_{1\alpha}^2 x_{2\alpha} - \lambda_1 a x_{2\alpha}$$

$$(5.253)\qquad A_{9\alpha} = x'_{9\alpha} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(1)} a_{j9} x'_{j\alpha} = x_{1\alpha} x_{2\alpha}^2 - \lambda_1 a x_{1\alpha}$$

$$(5.254)\qquad A_{10\alpha} = x'_{10\alpha} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(1)} a_{j10} x'_{j\alpha} = x_{2\alpha}^3 - 3\lambda_1 a x_{2\alpha}$$

$$(5.26)\qquad H_\alpha = \sum_{j=7}^{10} \beta'_j A_{j\alpha}$$

$$= \beta_{111}(x_{1\alpha}^3 - 3\lambda_1 a x_{1\alpha}) + \beta_{112}(x_{1\alpha}^2 x_{2\alpha} - \lambda_1 a x_{2\alpha})$$
$$+ \beta_{122}(x_{1\alpha} x_{2\alpha}^2 - \lambda_1 a x_{1\alpha})$$
$$+ \beta_{222}(x_{2\alpha}^2 - 3\lambda_1 a x_{2\alpha})$$

and hence

$$(5.271)\qquad L_1 = \sum_{\alpha=1}^{N} H_\alpha x'_{1\alpha} = \sum_{\alpha=1}^{N} H_\alpha = 0$$

$$(5.272)\qquad L_2 = \sum_{\alpha=1}^{N} H_\alpha x'_{2\alpha} = \sum_{\alpha=1}^{N} H_\alpha x_{1\alpha} = 0$$

$$L_3 = \sum_{\alpha=1}^{N} H_\alpha x'_{3\alpha} = \sum_{\alpha=1}^{N} H_\alpha x_{2\alpha} = 0 \tag{5.273}$$

$$L_4 = \sum_{\alpha=1}^{N} H_\alpha x'_{4\alpha} = \sum_{\alpha=1}^{N} H_\alpha x_{1\alpha}^2 = 0 \tag{5.274}$$

$$L_5 = \sum_{\alpha=1}^{N} H_\alpha x'_{5\alpha} = \sum_{\alpha=1}^{N} H_\alpha x_{2\alpha}^2 = 0 \tag{5.275}$$

$$L_6 = \sum_{\alpha=1}^{N} H_\alpha x'_{6\alpha} = \sum_{\alpha=1}^{N} H_\alpha x_{1\alpha} x_{2\alpha} = 0\,. \tag{5.276}$$

Therefore we have

$$\sum_{\alpha=1}^{N} e_\alpha^2 = \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)^2 - \sum_{i=1}^{6} \sum_{j=1}^{6} c_{ij}^{(1)} \eta_i \eta_j\,. \tag{5.28}$$

The Jacobi transformation enunciated in § 1 can be applied to the second term in the right-hand side of (5.27), which gives us

$$D_{1,1} = \frac{2\lambda_1}{(2\lambda_1 - 1)N}, \quad D_{1,4} = D_{1,5} = \frac{-1}{(2\lambda_1 - 1)aN}, \tag{5.291}$$
$$D_{1,2} = D_{1,3} = D_{1,6} = 0$$

$$D_{2,2} = \frac{2\lambda_1}{(2\lambda_1 - 1)aN^2}, \quad D_{2,3} = D_{2,4} = D_{2,5} = D_{2,6} = 0 \tag{5.292}$$

$$D_{3,3} = \frac{2\lambda_1}{(2\lambda_1 - 1)a^2 N^3}, \quad D_{3,4} = D_{3,5} = D_{3,6} = 0 \tag{5.293}$$

$$D_{4,4} = \frac{3}{4(2\lambda_1 - 1)a^4 N^4}, \quad D_{4,5} = \frac{-1}{4(2\lambda_1 - 1)a^4 N^4}, \quad D_{4,6} = 0 \tag{5.294}$$

$$D_{5,5} = \frac{1}{4\lambda_1(2\lambda_1 - 1)a^6 N^5}, \quad D_{5,6} = 0 \tag{5.295}$$

$$D_{6,6} = \frac{1}{4\lambda_1^2(2\lambda_1 - 1)a^8 N^6} \tag{5.296}$$

and consequently

$$\frac{Z_1}{\sqrt{D_{1,1}}} = \frac{1}{\sqrt{D_{1,1}}} \sum_{\alpha=1}^{N} \varepsilon_\alpha \left(\frac{2\lambda_1}{(2\lambda_1 - 1)N} - \frac{x_{1\alpha}^2 + x_{2\alpha}^2}{(2\lambda_1 - 1)aN} \right) \tag{5.301}$$

$$\frac{Z_2}{\sqrt{D_{1,1} D_{2,2}}} = \frac{1}{a^{1/2} N^{1/2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{1\alpha} \tag{5.302}$$

$$\frac{Z_3}{\sqrt{D_{2,2} D_{3,3}}} = \frac{1}{a^{1/2} N^{1/2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{2\alpha} \tag{5.303}$$

$$\frac{Z_4}{\sqrt{D_{3,3} D_{4,4}}} = \frac{1}{2(6\lambda_1)^{1/2} a N^{1/2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha (3x_{1\alpha}^2 - x_{2\alpha}^2) \tag{5.304}$$

$$\frac{Z_5}{\sqrt{D_{4,4} D_{5,5}}} = \frac{1}{(3\lambda_1)^{1/2} a N^{1/2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{2\alpha}^2 \tag{5.305}$$

$$(5.306) \qquad \frac{Z_6}{\sqrt{D_{5,5}D_{6,6}}} = \frac{1}{\lambda_1^{1/2} a N^{1/2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{1\alpha} x_{2\alpha} .$$

It follows that

$$(5.31) \qquad \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(1)} \zeta_i \zeta_j = \frac{Z_1^2}{D_{1,1}} + \frac{Z_2^2}{D_{1,1}D_{2,2}} + \cdots + \frac{Z_6^2}{D_{5,5}D_{6,6}}$$

and that six statistics $\{z_i\}$ $(i=1, 2, \cdots, 6)$ are independently distributed in the normal distribution $N(0, \sigma^2)$.

