On Limiting Distributions for One-dimensional Diffusion Processes

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ON LIMITING DISTRIBUTIONS FOR ONE-DIMENSIONAL DIFFUSION PROCESSES

By

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§ 1. **Introduction.** Let $P(t, x, \cdot)$ be the stationary transition probabilities of a one-dimensional diffusion process $X(t)$:

$$P(t, x, E) = P\{X(t+s) \in E/X(s)=x\}, \quad t, s \geqq 0.$$

The purpose of this paper is to show the existence of the limit of transition probabilities $P(t, x, \cdot)$ as $t \to \infty$. That the limiting distributions satisfy certain differential equations was stated by A. KOLMOGOROV [7].[1] S. BERNSTEIN [1] has discussed the existence of the limits for the processes determined by the stochastic difference equations under somewhat stronger conditions on diffusion coefficients which imply $E\{|X(t)|^{\lambda}\} < L$ for all large t with some positive constants λ and L. Theorem 1 asserts that if the mean of the first passage time from any one point to another is finite, then the limiting distribution exists which can be given explicitly as the solution of a certain differential equation. We use the strong Markov property and then apply a renewal theorem.

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§ 2. **Definitions, assumptions and notations.** Consider the backward diffusion equation

$$\text{(1)} \qquad \frac{\partial}{\partial t} u(t, x) = \Omega u(t, x),$$

$$\Omega = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \quad (-\infty \leqq r_1 < x < r_2 \leqq \infty),$$

where $a(x)(>0)$, $a'(x)$ and $b(x)$ are continuous in the open interval (r_1, r_2). Following FELLER [6], we say that $X(t)$ is a diffusion process in (r_1, r_2) obeying (1) if it has stationary transition probabilities $P(t, x, \cdot)$ satisfying the following three conditions:

$$\text{(A)} \qquad P(t, x, E) = P\{X(t+s) \in E/X(s)=x\} \qquad (t, s \geqq 0)$$

1) Numbers in brackets refer to references at the end of this paper.

is defined for all $x \in (r_1, r_2)$, all Borel sets $E \subset (r_1, r_2)$ and $t > 0$, and is for fixed E Borel measurable in both t and x,

(B) $$P(t+s, x, E) = \int_{r_1}^{r_2} P(t, x, dy) P(s, y, E),$$

(C) for every function $f(x)$ continuous and bounded in (r_1, r_2) the Laplace transform $F(x) = \int_0^\infty e^{-\lambda t} dt \int_{r_1}^{r_2} f(y) P(t, x, dy)$ $(\lambda > 0)$ satisfies

$$\lambda F(x) - \Omega F(x) = f(x), \quad r_1 < x < r_2.$$

Throughout this paper we consider the above $X(t)$ process in case of inaccessible boundaries,[2] and assume that almost all sample functions are continuous.

The process with initial position $a(\in (r_1, r_2))$ is denoted by $X^{(a)}(t)$. We define the first passage time $\tau_\rho^{(x)}$ from x to ρ by

$$\tau_\rho^{(x)} = \begin{cases} \inf\{t; X^{(x)}(t) = \rho\} & \text{if } X^{(x)}(t) \text{ reaches } \rho \text{ within finite time,} \\ +\infty & \text{otherwise,} \end{cases}$$

and put $F_\rho^{(x)}(t) = P\{\tau_\rho^{(x)} \leqq t\}$. For a fixed $x_0 \in (r_1, r_2)$ we put $B(x) = \exp\left\{\int_{x_0}^x b(t)/a(t)\,dt\right\}$.

§ 3. **Limits of transition probabilities.** First we state the conditions for $E(\tau_\rho^{(x)}) < \infty$ which are necessary and sufficient in case of inaccessible boundaries. These conditions are essential to the existence of the limiting distributions.

Lemma 1. (DOOB [3]). If the following two conditions are satisfied

(2) $$\begin{cases} \text{(i)} & \displaystyle\int_{r_1}^{x_0} 1/B(t)\,dt = \int_{x_0}^{r_2} 1/B(t)\,dt = \infty, \\ \text{(ii)} & \displaystyle\int_{r_1}^{r_2} B(t)/a(t)\,dt < \infty, \end{cases}$$

then we have $E(\tau_\rho^{(x)}) < \infty$ for any $x, \rho \in (r_1, r_2)$ and

(3) $$E(\tau_\rho^{(x)}) = \begin{cases} \displaystyle\int_\rho^x 1/B(t)\,dt \int_t^{r_1} B(s)/a(s)\,ds & \text{if } \rho > x, \\ \displaystyle\int_\rho^x 1/B(t)\,dt \int_t^{r_2} B(s)/a(s)\,ds & \text{if } \rho < x. \end{cases}$$

Proof. See DOOB [3] (Theorem 9.9, p. 202) or FELLER [6].

