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Kitagawa, Toshio
Kyushu University

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THE OPERATIONAL CALCULUS AND THE ESTIMATIONS OF FUNCTIONS OF PARAMETER ADMITTING SUFFICIENT STATISTICS

By

Tosio KITAGAWA

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§ 1. **Introduction.** The first purpose of this paper is to give a formulation upon which the operational calculus will be established in obtaining the unbiased minimum variance estimate of a function $\theta(\tau)$ of parameter τ when there exists a sufficient statistic U whose probability element can be written in the form

$$dF(u\,;\,\tau) = A(\tau)e^{-\tau u+v(u)}d\mu(u)\,, \tag{1.01}$$

where $\mu(u)$ is a measure for every finite range of the u space. Recently my colleagues Y. WASHIO, H. MORIMOTO and N. IKEDA [1] gave a detailed discussion about this problem concerning (1.01) including the various fundamental distributions as their examples. During their preparations I found out that the essential aspects of their results can be reviewed in virtue of the operational formulations connected with the linear translatable operations for which the present author contributed some general theory by the previous papers KITAGAWA [1] ~ [4], while T. ONOYAMA [1] discussed their generalizations to stochastic functional equations. For our present purpose some fundamental observations upon the bounded linear translatable operations will be given in § 2, and an introduction of some linear translatable operations which may not necessarily be bounded will be enunciated with reference to a sequence of bounded linear translatable operations. After these preparations a certain correspondence between a function $\theta(\tau)$ and a linear translatable operation Λ can be established which yields us the relation

$$\Lambda_u[e^{\tau u}] = \theta(\tau)e^{\tau u}\,, \tag{1.02}$$

for every τ belonging to a certain strip S depending upon Λ. Furthermore this correspondence shows us that under some fairly general conditions the unique unbiased minimum variance estimate $W(u)$ for the function $\theta(\tau)$ can be obtained by

$$W(u) = e^{-v(u)}\Lambda_u[e^{v(u)}]\,, \tag{1.03}$$

whose exact enunciation will be given in Theorem 3.1. Thus some of the conditions given in Theorem 1 in WASHIO, MORIMOTO and IKEDA [1] can be proved, and our correspondence enunciated just now will be seen to be more transparent and more convenient in some sense. Several Lemmas given in § 2 are also useful for deriving some of the operational calculus used in obtaining such estimates, as may be verified in the detailed discussions by WASHIO, MORIMOTO and IKEDA [1].

The second purpose of this paper is to discuss how to estimate a function $\theta(\tau)$ of the unknown parameter τ under the mentioned conditions. Even when the functional form of $\theta(\tau)$ may be known to us, it does not necessarily follow that the estimater $W(u)$ given in (1.03) should be used, because there are various problems of cost considerations in calculating $W(u)$ and also of the comparisons of the variances of the estimaters.

It is true that among the unbiased estimates the estimate with the minimum variance is given by the statistic $W(u)$. But there is a possibility of obtaining an estimate which may be a biased estimate for $\theta(\tau)$ and which may have the smallest mean square error in some sense. In § 4 a notion of the mean loss function is introduced, and Theorem 4.1 is established which determines the estimate yielding us the minimum mean loss among all the estimates of the form $g_N(u)=\sum_{k=0}^{N} a_k W_k(u)$, $N=1, 2, 3, \ldots, \infty$, when $\theta(\lambda)\sim\sum_{k=0}^{\infty} a_k e^{i\lambda k}$ and u is the unbiased sufficient statistic for λ with the minimum variance having the distribution of the form (4.07). The case when the statistic u is distributed in a discrete distribution such as Poisson and binomial ones can be treated quite similarly, although any detailed formulation will not be given here.