We have also

$$(5.32) \qquad \sum_{\alpha=1}^{N} H_\alpha^2 = a^3 N[3(5\lambda_2 - 3\lambda_1^2)(\beta_{111}^2 + \beta_{222}^2) + (3\lambda_2 - \lambda_1^2)(\beta_{112}^2 + \beta_{122}^2) + 6(\lambda_2 - \lambda_1^2)(\beta_{111}\beta_{122} + \beta_{222}\beta_{112})] .$$

§ **6. Example of two stage sequential design associated with response surface analysis in the two dimensions.** The object of this paragraph is to give an example of § 3 in connection with the rotatable design of the order two in response surface analysis. Let us consider an experimenter whose situation is the one described in § 5 [1]. Let us assume here that he will appeal to *two stage sequential design* in which

(1°) The first stage design consists of 4 points on a square of radius ρ_1 with 2 points in the center $(0, 0)$.

(2°) The second stage design consists of 4 points on a square of radius ρ_1, rotated $\pi/4$ from the square in (1°), with 2 points in the center $(0, 0)$, in addition to the 6 points of the first stage design.

Consequently for instance the design matrix of the first stage design may be given by

$$(6.01) \qquad D^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & \rho_1 & 0 \\ 1 & 0 & \rho_1 \\ 1 & -\rho_1 & 0 \\ 1 & 0 & -\rho_1 \end{pmatrix},$$

while that of the second stage design is given by

$$(6.02)\qquad D^{(2)}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \rho_1 & 0 & \rho_1^2 & 0 & 0 \\ 1 & 0 & \rho_1 & 0 & \rho_1^2 & 0 \\ 1 & -\rho_1 & 0 & 0 & \rho_1^2 & 0 \\ 1 & 0 & -\rho_1 & 0 & \rho_1^2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \rho_1/\sqrt{2} & \rho_1/\sqrt{2} & \rho_1^2/2 & \rho_1^2/2 & \rho_1^2/2 \\ 1 & -\rho_1/\sqrt{2} & \rho_1/\sqrt{2} & \rho_1^2/2 & \rho_1^2/2 & -\rho_1^2/2 \\ 1 & -\rho_1/\sqrt{2} & -\rho_1/\sqrt{2} & \rho_1^2/2 & \rho_1^2/2 & \rho_1^2/2 \\ 1 & \rho_1/\sqrt{2} & -\rho_1/\sqrt{2} & \rho_1^2/2 & \rho_1^2/2 & -\rho_1^2/2 \end{pmatrix}.$$

For the first design we have

$$(6.031)\qquad (a_{ij}^{(12)})=\begin{pmatrix} 6 & 0 & 0 & 2\rho_1^2 & 2\rho_1^2 & 0 \\ 0 & 2\rho_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\rho_1^2 & 0 & 0 & 0 \\ 2\rho_1^2 & 0 & 0 & \frac{3}{2}\rho_1^4 & \frac{1}{2}\rho_1^4 & 0 \\ 2\rho_1^2 & 0 & 0 & \frac{1}{2}\rho_1^4 & \frac{3}{2}\rho_1^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\rho_1^4}{2} \end{pmatrix},$$

for $i, j=1, 2, 3, 4, 5, 6$.

This is a special case of (5.06) when

$$(6.032)\qquad N_1=6, \quad a^{(1)}=\rho_1^2/3, \quad \lambda_1^{(1)}=3/4.$$

For the second design we have

$$(6.041)\qquad (a_{ij}^{(2)})=\begin{pmatrix} 12 & 0 & 0 & 4\rho_1^2 & 4\rho_1^2 & 0 \\ 0 & 4\rho_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\rho_1^2 & 0 & 0 & 0 \\ 4\rho_1^2 & 0 & 0 & 3\rho_1^4 & \rho_1^4 & 0 \\ 4\rho_1^2 & 0 & 0 & \rho_1^4 & 3\rho_1^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_1^4 \end{pmatrix},$$

which is also a special case of (5.06) when

$$(6.042)\qquad N_2=12, \quad a^{(2)}=\rho_1^2/3, \quad \lambda_1^{(2)}=3/4.$$

Since we have

$$(6.05)\qquad (c_{ij}^{(1)}) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2\rho_1^2 & 0 \\ 0 & 0 & 2\rho_1^2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2\rho_1^2} & 0 \\ 0 & 0 & \frac{1}{2\rho_1^2} \end{pmatrix},$$

it follows that, for $i=1, 2, 3$; $k=4, 5, 6$,

$$(6.06)\qquad (\Delta_{ik}^{(1)}) = (c_{ij}^{(1)})(a_{jk}^{(12)})$$

$$= \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2\rho_1^2} & 0 \\ 0 & 0 & \frac{1}{2\rho_1^2} \end{pmatrix} \begin{pmatrix} 2\rho_1^2 & 2\rho_1^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\rho_1^2}{3} & \frac{\rho_1^2}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and consequently that

$$(6.07)\qquad \begin{cases} \Delta_1^{(1)}(\beta\,;3, 6) = \beta_4'\Delta_{14}^{(1)} + \beta_5'\Delta_{15}^{(1)} + \beta_6'\Delta_{16}^{(1)} = \frac{\rho_1^2}{3}(\beta_{11}+\beta_{22}) \\ \Delta_2^{(1)}(\beta\,;3, 6) = \beta_4'\Delta_{24}^{(1)} + \beta_5'\Delta_{25}^{(1)} + \beta_6'\Delta_{26}^{(1)} = 0 \\ \Delta_3^{(1)}(\beta\,;3, 6) = \beta_4'\Delta_{34}^{(1)} + \beta_5'\Delta_{35}^{(1)} + \beta_6'\Delta_{36}^{(1)} = 0\,. \end{cases}$$

