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ON A NON-PARAMETRIC TEST IN LIFE TEST

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§ 1. **Introduction.** There are various topics in the theory of life test. The purpose of this paper is to treat the problem from the standpoint of "test of fit." In life test we start with M items and stop the test at a certain preassigned time T or stop when the s-th death has occurred, because if we wait till all the death occur we have to wait for a long time, c.f. B. Epstein [2], [3], G. Ishii [5]. In a previous paper, we had treated the test of fit in life test in the case that we stop when the s-th death has occurred, G. Ishii [4]. In the present paper we treat the case when we stop the life test at a certain preassigned time T.

Let X be a random variable having the continuous distribution function. In order to test the hypothesis H_0 that the distribution function is a known function $F(x)$, F. N. David [1] and M. Okamoto [6] have proposed the following non-parametric test:

Let x_i $(i=1, 2, \cdots, M)$ be M independent observation of a random variable X. There are real numbers $\{a_i\}$ $i=1, 2, \cdots, m-1$ such that $F(a_i) - F(a_{i-1}) = 1/m$, $i=1, 2, \cdots, m$ where $a_0 = -\infty$, $a_m = +\infty$. $(a_{i-1}, a_i]$ will be called "part." Let v be the number of parts which contain no x's. If v is too large, we reject H_0.

Now we shall apply the above non-parametric test to life test.

§ 2. **Distribution of v in life test.** In the following, we shall treat the life test in the case that we stop the test at a certain preassigned time T, where $F(T)=t$. We call T 'stop time.' We take the stop time T as is equal to a divided point a_n such that $F(T)=F(a_n)=t=n/m$. We start the life test with M items and stop at time T. Suppose that there are N deaths. Then N is a random variable which follows the binomial distribution.

$$P(\mathfrak{N}=N)=\binom{M}{N}t^N(1-t)^{M-N}. \tag{1}$$

N observations are situated in some $(-\infty, a_1]$, $(a_1, a_2]$, $\cdots$, $(a_{n-1}, T]$. Let v be the number of parts which contain no x's in the above n parts.

For fixed N, we have

$$P(v=\nu\,|\,\mathfrak{N}=N)=n^{-N}\binom{n}{\nu}\sum_{k=1}^{n-\nu}(-1)^{n-\nu-k}\binom{n-\nu}{k}k^N. \tag{2}$$

Then we have

$$(3) \qquad P(v=\nu, \mathfrak{N}=N) = P(v=\nu|\mathfrak{N}=N) \cdot P(\mathfrak{N}=N) .$$

We put $v^{(s)} = v(v-1) \cdots (v-s+1)$.

Then the s-th factorial moment is written as

$$(4) \qquad E(v^{(s)}) = \frac{n!}{(n-s)!}\left(1-\frac{s}{m}\right)^M .$$

Putting $s=1, 2$,

$$(5) \qquad E(v) = n(1-1/m)^M ,$$

$$(6) \qquad E(v(v-1)) = n(n-1)(1-2/m)^M .$$

If $M \to \infty$, $m \to \infty$, $n \to \infty$ under the restriction of $M = mr$ (r is a constant) and $n = mt$ (t is a constant),

$$(7) \qquad E(v/n) = e^{-r}(1-r/2m) + O(m^{-2}) ,$$

$$(8) \qquad D^2(v/n) = e^{-2r}(e^r - 1 - tr)/n + O(m^{-2}) .$$

Under the above conditions we have next theorem.

Theorem 1. *v/n is asymptotically normally distributed with mean e^{-r} and variance $e^{-2r}(e^r-1-tr)/n$, where $M=mr$, $n=mt$, (r and t are constants).*

Proof. It is sufficient to prove that the moments of $(n/c)^{1/2}(v/n - e^{-r})$ tend to the moments of the standard normal distribution, where $c = e^{-2r}(e^r - 1 - tr)$. The proof is almost parallel to that of theorem 1 of M. Okamoto [6], and theorem 2 of B. Sherman [7].