§ 2. **Bounded linear translatable operations.** Let $\mathfrak{L}^p(E)$ be a set of all measurable functions $f(u)$ defined over $E=(-\infty, \infty)$ such that, for a certain p in $1\leqq p<\infty$, $|f(u)|^p$ is integrable in the sense of Lebesque in any finite range of u. For each assigned bounded measurable set M belonging to the line E, let us define

$$\|f\|_M=\{\int_M |f(u)|^p du\}^{1/p}. \tag{2.01}$$

In what follows we shall denote by $\mathfrak{B}(E)$ the system of all bounded measurable sets of real numbers in E with the positive finite Lebesque measure. A mapping σ of $\mathfrak{B}(E)$ is defined as a correspondence of each element M out of $\mathfrak{B}(E)$ into a uniquely determined σM in the following manner: (1°) $M_1\supset M_2$ implies and is implied by $\sigma M_1\supset\sigma M_2$; (2°) for any assigned M out of $\mathfrak{B}(E)$ we can find M_1 such that $M_1\supset\sigma M$. A fundamental system denoted by $S=\{M_k\}$ $(k=1, 2, 3, \ldots)$ is a subsystem of $\mathfrak{B}(E)$ having the following two properties: (1°) the system is monotone: $M_k\subset M_{k+1}$ $(k=1, 2, 3, \ldots)$; (2°) to any given N out of $\mathfrak{B}(E)$ there corresponds a positive integer k such that $M_k\supset N$.

We shall now introduce

Definition 2.1.[1] An operation A, which transforms each element $f(u)$ of

1) Cf. Definition 3.1 in KITAGAWA [2].

$\mathfrak{L}^p(E)$ into an element $\Lambda_u[f(u)]$ of $\mathfrak{L}^p(E)$ as a function of u in $-\infty<u<\infty$, is said to be a *bounded linear translatable operation* if the following conditions are satisfied:

Condition (1°) It is *additive* in the sense that, for any assigned complex numbers c_1 and c_2, we have

$$\Lambda_u[c_1 f_1(u)+c_2 f_2(u)]=c_1\Lambda_u[f_1(u)]+c_2\Lambda_u[f_2(u)] \tag{2.02}$$

for all u in $-\infty<u<\infty$, except perhaps for a set of measure zero in u.

Condition (2°) It is *commutative* with all translations, that is, denoting by T_u^a the translation $T_u^a[f(u)]=f(u+a)$, we have, for almost all u in $-\infty<u<\infty$,

$$\Lambda_u[T_u^a[f(u)]]=T_u^a[\Lambda_u[f(u)]]. \tag{2.03}$$

Condition (3°) It is *bounded*, which means: (a) there is a mapping σ^Λ associated with Λ; (b) to each given M out of $\mathfrak{B}(E)$ there corresponds a positive C_M^Λ such that the relation holds

$$\|\Lambda[f]\|_M\leqq C_M^\Lambda\|f\|_{\sigma^\Lambda M} \tag{2.04}$$

for all f out of $\mathfrak{L}^p(E)$, where C_M^Λ may depend upon Λ, M, but is independent of f.

Example 1.[2] Let $\varphi(t)$ be L^q-integrable over the finite interval (α, β), where $p^{-1}+q^{-1}=1$. Then the additive operation defined over $\mathfrak{L}^p(E)$ by

$$\Lambda_u[f(u)]\equiv\int_\alpha^\beta f(u+t)\varphi(t)dt \tag{2.05}$$

is a bounded linear translatable operation.

The following four lemmas will be observed to be of use for our arguments in the sequel.

Lemma 2.1. (KITAGAWA [2] Lemma 3.2) *To each bounded linear translatable operation Λ defined in $\mathfrak{L}^p(E)$ there corresponds an integral function $G(\lambda)$ defined in the complex λ-plane such that $\Lambda_u[e^{\lambda u}]=G(\lambda)e^{\lambda u}$ in $-\infty<u<\infty$.*

Lemma 2.2. *Any two bounded linear translatable operations $\Lambda_u^{(1)}$ and $\Lambda_u^{(2)}$ defined in $\mathfrak{L}^p(E)$ are commutative with each other, that is, we have, for any f out of $\mathfrak{L}^p(E)$,*

$$\Lambda_u^{(1)}[\Lambda_u^{(2)}[f(u)]]=\Lambda_u^{(2)}[\Lambda_u^{(1)}[f(u)]] \tag{2.06}$$

for almost all u in $-\infty<u<\infty$.

Proof: Let M be any assigned set out of $\mathfrak{B}(E)$. Let us define $\sigma^{\Lambda^{(1)}\cdot\Lambda^{(2)}}M$

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2) For other examples, see Examples 3.1 and 3.2 in KITAGAWA [2]. These examples can be reduced to our present deterministic formulation.