Furthermore we have

$$(6.08)\qquad \begin{aligned} G_{1\alpha}^{(1)} &= c_{11}^{(1)}x_{1\alpha}'^{(1)} + c_{12}^{(1)}x_{2\alpha}'^{(1)} + c_{13}^{(1)}x_{3\alpha}'^{(1)} = \frac{1}{6} \\ G_{2\alpha}^{(1)} &= c_{21}^{(1)}x_{1\alpha}'^{(1)} + c_{22}^{(1)}x_{2\alpha}'^{(1)} + c_{23}^{(1)}x_{3\alpha}'^{(1)} = \frac{1}{2\rho_1^2}x_{2\alpha}'^{(1)} = \frac{x_{1\alpha}}{2\rho_1^2} \\ G_{3\alpha}^{(1)} &= c_{31}^{(1)}x_{1\alpha}'^{(1)} + c_{32}^{(1)}x_{2\alpha}'^{(1)} + c_{33}^{(1)}x_{3\alpha}'^{(1)} = \frac{1}{2\rho_1^2}x_{3\alpha}'^{(2)} = \frac{x_{2\alpha}}{2\rho_1^2} \end{aligned}$$

and hence

$$(6.09)\qquad \begin{aligned} \zeta_1^{(1)} &\equiv \sum_{\alpha=1}^{6} G_{1\alpha}^{(1)}\varepsilon_\alpha = \frac{1}{6}\sum_{\alpha=1}^{6}\varepsilon_\alpha \\ \zeta_2^{(1)} &\equiv \sum_{\alpha=1}^{6} G_{2\alpha}^{(1)}\varepsilon_\alpha = \frac{1}{2\rho_1^2}\sum_{\alpha=1}^{6} x_{1\alpha}\varepsilon_\alpha \\ \zeta_3^{(1)} &\equiv \sum_{\alpha=1}^{6} G_{3\alpha}^{(1)}\varepsilon_\alpha = \frac{1}{2\rho_1^2}\sum_{\alpha=1}^{6} x_{2\alpha}\varepsilon_\alpha\,. \end{aligned}$$

Now the application of (5.24) for the particular matrix (6.041) associated with the rotatable design of the order two, we shall have

$$(c_{ij}^{(2)}) = (a_{ij}^{(2)})^{-1} \tag{6.10}$$

$$= \begin{pmatrix} \frac{1}{4} & 0 & 0 & -\frac{1}{4\rho_1^2} & -\frac{1}{4\rho_1^2} & 0 \\ 0 & \frac{1}{4\rho_1^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4\rho_1^2} & 0 & 0 & 0 \\ -\frac{1}{4\rho_1^2} & 0 & 0 & \frac{5}{8\rho_1^4} & \frac{1}{8\rho_1^4} & 0 \\ -\frac{1}{4\rho_1^2} & 0 & 0 & \frac{1}{8\rho_1^4} & \frac{5}{8\rho_1^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_1^4} \end{pmatrix}$$

and consequently it follow that

$$\begin{cases} G_{1\theta}^{(2)} = \sum_{j=1}^{6} c_{1j}^{(2)} x'_{j\theta} = \frac{1}{4} - \frac{1}{4\rho_1^2}(x_{1\theta}^2 + x_{2\theta}^2) \\ G_{2\theta}^{(2)} = \sum_{j=1}^{6} c_{2j}^{(2)} x'_{j\theta} = \frac{1}{4\rho_1^2} x_{1\theta} \\ G_{3\theta}^{(2)} = \sum_{j=1}^{6} c_{3j}^{(2)} x'_{\theta} = \frac{1}{4\rho_1^2} x_{2\theta} \\ G_{4\theta}^{(2)} = \sum_{j=1}^{6} c_{4j}^{(2)} x'_{\theta} = -\frac{1}{4\rho_1^2} + \frac{1}{8\rho_1^4}(5x_{1\theta}^2 + x_{2\theta}^2) \\ G_{5\theta}^{(2)} = \sum_{j=1}^{6} c_{5j}^{(2)} x'_{j\theta} = -\frac{1}{4\rho_1^2} + \frac{1}{8\rho_1^4}(x_{1\theta}^2 + 5x_{2\theta}^2) \\ G_{6\theta}^{(2)} = \sum_{j=1}^{6} c_{6j}^{(2)} x'_{j\theta} = \frac{1}{\rho_1^4} x_{1\theta} x_{2\theta} \end{cases} \tag{6.11}$$

and that

$$\begin{cases} \zeta_1^{(2)} = \sum_{\theta=1}^{12} G_{1\theta}^{(2)} \varepsilon_\theta = \frac{1}{4} \sum_{\theta=1}^{12} \left(1 - \frac{1}{\rho_1^2}(x_{1\theta}^2 + x_{2\theta}^2)\right) \varepsilon_\theta \\ \zeta_2^{(2)} = \sum_{\theta=1}^{12} G_{2\theta}^{(2)} \varepsilon_\theta = \frac{1}{4\rho_1^2} \sum_{\theta=1}^{12} x_{1\theta} \varepsilon_\theta \\ \zeta_3^{(2)} = \sum_{\theta=1}^{12} G_{3\theta}^{(2)} \varepsilon_\theta = \frac{1}{4\rho_1^2} \sum_{\theta=1}^{12} x_{2\theta} \varepsilon_\theta \\ \zeta_4^{(2)} = \sum_{\theta=1}^{12} G_{4\theta}^{(2)} \varepsilon_\theta = -\frac{1}{4} \sum_{\theta=1}^{12} \varepsilon_\theta + \frac{1}{8\rho_1^4} \sum_{\theta=1}^{12} (5x_{1\theta}^2 + x_{2\theta}^2) \varepsilon_\theta \\ \zeta_5^{(2)} = \sum_{\theta=1}^{12} G_{5\theta}^{(2)} \varepsilon_\theta = -\frac{1}{4} \sum_{\theta=1}^{12} \varepsilon_\theta + \frac{1}{8\rho_1^4} \sum_{\theta=1}^{12} (x_{1\theta}^2 + 5x_{2\theta}^2) \varepsilon_\theta \\ \zeta_6^{(2)} = \sum_{\theta=1}^{12} G_{6\theta}^{(2)} \varepsilon_\theta = \frac{1}{\rho_1^4} \sum_{\theta=1}^{12} x_{1\theta} x_{2\theta} \varepsilon_\theta . \end{cases} \tag{6.12}$$

§ 7. **Example of three stage sequential design associated with response surface analysis in the two dimensions.** Let us consider an experimenter for whom the model is not completely specified as those enunciated in § 5 [2] and [3] and who will appeal to the three stage sequential design in which

(1°) The first stage design is the same with the one enunciated in § 6, (1°).

(2°) The second stage design is the same with the one enunciated in § 6, (2°).

(3°) The third stage design consists of 8 points distributed in successive equiangular distance on a circle of radius $\rho_2 = 2^{1/2}\rho_1/3^{1/2}$, in addition to the second stage design.