Denoting by $B_r^{(n)}$ the Bernoulli's number of order n and degree r.

$$(9) \quad E\left\{\left(\frac{n}{c}\right)^{1/2}\left(\frac{v}{n}-e^{-r}\right)\right\}^l = \left(\frac{n}{c}\right)^{l/2} \sum_{k=0}^{l} \binom{l}{k} (-e^{-r})^{l-k} E\left(\frac{v}{n}\right)^k$$

$$= \left(\frac{n}{c}\right)^{l/2} \sum_{k=0}^{l} \binom{l}{k} (-e^{-r})^{l-k} n^{-k} \sum_{q=0}^{k} \binom{k}{q} B_q^{(q-k)} E(v^{(k-q)})$$

$$= \left(\frac{n}{c}\right)^{l/2} \sum_{k=0}^{l} \sum_{q=0}^{k} \frac{l!\,(-1)^{l-k} e^{-r(l-k)}}{k!\,(l-k)!} \frac{k!}{q!\,(k-q)!} B_q^{(q-k)} \frac{n!\,(m-k+q)^M}{(n-k+q)!\,m^M n^k}$$

$$= \frac{(-1)^l n^{l/2} l!}{(e^r-1-tr)^{l/2}} \left\{a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_\alpha}{n^\alpha} + \cdots\right\} .$$

We have to evaluate these a_i. In the above expansion

$$(10) \quad \frac{n!\,(m-k+q)^M}{(n-k+q)!\,m^M n^k} = \frac{1}{n^k}\left(1-\frac{k-q}{m}\right)^M n(n-1)\cdots(n-k+q+1)$$

$$= n^{-q}\left(1-\frac{k-q}{m}\right)^M \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-q-1}{n}\right)$$

$$= x^q \{1-(k-q)tx\}^{M} \prod_{j=1}^{k-q-1}(1-jx)$$
$$= x^q \cdot G(x)\,,$$

where

$$x = 1/n$$
$$G(x) = \{1-(k-q)tx\}^{\frac{r}{xt}} \prod_{j=1}^{k-q-1}(1-jx)\,.$$

Expanding this, we have

$$G(x) = a_{kq0} + a_{kq1}x + a_{kq2}x^2 + \cdots,$$

where

(11) $a_{kq0} = e^{-(k-q)r}$

$$a_{kqp} = \frac{1}{p!}\left\{\frac{d^pG(x)}{dx^p}\right\}_{x=0}$$
$$= \frac{1}{p!}\sum_{s=0}^{p-1}\binom{p-1}{s}\frac{d^{p-1}G(x)}{dx^{p-1}}\frac{d^{s+1}\log G(x)}{dx^{s+1}}\bigg|_{x=0}$$
$$\log G(x) = -\sum_{j=1}^{k-q-1}\left(jx+\frac{j^2x^2}{2}+\frac{j^3x^3}{3}+\cdots+\frac{j^{s+1}x^{s+1}}{s+1}+\cdots\right)$$
$$-\frac{r}{xt}\left((k-q)tx+\frac{(k-q)^2t^2x^2}{2}+\cdots+\frac{(k-q)^{s+2}t^{s+2}x^{s+2}}{s+2}+\cdots\right).$$

Therefore

(12) $$\frac{d^{s+1}\log G(x)}{dx^{s+1}}\bigg|_{x=0} = -\sum_{j=1}^{k-q-1} s!\,j^{s+1} - \frac{(s+1)!\,(k-q)^{s+2}}{s+2}rt^{s+1}$$
$$= s!\,b_{kqs} \quad \text{(say)}$$
$$b_{kqs} = -\sum j^{s+1} - \frac{s+1}{s+2}rt^{s+1}(k-q)^{s+2}\,.$$

Then b_{kqs} is a polynomial in k of degree $s+2$.
Putting $s=0$,

$$b_{kq0} = -\frac{(k-q)(k-q-1)}{2} - \frac{1}{2}rt(k-q)^2$$
$$= -(1+rt)\frac{k^2}{2} + Ak + B \quad \text{(say)}\,.$$

From (11),

(13) $$a_{kpq} = \frac{1}{p!}\sum_{s=0}^{p-1}\frac{(p-1)!}{s!\,(p-s-1)!}a_{kq(p-1-s)}(p-s-1)!\,s!\,b_{kqs}$$
$$= \frac{1}{p}\sum_{s=0}^{p-1} a_{kq(p-1-s)}b_{kqs}\,.$$