$\equiv \sigma^{\Lambda^{(1)}}(\sigma^{\Lambda^{(2)}}M) \cup \sigma^{\Lambda^{(2)}}(\sigma^{\Lambda^{(1)}}M)$. Since f belongs to $\mathfrak{L}^p(E)$, we can find, for any assigned positive ε, an integer $N(\varepsilon, \sigma^{\Lambda^{(1)}\cdot\Lambda^{(2)}}M)$ and $s_n(u) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu u}$ such that

$$\| f(u) - \sum_{\nu=-n}^{n} a_\nu e^{i\nu u} \|_{\sigma^{\Lambda_1\cdot\Lambda_2}M} < \varepsilon . \tag{2.07}$$

for all $n \geq N(\varepsilon, \sigma^{\Lambda^{(1)}\cdot\Lambda^{(2)}}M)$.

Since we may and we shall assume without any loss of generality that

$$\sigma^{\Lambda^{(1)}}(\sigma^{\Lambda^{(2)}}M) \supseteqq \sigma^{\Lambda^{(i)}}M \supseteqq M \qquad (i=1, 2) \tag{2.08}$$

$$\sigma^{\Lambda^{(2)}}(\sigma^{\Lambda^{(1)}}M) \supseteqq \sigma^{\Lambda^{(i)}}M \supseteqq M \qquad (i=1, 2), \tag{2.09}$$

we have

$$\begin{aligned} &\| A_u^{(1)}[A_u^{(2)}[f(u)]] - A_u^{(1)}[A_u^{(2)}[S_n(u)]] \|_M \\ &\leqq C_M^{\Lambda^{(1)}} \| A_u^{(2)}[f(u)] - A_u^{(2)}[S_n(u)] \|_{\sigma^{\Lambda^{(1)}}M} \\ &\leqq C_M^{\Lambda^{(1)}} C_{\sigma^{\Lambda^{(1)}}M}^{\Lambda^{(2)}} \| f(u) - S_n(u) \|_{\sigma_{\sigma^{\Lambda^{(2)}}}(\sigma^{\Lambda^{(1)}}M)} \\ &\leqq C_M^{\Lambda^{(1)}} C_{\sigma^{\Lambda^{(1)}}M}^{\Lambda^{(2)}} \| f(u) - S_n(u) \|_{\sigma^{\Lambda^{(1)}\cdot\Lambda^{(2)}}M} \end{aligned} \tag{2.10}$$

and similarly

$$\begin{aligned} &\| A_u^{(2)}[A_u^{(1)}[f(u)]] - A_u^{(2)}[A_u^{(1)}[S_n(u)]] \|_M \\ &\leqq C_M^{\Lambda^{(2)}} C_{\sigma^{\Lambda^{(2)}}M}^{\Lambda^{(1)}} \| f(u) - S_n(u) \|_{\sigma^{\Lambda^{(1)}\cdot\Lambda^{(2)}}M} . \end{aligned} \tag{2.11}$$

On thè other hand Lemma 2.1 gives us

$$\begin{aligned} A_u^{(2)}[A_u^{(1)}[S_n(u)]] &= A_u^{(2)}[\sum_{\nu=-n}^{n} a_\nu G_1(i\nu) e^{i\nu u}] \\ &= \sum_{\nu=-n}^{n} a_\nu G_2(i\nu) G_1(i\nu) e^{i\nu u} = \sum_{\nu=-n}^{n} a_\nu G_1(i\nu) G_2(i\nu) e^{i\nu u} \\ &= A_u^{(1)}[A_u^{(2)}[S_n(u)]] . \end{aligned} \tag{2.12}$$

The combination of (2.10), (2.11) and (2.12) will give us

$$\begin{aligned} &\| A_u^{(1)}[A_u^{(2)}[f(u)]] - A_u^{(2)}[A_u^{(1)}[f(u)]] \|_M \\ &\leqq (C_M^{\Lambda^{(1)}} C_{\sigma^{\Lambda^{(1)}}M}^{\Lambda^{(2)}} + C_M^{\Lambda^{(2)}} C_{\sigma^{\Lambda^{(2)}}M}^{\Lambda^{(1)}}) \varepsilon . \end{aligned} \tag{2.13}$$