In this case we have a sequence of design matrices $D^{(1)}$, $D^{(2)}$ and $D^{(3)}$, and the corresponding sequence of matrices $(a_{ij}^{(1)})$, $(a_{ij}^{(2)})$ and $(a_{ij}^{(3)})$, which are of the types 3×3, 6×6 and 10×10 respectively. Since each stage design is an expansion of the former ones and each of the values $x_{1\alpha}^k x_{2\alpha}^l$ for $k, l \geqq 0$ can be available through the values of $x_{1\alpha}$ and $x_{2\alpha}$, we can form the enlarged matrices $(a_{ij}^{*(1)})$, $(a_{ij}^{*(2)})$ and $(a_{ij}^{*(3)}) \equiv (a_{ij}^{(3)})$ of the type 10×10 in

(7.10) $$a_{ij}^{*(1)} = \sum_{\alpha=1}^{N_1} x_{i\alpha}'^{(1)} x_{j\alpha}'^{(1)} = \sum_{\alpha=1}^{N_1} x_{i\alpha}'^{(2)} x_{j\alpha}^{(2)} = \sum_{\alpha=1}^{M} x_{i\alpha}'^{(3)} x_{j\alpha}'^{(3)}$$

(7.02) $$a_{ij}^{*(2)} = \sum_{\theta=1}^{N_t} x_{i\theta}'^{(2)} x_{j\theta}'^{(2)} = \sum_{\theta=1}^{N_2} x_{i\theta}^{(3)} x_{j\theta}^{(3)}$$

(7.03) $$a_{ij}^{*(3)} = \sum_{\psi=1}^{N_3} x_{i\theta}'^{(3)} x_{j\theta}'^{(3)} = a_{ij}^{(3)},$$

for $i, j = 1, 2, 3, \cdots, 10$. This means that

(7.041) $$a_{ij}^{(12)} = a_{ij}^{*(1)} \qquad (i=1, 2, 3;\ j=4, 5, 6)$$

(7.042) $$a_{ij}^{(13)} = a_{ij}^{*(1)} \qquad (i=1, 2, 3;\ j=4, 5, 6, 7, 8, 9, 10)$$

(7.043) $$a_{ij}^{(23)} = a_{ij}^{*(2)} \qquad (i=1, 2, 3, 4, 5, 6;\ j=7, 8, 9, 10),$$

in connection with the notations defined in § 3.

In the consequence we have

(7.05) $$(\Delta_{ik}^{(1)}) = (c_{ij}^{(1)})(a_{jk}^{(13)})$$
$$= \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2\rho_1^2} & 0 \\ 0 & 0 & \frac{1}{2\rho_1^2} \end{pmatrix} \begin{pmatrix} 2\rho_1^2 & 2\rho_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2}\rho_1^4 & 0 & \frac{\rho_1^4}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho_1^4}{2} & 0 & \frac{3}{2}\rho_1^4 \end{pmatrix}$$

$$
= \begin{pmatrix} \frac{\rho_1^2}{3} & \frac{\rho_1^2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4}\rho_1^2 & 0 & \frac{\rho_1^2}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho_1^2}{4} & 0 & \frac{3}{4}\rho_1^2 \end{pmatrix},
$$

and hence

$$
(7.06) \quad \begin{cases} \Delta_1^{(1)}(\beta\,;\,3,10) \equiv \sum_{k=4}^{10} \beta'_k \Delta_{1k}^{(1)} = \frac{\rho_1^2}{3}(\beta_{11}+\beta_{22}) \\ \Delta_2^{(1)}(\beta\,;\,3,10) \equiv \sum_{k=4}^{10} \beta'_k \Delta_{2k}^{(1)} = \frac{\rho_1^2}{4}(3\beta_{111}+\beta_{122}) \\ \Delta_3^{(1)}(\beta\,;\,3,10) \equiv \sum_{k=4}^{10} \beta'_k \Delta_{3k}^{(1)} = \frac{\rho_1^2}{4}(\beta_{112}+3\beta_{222})\,. \end{cases}
$$

Furthermore (7.05) gives us

$$
(7.07) \quad \begin{cases} A_{4\alpha}^{(1)} = x_{4\alpha}^{\prime(2)} - \sum_{i=1}^{3} x_{i\alpha}^{\prime(2)} \Delta_{i4}^{(1)} = x_{1\alpha}^2 - \frac{\rho^2}{3} \\ A_{5\alpha}^{(1)} = x_{5\alpha}^{\prime(2)} - \sum_{i=1}^{3} x_{i\alpha}^{(2)} \Delta_{i5}^{(1)} = x_{2\alpha}^2 - \frac{\rho^2}{3} \\ A_{6\alpha}^{(1)} = x_{6\alpha}^{\prime(2)} - \sum_{i=1}^{3} x_{i\alpha}^{\prime(2)} \Delta_{i6}^{(1)} = x_{1\alpha} x_{2\alpha} \\ A_{7\alpha}^{(1)} = x_{7\alpha}^{\prime(2)} - \sum_{i=1}^{3} x_{i\alpha}^{\prime(2)} \Delta_{i7}^{(1)} = x_{1\alpha}^3 - \frac{3}{4}\rho_1^2 x_{1\alpha} \\ A_{8\alpha}^{(1)} = x_{8\alpha}^{\prime(2)} - \sum_{i=1}^{3} x_{i\alpha}^{\prime(2)} \Delta_{i8}^{(1)} = x_{1\alpha}^2 x_{2\alpha} - \frac{\rho_1^2}{4} x_{2\alpha} \\ A_{9\alpha}^{(1)} = x_{9\alpha}^{\prime(2)} - \sum_{i=1}^{3} x_{i\alpha}^{\prime(2)} \Delta_{i9}^{(1)} = x_{1\alpha} x_{2\alpha}^2 - \frac{\rho_1^2}{4} x_{1\alpha} \\ A_{10\alpha}^{(1)} = x_{10\alpha}^{\prime(2)} - \sum_{i=1}^{3} x_{i\alpha}^{\prime(2)} \Delta_{i10}^{(1)} = x_{1\alpha}^3 - \frac{3}{4}\rho_1^2 x_{2\alpha}\,, \end{cases}
$$