Lemma 2. Let $F(t)$ be a distribution function of non-lattice type such that $F(+0) = 0$ and $\mu = \int_0^\infty t\,dF(t) \leqq \infty$, and let $\varphi(t)$ be a continuous func-

2) For the classification of boundaries, see FELLER [5] and [6].

tion defined in $[0, \infty)$ with $\varphi(t)=O(t^{-1-\alpha})$ $(t\to\infty)$ for some positive constant α. Put

$$M(t)=F(t)+F*F(t)+F*F*F(t)+\cdots.^{3)}$$

Then we have

$$\lim_{t\to\infty}\int_0^t \varphi(t-u)\,dM(u)=(1/\mu)\cdot\int_0^\infty \varphi(u)\,du.^{4)}$$

Proof. Put $\Phi(u,t)=M(t)-M(t-u)$. Then by BLACKWELL's theorem ([2], p. 145) we have for any fixed $u>0$

$$\lim_{t\to\infty}\Phi(u,t)=u/\mu.^{4)} \tag{4}$$

Since $M(t)$ is the expected number of occurrences of a recurrent event with the distribution of recurrence time $F(\cdot)$ before time t, we have

$$M(t+s)\leqq M(t)+M(s)+1, \qquad t,s\geqq 0.$$

Therefore

$$\Phi(u+1,t)-\Phi(u,t)<c_1 \qquad \text{for any } 0\leqq u\leqq t-1, \tag{5}$$

with some constant c_1.

Next let us write

$$\begin{aligned}\int_0^t \varphi(t-u)\,dM(u) &= \int_0^t \varphi(u)\,d_u\Phi(u,t)\\ &= \int_0^{n_0}\varphi(u)\,d_u\Phi(u,t)+\int_{n_0}^t \varphi(u)\,d_u\Phi(u,t)\\ &= I_1+I_2,\end{aligned}$$

and for $\varepsilon>0$, take an integer n_0 such that

$$\begin{cases}\sum_{n=n_0}^{\infty} 1/n^{1+\alpha}<\varepsilon,\\ |\varphi(u)|<c_2/u^{1+\alpha} \qquad \text{for all } u\geqq n_0 \ (c_2\text{: constant}).\end{cases}$$

Then by (5) we have

$$\begin{aligned}|I_2| &= \left|\sum_{n=n_0}^{[t]-1}\int_n^{n+1}\varphi(u)\,d_u\Phi(u,t)+\int_{[t]}^t \varphi(u)\,d_u\Phi(u,t)\right|\\ &\leqq c_1\cdot\sum_{n=n_0}^{\infty} c_2/n^{1+\alpha}<c_1\cdot c_2\cdot\varepsilon \qquad \text{for all } t>n_0.\end{aligned} \tag{6}$$

On the other hand by (5) and the continuity of $\varphi(u)$

$$\lim_{t\to\infty} I_1=(1/\mu)\int_0^{n_0}\varphi(u)\,du \qquad \text{for fixed } n_0. \tag{7}$$

By (6) and (7), the lemma is proved.

3) $*$ means convolution.
4) When μ is infinite, we interprete this as 0.

Using the above lemmas we prove the following

Theorem 1. *Let $X(t)$ be a diffusion process obeying* (1) *and $P(t, x, \cdot)$ be its transition probabilities. Then if the condition* (2) *is satisfied, $P(t, x, \cdot)$ has the limit distribution as $t \to \infty$ independent of x, with density*

$$p(y) = \left\{ \int_{r_1}^{r_2} B(t)/a(t)\,dt \right\}^{-1} \cdot B(y)/a(y),$$

which is a solution of the differential equation

$$\frac{1}{p(y)} \cdot \frac{dp(y)}{dy} = \{b(y) - a'(y)\}/a(y),$$

satisfying $\int_{r_1}^{r_2} p(y)\,dy = 1.$

Proof. Take ρ_1, ρ_1', ρ_2 and ρ_2' such that $r_1 < \rho_1 < \rho_1' < \rho_2' < \rho_2 < r_2$, and let E be a Borel subset of the open interval (ρ_1', ρ_2'). In the following arguments, E and ρ's are arbitrary but fixed under the above conditions.