In order to enunciate a lemma concerning the commutativity of two bounded linear operations applied to a function of two variables, let us introduce the following notions. Let $E_1 = [u; -\infty < u < \infty]$ and $E_2 = [v; -\infty < v < \infty]$, and let $\mathfrak{B}(E_1)$ and $\mathfrak{B}(E_2)$ be defined as for $\mathfrak{B}(E)$. Let $\mathfrak{L}^p(E_1 \times E_2)$ be a set of all functions $h(u, v)$, which are measurable over the product space $(E_1 \times E_2)$, and which are L^p-integrable over the product set $(M \times N)$ for every M out of $\mathfrak{B}(E_1)$ and every N out of $\mathfrak{B}(E_2)$, that is,

$$\iint_{M \times N} |h(u, v)|^p \, du dv < \infty . \tag{2.141}$$

and we shall define

(2.142) $$\|h(u, v)\|_{M\times N} \equiv \left\{ \iint_{M\times N} |h(u, v)|^p \, dudv \right\}^{1/p}.$$

Definition 2.2. An operation Λ_u is said to be a *bounded linear translatable operation defined over the space* $\mathfrak{L}^p(E_1 \times E_2)$, provided that Λ_u transforms every function out of $\mathfrak{L}^p(E_1 \times E_2)$ into another one belonging to $\mathfrak{L}^p(E_1 \times E_2)$ and satisfies the following conditions:

(1°) It is *additive*, that is, for each assigned c_1 and c_2 we have

(2.15) $$\Lambda_u[c_1 h_1(u, v) + c_2 h_2(u, v)] = c_1 \Lambda_u[h_1(u, v)] + c_2 \Lambda_u[h_2(u, v)].$$

(2°) It is *translatable* with every translation defined over $\mathfrak{L}^p(E_1 \times E_2)$, that is

(2.16) $$\Lambda_u[T_u^\alpha[h(u, v)]] \equiv \Lambda_u[h(u+\alpha, v)] = \Lambda_{u+\alpha}[h(u, v)] \equiv T_u^\alpha[\Lambda_u[h(u, v)]].$$

(3°) It is *bounded* in the space $\mathfrak{L}^p(E_1 \times E_2)$ in the sense that, for every M out of $\mathfrak{B}(E_1)$ and every N out of $\mathfrak{B}(E_2)$, we have

(2.171) $$\iint_{M\times N} |\Lambda_u[h(u, v)]|^p dudv \leqq (C_M^\Lambda)^p \iint_{\sigma^\Lambda M\times N} |h(u, v)|^p dudv,$$

that is,

(2.172) $$\|\Lambda_u[h(u, v)]\|_{M\times N} \leqq C_M^\Lambda \|h(u, v)\|_{\sigma^\Lambda M\times N}.$$

(4°) Specially for a function $h(u, v) = f(u)g(v)$, where $f(u)$ and $g(v)$ belong to $\mathfrak{L}^p(E_1)$ and $\mathfrak{L}^p(E_2)$ respectively, we have

(2.18) $$\Lambda_u[f(u)g(v)] = \Lambda_u[f(u)]g(v).$$

Similarly an operation Γ_v is said to be a bounded linear translatable operation defined over the space $\mathfrak{L}^p(E_1 \times E_2)$, provided that it satisfies four conditions similar to those (1°), (2°), (3°) and (4°), replacing (2.16), (2.171), (2.172) and (2.18) by the following (2.19), (2.201), (2.202) and (2.21) respectively:

(2.19) $$\Gamma_v[T_v^\alpha[h(u, v)]] \equiv \Gamma_v[h(u, v+\alpha)] = \Gamma_{v+\alpha}[h(u, v)] \equiv T_v^\alpha[\Gamma_v[h(u, v)]]$$

(2.201) $$\iint_{M\times N} |\Gamma_v[h(u, v)]|^p dudv \leqq (C_N^\Gamma)^p \int_{M\times\sigma^\Gamma N} |h(u, v)|^p dudv,$$

that is,

$$\|\Gamma_v[h(u,v)]\|_{M\times N} \leqq C_N^{\Gamma}\|h(u,v)\|_{M\times\sigma^{\Gamma}N} \tag{2.202}$$

and

$$\Gamma_v[h(u,v)] = \Gamma_u[f(u)g(v)] = f(u)\Gamma_v[g(v)]\,. \tag{2.21}$$

Lemma 2.3. *Let Λ_u and Γ_v be the bounded linear translatable operations defined over $\mathfrak{L}^p(E_1\times E_2)$ satisfying the conditions enunciated in Definition 2.2.*

Then we have

$$\Lambda_u[\Gamma_v[h(u,v)]] = \Gamma_v[\Lambda_u[h(u,v)]]\,, \tag{2.22}$$

for almost all u and v.