which yields us

$$
(7.08) \quad \begin{aligned} H_\alpha^{(1)} &\equiv \sum_{k=4}^{10} \beta'_k A_{k\alpha}^{(1)} \\ &= \beta_{11}\left(x_{1\alpha}^2 - \frac{\rho_1^2}{3}\right) + \beta_{22}\left(x_{2\alpha}^2 - \frac{\rho^2}{3}\right) + \beta_{12} x_{1\alpha} x_{2\alpha} \\ &\quad + \beta_{111}\left(x_{1\alpha}^3 - \frac{3}{4}\rho_1^2 x_{1\alpha}\right) + \beta_{112}\left(x_{1\alpha}^2 x_{2\alpha} - \frac{\rho_1^2}{4} x_{2\alpha}\right) \\ &\quad + \beta_{122}\left(x_{1\alpha} x_{2\alpha}^2 - \frac{\rho_1^2}{4} x_{1\alpha}\right) + \beta_{222}\left(x_{2\alpha}^3 - \frac{3}{4}\rho_1^2 x_{2\alpha}\right). \end{aligned}
$$

$$(7.09)\qquad L_1^{(1)} \equiv \sum_{\alpha=1}^{6} H_\alpha^{(1)} x'_{1\alpha} = 0$$

$$L_2^{(1)} = \sum_{\alpha=1}^{6} H_\alpha^{(1)} x'_{2\alpha} = 0$$

$$L_3^{(1)} = \sum_{\alpha=1}^{6} H_\alpha^{(1)} x'_{3\alpha} = 0$$

and

$$(7.10)\qquad \sum_{\alpha=1}^{6} (H_\alpha^{(1)})^2 = \frac{5\rho_1^4}{6}(\beta_{11}^2 + \beta_{22}^2) + \frac{\rho_1^4}{4}\beta_{12}^2 + \frac{\rho_1^6}{8}(\beta_{111} - \beta_{122})^2 + \frac{\rho_1^6}{8}(\beta_{222} - \beta_{112})^2 .$$

These quantities given in (7.05)-(7.10) are concerned with the steps given in § 4. (1°). The following ones are then associated with the steps given in § 4 (2°)-(3°).

We have

$$(7.11)\qquad (\Delta_{ik}^{(2)}) = (c_{ij}^{(2)})(a_{jk}^{(23)})$$

$$= \begin{pmatrix} \frac{1}{4} & 0 & 0 & -\frac{1}{4\rho_1^2} & -\frac{1}{4\rho_1^2} & 0 \\ 0 & \frac{1}{4\rho_1^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4\rho_1^2} & 0 & 0 & 0 \\ -\frac{1}{4\rho_1^2} & 0 & 0 & \frac{5}{8\rho_1^4} & \frac{1}{8\rho_1^4} & 0 \\ -\frac{1}{4\rho_1^2} & 0 & 0 & \frac{1}{8\rho_1^4} & \frac{5}{8\rho_1^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_1^4} \end{pmatrix}$$

$$\cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3\rho_1^4 & 0 & \rho_1^4 & 0 \\ 0 & \rho_1^4 & 0 & 3\rho_1^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{3}{4}\rho_1^2 & 0 & \frac{1}{4}\rho_1^2 & 0 \\ 0 & \frac{\rho_1^2}{4} & 0 & \frac{3}{4}\rho_1^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

and hence

$$
(7.12) \qquad \begin{cases} \Delta_i^{(2)}(\beta, 6, 10) = 0 \qquad (i = 1, 4, 5, 6) \\ \Delta_2^{(2)}(\beta ; 6, 10) = \frac{\rho^2}{4}(3\beta_{111} + \beta_{122}) \\ \Delta_3^{(2)}(\beta ; 6, 10) = \frac{\rho^2}{4}(\beta_{112} + 3\beta_{222}) \,. \end{cases}
$$

We have

$$
(7.13) \qquad \begin{cases} A_{7\theta}^{(2)} = x'_{7\theta} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(2)} a_{j8}^{(23)} x'^{(3)}_{i\theta} = x_{1\theta}^3 - \frac{3}{4}\rho_1^2 x_{2\theta} \\ A_{8\theta}^{(2)} = x'_{8\theta} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(2)} a_{8}^{(23)} x'^{(3)}_{i\theta} = x_{1\theta}^2 x_{2\theta} - \frac{1}{4}\rho_1^2 x_{2\theta} \\ A_{9\theta}^{(2)} = x'_{9\theta} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(2)} a_{j9}^{(23)} x'^{(3)}_{i\theta} = x_{1\theta} x_{2\theta}^2 - \frac{1}{4}\rho_1^2 x_{1\theta} \\ A_{10\theta}^{(2)} = x'_{10\theta} - \sum_{i=1}^{6}\sum_{j=1}^{6} c_{ij}^{(2)} a_{j10}^{(23)} x'^{(3)}_{i\theta} = x_{2\theta}^3 - \frac{3}{4}\rho_1^2 x_{2\theta} \,, \end{cases}
$$

which yields us

$$
(7.14) \qquad \begin{aligned} H_\theta^{(3)} &\equiv \sum_{k=7}^{10} \beta'_k A_{k\theta}^{(2)} \\ &= \beta_{111}\left(x_{1\theta}^3 - \frac{3}{4}\rho_1^2 x_{1\theta}\right) + \beta_{112}\left(x_{1\theta}^2 x_{2\theta} - \frac{1}{4}\rho_1^2 x_{2\theta}\right) \\ &\quad + \beta_{122}\left(x_{1\theta} x_{2\theta}^2 - \frac{1}{4}\rho_1^2 x_{1\theta}\right) + \beta_{222}\left(x_{2\theta}^3 - \frac{3}{4}\rho_1^2 x_{2\theta}\right) \end{aligned}
$$

$$
\begin{cases} L_1^{(2)} \equiv \sum_{\theta=1}^{12} H_\theta^{(2)} x'_{1\theta} = \sum_{\theta=1}^{12} H_\theta^{(2)} = 0 \\ L_2^{(2)} \equiv \sum_{\theta=1}^{12} H_\theta^{(2)} x'_{2\theta} = \sum_{\theta=1}^{12} H_\theta^{(2)} x_{1\theta} = 0 \end{cases}
$$

$$(7.15)\quad \begin{cases} L_3^{(2)} \equiv \sum_{\theta=1}^{12} H_\theta^{(2)} x'_{3\theta} = \sum_{\theta=1}^{12} H_\theta^{(2)} x_{2\theta} = 0 \\ L_4^{(2)} \equiv \sum_{\theta=1}^{12} H_\theta^{(2)} x'_{4\theta} = \sum_{\theta=1}^{12} H_\theta^{(2)} x_{1\theta}^2 = 0 \\ L_5^{(2)} \equiv \sum_{\theta=1}^{12} H_\theta^{(2)} x'_{5\theta} = \sum_{\theta=1}^{12} H_\theta^{(2)} x_{2\theta}^2 = 0 \\ L_6^{(2)} \equiv \sum_{\theta=1}^{12} H_\theta^{(2)} x'_{6\theta} = \sum_{\theta=1}^{12} H_\theta^{(2)} x_{1\theta} x_{2\theta} = 0 . \end{cases}$$