Then we have next equations

$$a_{kq0} = e^{-(k-q)s}$$
$$a_{kq0} b_{kq0} - a_{kq1} = O$$
$$a_{kq0} b_{kq1} + a_{kq1} b_{kq0} - 2a_{kq2} = O$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{kq0} b_{kq(i-q-1)} + a_{kq1} b_{kq(i-q-2)} + \cdots + a_{kq(i-q-1)} b_{kq0} - (i-q) a_{kq(i-q)} = O .$$

Hence we have

$$a_{kq(i-q)} = e^{-(k-q)r} B_{kqi} , \tag{14}$$

where

$$B_{kqi} = \frac{(b_{kq0})^{i-q}}{(i-q)!} + (\text{terms of lower degree in } k)$$
$$= \frac{1}{(i-q)!}\left(-\frac{1+rt}{2}\right)^{i-q} k^{2(i-q)} + \sum_{j=0}^{2(i-q)-1} A_j k^j .$$

In (9) we put

$$\sum_{q=0}^{l} \sum_{k=q}^{l} \frac{(-1)^k e^{rk}}{q!\,(l-k)!\,(k-q)!} B_q^{(q-k)} \frac{n!\,(m-k+q)^M}{(n-k+q)!\,m^M n^k} \tag{15}$$
$$= a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_\alpha}{n^\alpha} + \cdots .$$

If we denote by a_{iq} the coefficient of n^{-i} in the expansion in the power of n^{-1} of

$$\sum_{k=q}^{l} \frac{(-1)^k e^{rk}}{(l-k)!\,(k-q)!} B_q^{(q-k)} \frac{n!\,(m-k+q)^M}{(n-k+q)!\,m^M n^k}$$
$$= \sum_{k=q}^{l} \frac{(-1)^k e^{rk}}{(l-k)!\,(k-q)!} B_q^{(q-k)} x^q \{a_{kq0} + a_{kq1} x + \cdots + a_{kq(i-q)} x^{i-q} + \cdots\} ,$$

we have

$$a_i = \sum_{q=0}^{l} \frac{a_{iq}}{q!} \tag{16}$$

and

$$a_{iq} = \sum_{k=q}^{l} \frac{(-1)^k e^{rk}}{(l-k)!\,(k-q)!} B_q^{(q-k)} a_{kq(i-q)}$$
$$= e^{qr} \sum_{k=q}^{l} \frac{(-1)^k}{(l-k)!\,(k-q)!} B_q^{(k-b)} B_{kqi} .$$

$B_q^{(q-k)}$ is the coefficient of the expansion of $(q/(e^t-1))^{q-k}$, then $B_q^{(q-k)}$ is the polynomial in k of degree q with 2^{-q} as the coefficient of the term of highest degree. Therefore

$$(17)\quad a_{iq}=e^{qr}\sum_{k=q}^{l}\frac{(-1)^k}{(l-k)!\,(k-q)!}\left\{\frac{k^q}{2^q}+\cdots\right\}\left\{\frac{k^{2(i-q)}}{(i-q)!}\left(-\frac{1+tr}{2}\right)^{i-q}+\sum A_jk^j\right\}$$
$$=\frac{e^{qr}}{2^q(i-q)!}\left(-\frac{1+tr}{2}\right)^{i-q}\sum_{k=q}^{l}\frac{(-1)^k}{(l-k)!\,(k-q)!}\left\{k^{2i-q}+\begin{matrix}\text{terms of lower}\\ \text{degree in } k.\end{matrix}\right\}.$$

As

$$\sum_{k=q}^{l}\frac{(-1)^l k^{l'}}{(l-k)!\,(k-q)!}=\begin{cases}0 & l'<l-q\\ 1 & l'=l-q\end{cases}$$

we obtain

$$a_{iq}=0 \qquad i<\frac{l}{2}$$

$$a_{hq}=\frac{e^{qr}\left(-\frac{1+rt}{2}\right)^{i-q}}{2^q(h-q)!} \qquad i=\frac{l}{2}=h \quad (\text{when } l \text{ is even}).$$

Thus from (16)

$$a_i=0 \qquad \text{when} \quad i<\frac{l}{2}$$

$$a_h=\sum_{q=0}^{2h}\frac{a_{hq}}{q!}=\sum_{q=0}^{2h}\frac{1}{q!\,(h-q)!}\left(\frac{e^r}{2}\right)^q\left(-\frac{1+tr}{2}\right)^{h-q}$$
$$=\frac{1}{h!}\frac{(e^r-1-tr)^h}{2^h}.$$