We say that the "event A_i" occurs at the moment when a path starting from ρ_i' returns for the first time to ρ_i' after reaching ρ_i $(i = 1, 2)$, and define $P_0(t, x, E)$; $P_n(t, \rho_i', E)$ as follows $(i = 1, 2)$:

(i) $P_0(t, x, E)$ is the probability that $X^{(x)}(t) \in E$ neither reaching ρ_1 nor ρ_2 up to time t $(x \in (\rho_1, \rho_2))$,

(ii) $P_n(t, \rho_i', E)$ is the probability that up to time t exactly n A_i's occur and $X^{(\rho_i')}(t) \in E$ neither reaching ρ_1 nor ρ_2 after the n-th occurrence of A_i $(n = 1, 2, \cdots)$. Also put

(iii) $F_i(t) = F_{\rho_i}^{(\rho_i')} * F_{\rho_i'}^{(\rho_i)}(t)$.

Then by the strong Markov property ([8], p. 457) of the $X(t)$ process, we have

$$\begin{aligned} P_n(t, \rho_i', E) &= \int_0^t P_{n-1}(t-u, \rho_i', E)\,dF_i(u) \\ &= \int_0^t P_0(t-u, \rho_i', E)\,dF_i^{*n}(u)^{5)} \qquad (i = 1, 2;\ n = 1, 2, \cdots), \end{aligned}$$

and $P(t, \rho_1', E)$ can be written in the form

$$\begin{aligned} P(t, \rho_1', E) &= \sum_{n=0}^{\infty} P_n(t, \rho_1', E) + \int_0^t \{\sum_{n=1}^{\infty} P_n(t-u, \rho_2', E)\}\,dF_{\rho_2'}^{(\rho_1')}(u) \\ &= P_0(t, \rho_1', E) + \int_0^t P_0(t-u, \rho_1', E)\,dM_1(u) \qquad (8) \\ &\quad + \int_0^t \left\{ \int_0^{t-u} P_0(t-u-s, \rho_2', E)\,dM_2(s) \right\} dF_{\rho_2'}^{(\rho_1')}(u), \end{aligned}$$

where $M_i(u) = F_i(u) + F_i^{*2}(u) + F_i^{*3}(u) + \cdots$, $(i = 1, 2)$.

5) F^{*n} means the n-th iterated convolution of F.

Next, let $f(x)$ be any continuous function with the support entirely contained in the interior of (ρ_1', ρ_2') and put

$$\varphi_i(t)=\int_{\rho_1'}^{\rho_2'} f(y)P_0(t, \rho_i', dy), \quad t\geqq 0, \quad (i=1,2).$$

Then by (8) we have

$$(9) \qquad \int_{\rho_1'}^{\rho_2'} f(y)P(t, \rho_1', dy)=\varphi_1(t)$$
$$+\int_0^t \varphi_1(t-u)dM_1(u)+\int_0^t dF_{\rho_2'}^{(\rho_1')}(u)\int_0^{t-u}\varphi_2(t-u-s)dM_2(s).$$

Here, $\varphi_i(t)$ $(i=1,2)$ are continuous[6] in $(0,\infty)$ and $\varphi_i(t)=O(e^{-ct})$[6] as $t\to\infty$ with some positive constant c. For, we have for any $x\in(\rho_1,\rho_2)$ $P_0(t_0, x, (\rho_1, \rho_2))<c'<1$ with some positive constant c' and $t_0>0$. Therefore, we have $P(nt_0, x, (\rho_1,\rho_2))<c'^n$, hence $P_0(t, x, (\rho_1,\rho_2))=O(e^{-ct})$ $(t\to\infty)$ for some positive constant c and so is $\varphi_i(t)$.