We have

$$(7.16)\quad \sum_{\theta=1}^{12} (H_\theta^{(2)})^2 = \frac{\rho^6}{4} \{ (\beta_{111} - \beta_{122})^2 + (\beta_{222} - \beta_{112})^2 \} .$$

Furthermore the relation

$$(7.17)\quad G_{i\theta} = \sum_{j=1}^{6} c_{ij}^{(2)} x_{j\theta}^{(2)} \qquad (i = 1, 2, \cdots, 6)$$

gives us that

$$(7.18)\quad \begin{cases} G_{1\theta}^{(2)} = \frac{1}{4} - \frac{1}{4\rho_1^2} (x_{1\theta}^{(2)2} + x_{2\theta}^{(2)2}) \\ G_{2\theta}^{(2)} = \frac{1}{4\rho_1^2} x_{1\theta}^{(2)} \\ G_{3\theta}^{(2)} = \frac{1}{4\rho_1^2} x_{2\theta}^{(2)} \\ G_{4\theta}^{(2)} = \frac{-1}{4\rho_1^2} + \frac{1}{8\rho_1^4} (5x_{1\theta}^{(2)2} + x_{2\theta}^{(2)2}) \\ G_{5\theta}^{(2)} = \frac{-1}{4\rho_1^2} + \frac{1}{8\rho_1^4} (x_{1\theta}^{(2)2} + 5x_{2\theta}^{(2)2}) \\ G_{6\theta}^{(2)} = \frac{1}{\rho_1^4} x_{1\theta}^{(2)} x_{2\theta}^{(2)} , \end{cases}$$

and hence that

$$(7.19)\quad \begin{cases} \zeta_1^{(2)} = \frac{1}{4} \sum_{\theta=1}^{12} \left(1 - \frac{1}{4\rho_1^2} (x_{1\theta}^{(2)2} + x_{2\theta}^{(2)2}) \right) \varepsilon_\theta \\ \zeta_2^{(2)} = \frac{1}{4\rho_1^2} \sum_{\theta=1}^{12} x_{1\theta}^{(2)} \varepsilon_\theta \\ \zeta_3^{(2)} = \frac{1}{4\rho_1^2} \sum_{\theta=1}^{12} x_{2\theta}^{(2)} \varepsilon_\theta \\ \zeta_4^{(2)} = -\frac{1}{4\rho_1^2} \sum_{\theta=1}^{12} \varepsilon_\theta + \frac{1}{8\rho_1^4} \sum_{\theta=1}^{12} (5x_{1\theta}^{(2)2} + x_{2\theta}^{(2)2}) \varepsilon_\theta \\ \zeta_5^{(2)} = -\frac{1}{4\rho_1^2} \sum_{\theta=1}^{12} \varepsilon_\theta + \frac{1}{8\rho_1^4} \sum_{\theta=1}^{12} (x_{1\theta}^{(2)2} + 5x_{2\theta}^{(2)2}) \varepsilon_\theta \\ \zeta_6^{(2)} = \frac{1}{\rho_1^4} \sum_{\theta=1}^{12} x_{1\theta}^{(2)} x_{2\theta}^{(2)} \varepsilon_\theta . \end{cases}$$

Let us now proceed to the design $D^{(3)}$. The inverse matrix $(c_{ij}^{(3)})$ of the matrix $(a_{ij}^{(3)})$ in (5.17) is given by

$$(7.20)\quad (c_{ij}^{(3)}) = \begin{pmatrix} g_1 & 0 & 0 & g_2 & g_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 & 0 & 0 & k_2 & 0 & k_2 & 0 \\ 0 & 0 & k_1 & 0 & 0 & 0 & 0 & k_2 & 0 & k_2 \\ g_2 & 0 & 0 & g_3 & g_4 & 0 & 0 & 0 & 0 & 0 \\ g_2 & 0 & 0 & g_4 & g_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 & k_3 & 0 & k_4 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 0 & 0 & k_5 & 0 & k_4 \\ 0 & k_2 & 0 & 0 & 0 & 0 & k_4 & 0 & k_5 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 0 & 0 & k_4 & 0 & k_3 \end{pmatrix},$$

where

$$(7.21)\quad \begin{cases} g_1 = \dfrac{2\lambda_1}{(2\lambda_1-1)N}, & g_2 = \dfrac{-1}{2(2\lambda_1-1)aN} \\ g_3 = \dfrac{3\lambda_1-1}{4\lambda_1(2\lambda_1-1)a^2N}, & g_4 = \dfrac{1-\lambda_1}{4\lambda_1(2\lambda_1-1)a^2N} \\ h = \dfrac{1}{\lambda_1 a^2 N}, \end{cases}$$

and

$$(7.22)\quad \begin{cases} k_1 = \dfrac{3\lambda_2}{3\lambda_2-2\lambda_1^2}\cdot\dfrac{1}{aN} \\ k_2 = \dfrac{-\lambda_1}{2(3\lambda_2-2\lambda_1^2)}\cdot\dfrac{1}{a^2N} \\ k_3 = \dfrac{3-\lambda_1^2\lambda_2^{-1}}{12(3\lambda_2-2\lambda_1^2)}\cdot\dfrac{1}{a^3N} \\ k_4 = \dfrac{\lambda_1^2\lambda_2^{-1}-1}{4(3\lambda_2-2\lambda_1^2)}\cdot\dfrac{1}{a^3N} \\ k_5 = \dfrac{5-3\lambda_1^2\lambda_2^{-1}}{4(3\lambda_2-2\lambda_1^2)}\cdot\dfrac{1}{a^3N}. \end{cases}$$