Proof: This is immediate from the following three observations.

(a) Let $h_n(u,v)$ be a simple function which can be written

$$h_n(u,v) = \sum_{j=1}^{n} a_j^{(n)} C_{A_j}(u) C_{B_j}(v)\,, \tag{2.23}$$

where $A_1, A_2, \ldots, A_{n-1}$ and A_n are finite intervals belonging to E_1, while $B_1, B_2, \ldots, B_{n-1}$ and B_n are finite intervals belonging to E_2, $a_j^{(n)}$ being constants independent of u and v.

Then, in virtue of the additivity, (2.18) and (2.21), we shall have

$$\begin{aligned} \Lambda_u[\Gamma_v[h(u,v)]] &= \Gamma_v[\Lambda_u[h(u,v)]] \\ &= \sum_{j=1}^{n} a_j^{(n)} \Lambda_u[C_{A_j}(u)]\,\Gamma_v[C_{B_j}(v)]\,. \end{aligned} \tag{2.24}$$

(b) Let $h(u,v)$ be any given function belonging to $\mathfrak{L}^p(E_1\times E_2)$. Let M out of $\mathfrak{B}(E_1)$ and N out of $\mathfrak{B}(E_2)$ be arbitrarily assigned. Then we can find a sequence of simple functions $\{h_n(u,v)\}$ such that

$$\lim_{n\to\infty}\|h_n(u,v) - h(u,v)\|_{\sigma^{\Lambda}M\times\sigma^{\Gamma}N} = 0\,. \tag{2.25}$$

(c) For any assigned M and N out of $\mathfrak{B}(E_1)$ and $\mathfrak{B}(E_2)$ respectively, we have

$$\begin{aligned} &\|\Lambda_u[\Gamma_v[h(u,v)]] - \Lambda_u[\Gamma_v[h_n(u,v)]]\|_{M\times N} \\ &\leqq C_M^{\Lambda} C_N^{\Gamma}\|h(u,v) - h_n(u,v)\|_{\sigma^{\Lambda}M\times\sigma^{\Gamma}N}\,, \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\|\Gamma_v[\Lambda_u[h(u,v)]] - \Gamma_v[\Lambda_u[h_n(u,v)]]\|_{M\times N} \\ &\leqq C_N^{\Gamma} C_M^{\Lambda}\|h(u,v) - h_n(u,v)\|_{\sigma^{\Lambda}M\times\sigma^{\Gamma}N}\,. \end{aligned} \tag{2.27}$$

Consequently we have

$$\begin{aligned} &\|\Lambda_u[\Gamma_v[h(u,v)]] - \Gamma_v[\Lambda_u[h(u,v)]]\|_{M\times N} \\ &\leqq \|\Lambda_u[\Gamma_v[h(u,v)]] - \Lambda_u[\Gamma_v[h_n(u,v)]]\|_{M\times N} \\ &+ \|\Lambda_u[\Gamma_v[h_n(u,v)]] - \Gamma_v[\Lambda_u[h_n(u,v)]]\|_{M\times N} \end{aligned} \tag{2.28}$$

$$+\|\Gamma_v[A_u[h_n(u,v)]]-\Gamma_v[A_u[h(u,v)]]\|_{M\times N}$$
$$<2C^{\Lambda}_{M}C^{\Gamma}_{N}\|h(u,v)-h_n(u,v)\|_{\sigma^{\Lambda}M\times\sigma^{\Gamma}N}<2\varepsilon$$

for $n\geqq n(\varepsilon, \sigma^{\Lambda}M, \sigma^{\Gamma}N)$, which will prove what was to be proved.

The similar arguments give us the following Lemma which will be used in § 3.