Moreover, $F_i(t)$ $(i=1,2)$ are obviously of non-lattice type with $F_i(+0)=0$ and $\mu_i=\int_0^\infty t dF_i(t)<\infty$ by Lemma 1. Hence by Lemma 2 we have

$$\lim_{t\to\infty}\int_0^t \varphi_i(t-u)dM_i(u)=(1/\mu_i)\int_0^\infty \varphi_i(u)du.$$

Inserting this into (9)

$$(10) \qquad \lim_{t\to\infty}\int_{\rho_1'}^{\rho_2'} f(y)P(t, \rho_1', dy)=\sum_{i=1,2}(1/\mu_i)\int_0^\infty \varphi_i(u)du.$$

To calculate the right hand side of (10), first we note that $\int_0^\infty dt\int_{\rho_1'}^{\rho_2'} f(y) \times P_0(t, x, dy)\equiv y(x)$ is the solution of $-\Omega y=f$ with the boundary conditions $y(\rho_1)=y(\rho_2)=0$, thus having

$$(11) \qquad \int_0^\infty \varphi_i(u)du$$
$$=\begin{cases}\left\{\int_{\rho_1}^{\rho_2} 1/B(t)dt\right\}^{-1}\left\{\int_{\rho_1}^{\rho_1'} 1/B(t)dt\right\}\int_{\rho_1'}^{\rho_2'} f(y)B(y)/a(y)\int_y^{\rho_2} 1/B(t)dtdy \\ \left\{\int_{\rho_1}^{\rho_2} 1/B(t)dt\right\}^{-1}\left\{\int_{\rho_2'}^{\rho_2} 1/B(t)dt\right\}\int_{\rho_1'}^{\rho_2'} f(y)B(y)/a(y)\int_{\rho_1}^{y} 1/B(t)dtdy \end{cases}$$
$$\text{for } \quad i=2,$$

and secondly by Lemma 1,

$$(12) \qquad \mu_i=(-1)^i\int_{r_1}^{r_2} B(t)/a(t)dt\int_{\rho_i'}^{\rho_i} 1/B(t)dt \qquad (i=1,2).$$

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6) $P_0(t, x, \cdot)$ are the transition probabilities of the process with ρ_1 and ρ_2 as absorbing barriers which correspond to the strongly continuous and norm decreasing semi-group ([6]).

Combinig (10), (11) and (12) we finally obtain

$$\lim_{t\to\infty}\int_{\rho_1'}^{\rho_2'} f(y)P(t,\rho_1',dy)=\left\{\int_{r_1}^{r_2} B(t)/a(t)dt\right\}^{-1}\int_{\rho_1'}^{\rho_2'} f(y)B(y)/a(y)dy\,.$$

The above formula holds for any initial position $x\in(r_1,r_2)$, for, putting $\tilde{P}(t,x,E)=P\{X^{(x)}(t)\in E$ without reaching ρ_1' before $t\}$ we have

$$\begin{aligned}&\lim_{t\to\infty}\int_{\rho_1'}^{\rho_2'} f(y)P(t,x,dy)\\&=\lim_{t\to\infty}\left[\int_0^t\int_{\rho_1'}^{\rho_2'} f(y)P(t-u,\rho_1',dy)dF_{\rho_1'}^{(x)}(u)+\int_{\rho_1'}^{\rho_2'} f(y)\tilde{P}(t,x,dy)\right]\\&=\left\{\int_{r_1}^{r_2} B(t)/a(t)dt\right\}^{-1}\int_{\rho_1'}^{\rho_2'} f(y)B(y)/a(y)dy\,.\end{aligned}$$

Since ρ's are arbitrary, this is true for any continuous function $f(x)$ with the support entirely contained in the open interval (r_1,r_2). This means that $P(t,x,E)$ has the limit distribution independent of x, as $t\to\infty$, as was to be proved.

Remark 1. If $E(\tau_\rho^{(x)})=\infty$, then applying Lemma 2 to (9) we have $\lim_{t\to\infty} P(t,x,E)=0$ for any Borel set E with the closure contained in the open interval (r_1,r_2).

Remrak 2. If a boundary is accesible, we can let a path jump into the interior of (r_1,r_2) when it reaches the boundary. For this process, the same problem can be treated by a similar method. In connection with the statistical quality control problems, this will be discussed by a joint paper of T. KITAGAWA and the author.

Example 1.[7] For r_j finite put $a(x)\equiv 1$, and $b(x)=k/(x-r_1)-l/(r_2-x)$ with $k\geqq 1$ and $l\geqq 1$. In this case the condition (2) is satisfied, so that the limiting distribution exists. The density $p(y)$ is given by

$$p(y)=A(x-r_1)^k(r_2-x)^l$$

with the normalizing constant A.