Then we have

$$\lim_{n\to\infty}E\left\{\left(\frac{n}{c}\right)^{1/2}\left(\frac{v}{n}-e^{-r}\right)\right\}^l=0 \qquad (\text{when } l \text{ is odd})$$

$$\lim_{n\to\infty}E\left\{\left(\frac{n}{c}\right)^{1/2}\left(\frac{v}{n}-e^{-r}\right)\right\}^{2h}=\frac{a_h(2h)!}{(e^r-1-tr)^h}$$
$$=\frac{(2h)!}{2^h h!} \qquad \left(\begin{matrix}\text{when } l \text{ is even,}\\ l=2h\end{matrix}\right).$$

This completes the proof.

§ 3. **The power function.** The power function of the test with respect to the alternative hypothesis H_1 is

$$P=P(v\geqq l\,|\,H_1)\,.$$

Under H_1: distribution function is $F_1(x)$, and the range of distribution is equal to that of $F(x)$, where $F(T)=\tau$

$$\int_{i-1/m}^{i/m}dH(x)=p_i\,, \qquad i=1,2,\cdots,n$$

$$\text{where}\quad H(x)=F_1(F^{-1}(x))\,, \qquad \sum_{i=1}^{n}p_i=\tau\,.$$

We put $p_i/\tau = q_i$, then $\sum_{i=1}^n q_i = 1$

$$P(\mathfrak{N} = N|H_1) = \binom{M}{N} \tau^N (1-\tau)^{M-N}$$

$$P(v = k|\mathfrak{N} = N|H_1) = \sum_{j=1}^{n-k} (-1)^{n-k-j} \binom{n-j}{k} \sum_{(i_1\cdots i_j)}^{n} (q_{i_1} + \cdots + q_{i_j})^N ,$$

where $\sum_{(t_1\cdots t_l)}^{j}$ denotes the summation over all combinations $(t_1, \cdots, t_l)$ drawn from $(1, 2, \cdots, j)$.

$$P(v = k, \mathfrak{N} = N|H_1) = P(v = k|\mathfrak{N} = N|H_1) \cdot P(\mathfrak{N} = N|H_1)$$

$$E(v^{(s)}|H_1) = \sum_N s! \sum_{(i_1\cdots i_s)}^{n} (1 - q_{i_1} - \cdots - q_{i_s})^N \binom{M}{N} \tau^N (1-\tau)^{M-N}$$

$$= s! \sum_{(i_1\cdots i_s)}^{n} (1 - p_{i_1} - \cdots - p_{i_s})^M .$$

Putting $s = 1, 2$, we get

$$E(v|H_1) = \sum_{i=1}^{n} (1 - p_i)^M$$

$$E(v(v-1)|H_1) = \sum_{i \neq j}^{n} (1 - p_i - p_j)^M .$$

If $F_1(x)$ is absolutely continuous with respect to $F(x)$ and its relative density is differentiable, putting $H'(x) = h(x)$,

$$p_i = \frac{1}{m} h\left(\frac{i}{m}\right) - \frac{1}{2m^2} h'\left(\frac{i}{m}\right) + O(m^{-3})$$

$$(1 - p_i)^M = e^{-rh\left(\frac{i}{m}\right)} \left[1 + \frac{r}{2m} \left\{h'\left(\frac{i}{m}\right) - h^2\left(\frac{i}{m}\right)\right\} + O(m^{-2})\right]$$

$$E(v/n|H_1) = \frac{1}{t} \int_0^t e^{-rh(x)} dx - \frac{r}{2mt} \int_0^t e^{-rh(x)} h^2(x)\, dx + O(m^{-2}) .$$

The power function

$$P = \sum_{k \geqq l} \sum_{N=0}^{M} P(v = k, \mathfrak{N} = N|H_1)$$

satisfies the next condition when $\tau \leqq t$

$$P \geqq P(v \geqq l|H_0) .$$

This inequality can be proved by the same method in theorem 3 of Okamoto [6].

Thus we have the next theorem.

Theorem 2. *The test based on v is unbiased against the class of alternative hypotheses, $\tau \leqq t$.*

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