For our present example we have

$$(7.23)\quad aN_3 = \sum_{\psi=1}^{N_3} x_{1\psi}^2 = \rho_1^2 \sum_{k=1}^{8}\cos^2\left(\frac{2\pi}{8}k\right) + \rho_2^2\sum_{k=1}^{8}\cos^2\left(\frac{2\pi}{8}k+\theta\right)$$
$$= \rho_1^2\frac{8}{2} + \rho_2^2\frac{8}{2} = 4(\rho_1^2+\rho_2^2),$$

which gives us, since $N = N_3 = 20$, $\rho_2^2 = 2\rho_1^2/3$,

$$a = \frac{4(\rho_1^2 + \rho_2^2)}{20} = \frac{\rho_1^2}{3}. \tag{7.24}$$

Now the relation

$$3\lambda_1 a^2 N_3 = \sum_{\psi=1}^{N_3} x_{1\psi}^4, \tag{7.25}$$

gives us similarly

$$3\lambda_1\left(\frac{\rho_1^2}{3}\right)^2 \cdot 20 = 3(\rho_1^4 + \rho_2^4) = \frac{13}{3}\rho_1^4, \tag{7.26}$$

and hence

$$\lambda_1 = 13/20. \tag{7.27}$$

Furthermore the relation

$$15\lambda_2 a^3 N_3 = \sum_{\psi=1}^{N_3} x_{1\psi}^6 = \frac{5}{2}(\rho_1^2 + \rho_2^2), \tag{7.28}$$

gives us similarly

$$\lambda_2 = 7/24. \tag{7.29}$$

We have

$$\left\{\begin{aligned}
G_{1\psi}^{(3)} &= \sum_{j=1}^{10} c_{1j}^{(3)} x_{j\psi}^{\prime(3)} = g_1 + g_2\left(x_{1\psi}^{(3)^2} + x_{2\psi}^{(3)^2}\right)\\
G_{2\psi}^{(3)} &= \sum_{j=1}^{10} c_{2j}^{(3)} x_{j\psi}^{\prime(3)} = k_1 x_{1\psi}^{(3)} + k_2 x_{1\psi}^{(3)^3} + k_2 x_{1\psi}^{(3)} x_{2\psi}^{(3)2}\\
G_{3\psi}^{(3)} &= \sum_{j=1}^{10} c_{3j}^{(3)} x_{j\psi}^{\prime(3)} = k_1 x_{2\psi}^{(3)} + k_2 x_{1\psi}^{(3)2} x_{2\psi}^{(3)} + k_2 x_{2\psi}^{(3)3}\\
G_{4\psi}^{(4)} &= \sum_{j=1}^{10} c_{4j}^{(3)} x_{j\psi}^{\prime(3)} = g_2 + g_3 x_{1\psi}^{(3)^2} + g_4 x_{2\psi}^{(3)^2}\\
G_{5\psi}^{(3)} &= \sum_{j=1}^{10} c_{5j}^{(3)} x_{i\psi}^{\prime(3)} = g_2 + g_4 x_{1\psi}^{(3)^2} + g_3 x_{2\psi}^{(3)^2}\\
G_{6\psi}^{(3)} &= \sum_{j=1}^{10} c_{6j}^{(3)} x_{j\psi}^{\prime(3)} = h x_{1\psi}^{(3)} x_{2\psi}^{(3)}\\
G_{7\psi}^{(3)} &= \sum_{j=1}^{10} c_{7j}^{(3)} x_{j\psi}^{\prime(3)} = k_2 x_{1\psi}^{(3)} + k_3 x_{1\psi}^{(3)^3} + k_4 x_{1\psi}^{(3)} x_{2\psi}^{(3)^2})\\
G_{8\psi}^{(3)} &= \sum_{j=1}^{10} c_{8j}^{(3)} x_{j\psi}^{\prime(3)} = k_3 x_{2\psi}^{(3)} + k_5 x_{1\psi}^{(3)^2} x_{2\psi}^{(3)} + k_4 x_{2\psi}^{(3)^3}\\
G_{9\psi}^{(3)} &= \sum_{j=1}^{10} c_{9j}^{(3)} x_{j\psi}^{\prime(3)} = k_3 x_{1\psi}^{(3)} + k_4 x_{1\psi}^{(3)^3} + k_5 x_{1\psi}^{(3)} x_{2\psi}^{(3)^2}\\
G_{10\psi}^{(3)} &= \sum_{j=1}^{10} c_{1j}^{(3)} x_{j\psi}^{\prime(3)} = k_2 x_{2\psi}^{(3)} + k_3 x_{2\psi}^{(3)^3} + k_4 x_{1\psi}^{(3)^2} x_{2\psi}.
\end{aligned}\right. \tag{7.30}$$

and hence we can write explicitly

$$\zeta_i^{(3)} = \sum_{\psi=1}^{20} G_{i\psi}^{(3)} \varepsilon_\psi \qquad (i = 1, 2, \cdots, 10) . \tag{7.31}$$

§ 8. **The evaluations of the probabilities of the events and the mean values of the stochastic variables associated with the estimations after preliminary test of significance.** In view of Theorems 1 and 2 and various examples discussed in § 5-7, our next task is to evaluate the probabilities such as Pr.$\{A_1\}$, Pr.$\{R_1\}$, Pr.$\{R_1A_2\}$ and Pr.$\{R_1R_2\}$ and the mean values of the types $E_{R_1}\{\cdot\}$, $E_{R_1A_2}\{\cdot\}$ and $E_{R_1R_2}\{\cdot\}$. Although there do not seem to exist any theoretical difficulties for this task under our formulation, some systematic approach may be profitable for the command of technical difficulties associated with the complexities of integrals. In this paragraph the uses of orthogonal vectors in connection with the rotatable design in the two dimensions will be enunciated as one of the possible approaches to our goal, whose systematic development is however postponed to another occasion.