Example 2.[7] Put $a(x)=A'x^2+B'x+C'$, $b(x)=Dx+E$ and let (r_1,r_2) be the interval in which $A'x^2+B'x+C'>0$. Under the condition (2), the limit distribution exists and the density $p(y)$ is a Pearsonian curve (A. KOLMOGOROV [7] and S. BERNSTEIN [1]):

$$\frac{1}{p(y)}\cdot\frac{dp(y)}{dy}=\frac{(D-2A')y+(E-B')}{A'y^2+B'y+C'}\,.$$

7) For the preparation of these examples the author owes to the suggestion of Prof. T. KITAGAWA.

The following table illustrates the condition (2) more intuitively.

	$a(x)$	(r_1, r_2)	the condition (2)	$p(y)$
(i)	$-A(x-\lambda_1)(\lambda_2-x)$ $(A<0,\ \lambda_1<\lambda_2)$	(λ_1, λ_2)	$D\lambda_1+E \geqq A(\lambda_1-\lambda_2)$ $D\lambda_2+E \leqq A(\lambda_2-\lambda_1)$	(I)
(ii)	$A(x-\lambda_1)(x-\lambda_2)$ $(A>0,\ \lambda_1<\lambda_2)$	(λ_2, ∞)	$D\lambda_2+E \geqq A(\lambda_2-\lambda_1)$ $D<A$	(II)
(iii)	$A(x-B)^2$ $(A>0)$	(B, ∞)	$DB+E>0$ $D<A$	(III)
(iv)	$A(x-B)^2+C$ $(A\geqq 0,\ C>0)$	$(-\infty, \infty)$	$D<A$	(IV)
(v)	$B(x-C)$ $(B>0)$	(C, ∞)	$DC+E\geqq B$ $D<0$	(V)

(I)$\sim$(V) in the table correspond to the followings:

$$\text{(I)} : \quad p(y) = \text{const.}\ (y-\lambda_1)^{\frac{D\lambda_1+E}{A(\lambda_1-\lambda_2)}-1}(\lambda_2-y)^{\frac{D\lambda_2+E}{A(\lambda_2-\lambda_1)}-1}, \quad \lambda_1<y<\lambda_2,$$

$$\text{(II)} : \quad p(y) = \text{const.}\ (y-\lambda_1)^{\frac{D\lambda_1+E}{A(\lambda_1-\lambda_2)}-1}(y-\lambda_2)^{\frac{D\lambda_2+E}{A(\lambda_2-\lambda_1)}-1}, \quad \lambda_2<y<\infty,$$

$$\text{(III)} : \quad p(y) = \text{const.}\ (y-B)^{\frac{D}{A}-2}\exp\left\{-\frac{DB+E}{A}\cdot\frac{1}{y-B}\right\}, \quad B<y<\infty,$$

$$\text{(IV)} : \quad p(y) = \begin{cases} \text{const.}\ \{A(x-B)^2+C\}^{\frac{D}{2A}-1}\exp\left\{\frac{DB+E}{\sqrt{AC}}\tan^{-1}\sqrt{\frac{A}{C}}(x-B)\right\} & \text{for } A>0, \\ \text{const.}\ \exp\left\{\frac{D}{2C}x^2+\frac{E}{C}x\right\} & \text{for } A=0, \end{cases}$$
$$-\infty<y<\infty,$$

$$\text{(V)} : \quad p(y) = \text{const.}\ (x-C)^{\frac{DC+B}{E}-1}\exp\left\{\frac{D}{B}x\right\}, \quad C<y<\infty.$$

In case (i) we have $b(r_1)>0$ and $b(r_2)<0$. Therefore if a path of the particle approaches a boundary, it is more likely to move towards the center, and hence to imply the existence of the limiting distribution.

In case (iii) we can construct the diffusion process $X(t)$ in $(-\infty, \infty)$ by solving Ito's stochastic integral equation ([4], p. 273). For this process, a path starting from $x \in (-\infty, B)$ enter the interval (B, ∞) and thereafter never returns to B with probability one because B is an isolated zero of $a(x)$ and $b(B)>0$ (DOOB [3], Theorem 9.10). Hence we have also

$$\lim_{t\to\infty} P\{X(t)\in E\} = \begin{cases} \int_E p(y)\,dy & \text{for } E\subset(B,\infty) \\ 0 & \text{for } E\subset(-\infty, B). \end{cases}$$

with $p(y)$ given by (III).

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