Let us consider the construction of the right-hand side of (5.12). Let us define a system of indepenpent normalised stochastic variables:

$$\begin{cases} \chi_1 \equiv N^{-\frac{1}{2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha \\ \chi_2 \equiv (aN)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{1\alpha} \\ \chi_3 \equiv (aN)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{2\alpha} \\ \chi_4 \equiv (4\lambda_1 a^2 N)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha (x_{1\alpha}^2 - x_{2\alpha}^2) \\ \chi_5 \equiv ((2\lambda_1 - 1) N)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha \left(1 - \frac{1}{2a}(x_{1\alpha}^2 + x_{2\alpha}^2)\right) \\ \chi_6 \equiv (\lambda_1 a^2 N)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} \varepsilon_\alpha x_{1\alpha} x_{2\alpha} , \end{cases} \tag{8.01}$$

which have the properties:

$$\begin{cases} (1^\circ) & E\{\chi_i\} = 0 & (i = 1, 2, 3, \cdots, 6) \\ (2^\circ) & E\{\chi_i^2\} = \sigma^2 & (i = 1, 2, 3, \cdots, 6) \\ (3^\circ) & E\{\chi_i \chi_j\} = 0 & (i \neq j;\ i, j = 1, 2, \cdots, 6) \\ (4^\circ) & \text{Each } \chi_i \text{ is normally distributed.} \end{cases} \tag{8.02}$$

Now let us define

$$\begin{cases} \chi_1' \equiv N^{-\frac{1}{2}} \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha) = \chi_1 \\ \chi_2' \equiv (aN)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha) x_{1\alpha} = \chi_2 \end{cases}$$

$$(8.03)\quad \begin{cases} \chi_3' \equiv (aN)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)\, x_{2\alpha} = \chi_3 \\ \chi_4' \equiv (4\lambda_1 a^2 N)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)(x_{1\alpha}^2 - x_{2\alpha}^2) \\ \chi_5' \equiv ((2\lambda_1 - 1)N)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)\left(1 - \frac{1}{2a}(x_{1\alpha}^2 + x_{2\alpha}^2)\right) \\ \chi_6' \equiv (\lambda_1 a^2 N)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)\, x_{1\alpha} x_{2\alpha}\,. \end{cases}$$

In view of (5.12) we have

$$(8.04)\quad \begin{aligned} \sum_{\alpha=1}^{N} e_\alpha^2 &= \left(\sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha)^2 - \sum_{h=1}^{6} \chi_h'^2\right) \\ &\quad + \sum_{h=1}^{6} \chi_h'^2 - \sum_{j=1}^{3} \chi_j^2 \\ &= \left(\sum_{\alpha=1}^{N} (H_\alpha + \varepsilon_\alpha) - \sum_{h=1}^{6} \chi_h'^2\right)^2 + \sum_{h=4}^{6} \chi_h'^2\,, \\ &\equiv e_{(1)}^2 + e_{(2)}^2\,, \quad \text{say,} \end{aligned}$$

which shows that (1°) $e_{(1)}^2/\sigma^2$ and $e_{(2)}^2/\sigma^2$ are independently distributed in the non-central chi-square distributions with their respective $(N-6)$ and 3 degrees of freedom, (2°) $e_{(1)}^2$, $e_{(2)}^2$, $\chi_2^1 + \chi_2^2 + \chi_3^2$ and u_ν^2 are mutually independent.

Consequently we have

$$(8.05)\quad \begin{aligned} \Pr.\{A\} &\equiv \Pr.\left\{\sum_{\alpha=1}^{N} e_\alpha^2 / (N-p)\, u_\nu^2 \leqq F_\nu^{N-p}(\alpha)\right\} \\ &= \Pr.\{\chi_{((N-p))}'^2 / (N-p)\, u_\nu^2 \leqq F_\nu^{N-p}(\alpha)\}\,, \end{aligned}$$

where $\chi_{((N-p))}'^2/\sigma^2$ denotes the stochastic variable distributed in the non-central *chi*-square distribution with the $(N-p)$ degrees of freedom.

In order to evaluate the mean value such as

$$(8.06)\quad \begin{aligned} S_{R_1} &\equiv \sum_{i=1}^{q} \sum_{j=1}^{q} \sum_{\alpha=1}^{N_1} \sum_{\beta=1}^{N_1} G_{i\alpha}^{(2)} G_{j\beta}^{(2)} E_{R_1}\{\varepsilon_\alpha \varepsilon_\beta\} \\ &= \sum_{i=1}^{q} \sum_{j=1}^{q} E_{R_1}\left\{\left(\sum_{\alpha=1}^{N_1} G_{i\alpha}^{(2)} \varepsilon_\alpha\right)\left(\sum_{\beta=1}^{N_1} G_{i\beta}^{(2)} \varepsilon_\beta\right)\right\}\,, \end{aligned}$$

in (3.31), let us write out

$$(8.07)\quad \sum_{\alpha=1}^{N_1} G_{i\alpha}^{(2)} \varepsilon_\alpha = \sum_{h=1}^{N_1} S_{ih} \chi_h' \qquad (i = 1, 2, \cdots, q)\,,$$

by means of the system $\{\chi_h'\}$ $(h = 1, 2, \cdots, N_1)$ with the deterministic coefficients $\{S_{ih}\}$. In the consequence it suffices us now to evaluate

$$(8.08)\quad E_{R_1}\{\chi_h'^2\} \qquad (h = 1, 2, \cdots, N_1)$$

$$(8.09)\quad E_{R_1}\{\chi_h' \chi_l'\} \qquad (h \neq l\,;\ h, j = 1, 2, \cdots, N_1)\,,$$

where the condition R_1 is defined by

$$\frac{\chi_{N_1}'^2 + \chi_{N_1-1}'^2 + \cdots + \chi_4'^2}{(N_1 - 4)\, u_\nu^2} \geqq F_\nu^{N_1-3}(\alpha_1)\,. \tag{8.10}$$

Since we have

$$\chi_h' \chi_l' = \left(\frac{\chi_h' + \chi_l'}{\sqrt{2}}\right)^2 + \left(\frac{\chi_h' - \chi_l'}{\sqrt{2}}\right)^2, \tag{8.11}$$

and $(\chi_h' + \chi_l')/\sqrt{2}$ and $(\chi_h' - \chi_l')/\sqrt{2}$ are independently distributed in the normal distribution $N(0, 1)$, our task is reduced to evaluate (8.07), for which no theretical difficulties exist.

When more than one preliminary tests should be applied, the evaluations of the probabilies Pr. $\{R_1 A_2\}$ and Pr. $\{R_1 R_2\}$ and of the mean values $E_{R_1 A_2}\{\cdot\}$ and $E_{R_1 R_2}\{\cdot\}$ can be treated in the similar way in its principle by appealing to the orthogonal system of vectors, that is, the mutually independent stochastic variables each of which is distributed in the normal distribution $N(0, \sigma^2)$.

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