Lemma 2.4. *Let A_u be a bounded linear translatable operation defined over $\mathfrak{L}^p(E_1\times E_2)$. Let $\varphi(v)$ be a measurable function such that $\varphi(v)^q$ is integrable in the sense of Lebesgue over a finite interval (α,β), where $p^{-1}+q^{-1}=1$.*

Then we have, for each h out of $\mathfrak{L}^p(E_1\times E_2)$,

$$A_u\Big[\int_\alpha^\beta h(u,v)\varphi(v)\,dv\Big]=\int_\alpha^\beta A_u[h(u,v)]\varphi(v)\,dv \tag{2.29}$$

in $-\infty<u<\infty$, except possibly a set of measure zero.

§ 3. **Limit and operational calculus of a sequence of bounded linear translatable operations.** Henceforth we shall be concerned with the space of $\mathfrak{L}^2(E)$, and we shall proceed to discuss how to define a linear translatable operation which is not necessary bounded.

Lemma 3.1. *Let $\{A_u^{(n)}\}$ $(n=1,2,3,\dots)$ be a sequence of the bounded linear translatable operations each of whose generating functions $\theta_n(\lambda)$ is defined by*

$$A_u^{(n)}[e^{\lambda u}]=\theta_n(\lambda)e^{\lambda u} \tag{3.01}$$

for all complex number λ.

Let T be any assigned positive number. Assume that $1/A(s+it)$ is belonging to $L^2(-\infty,\infty)$ as a function of t for each s such that $\alpha+\varepsilon\leqq s\leqq\beta-\varepsilon$, where ε is any assigned positive number.[3]

Then we have, for every s in $\alpha<s<\beta$,

$$A_u^{(n)}\Big[\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{(s+it)u}}{A(s+it)}dt\Big]=\frac{1}{2\pi}\int_{-T}^{T}\frac{\theta_n(s+it)e^{(s+it)u}}{A(s+it)}dt, \tag{3.02}$$

in almost all u.

Proof: This can be observed from the fact that the operation A_u and the integration over the interval $(-T,T)$ with respect to t is commutative, that is,

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3) The cases when $\alpha=-\infty$ and/or $\beta=+\infty$ are included as special cases.

$$(3.03) \qquad \Lambda_u^{(n)}\left[\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{(s+it)u}}{A(s+it)}dt\right] = \frac{1}{2\pi}\int_{-T}^{T}\Lambda_u^{(n)}\left[\frac{e^{(s+it)u}}{A(s+it)}\right]dt,$$

which can be verified by Lemma 2.4.

The remaining part follows from (3.01), in view of the fact that $\theta_n(\lambda)$ is an integral function of λ and hence $\theta_n(s+it)$ is bounded and continuous over any finite interval $(-T, T)$.

Lemma 3.2. *In addition to the assumptions to Lemma* 3.1, *let us assume that, for* $n=1, 2, 3, \dots,$

$$(3.04) \qquad \int_{-\infty}^{\infty}\left|\frac{\theta_n(s+it)}{A(s+it)}\right|^p dt < \infty.$$

Then we have, for every s *in* $\alpha < s < \beta$,

$$(3.05) \qquad \Lambda_u^{(n)}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{(s+it)u}}{A(s+it)}du\right] = \lim_{T\to\infty}\Lambda_u^{(n)}\left[\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{(s+it)u}}{A(s+it)}du\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\theta_n(s+it)e^{(s+it)u}}{A(s+it)}dt.$$

in almost all u *in* $-\infty < u < \infty$.

Proof: In virtue of L^p-convergence in the whole u line E_2, we have

$$(3.06) \qquad \lim_{T\to\infty}\int_{-\infty}^{\infty}\left|\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{itu}}{A(s+it)}dt - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{itu}}{A(s+it)}dt\right|^p du = 0.$$

Consequently we have, for any assigned set M out of $\mathfrak{B}(E_2)$ and for any assigned positive number ε,

$$(3.07) \qquad \int_{\sigma^\Lambda M} e^{psu}\left|\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{itu}}{A(s+it)}dt - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{itu}}{A(s+it)}dt\right|^p du < \varepsilon,$$

provided that $T \geqq T_0(\varepsilon, \sigma^\Lambda M)$.

Since each $\Lambda^{(n)}$ is a bounded linear translatable operation we have

$$(3.08) \qquad \left\|\Lambda_u^{(n)}\left[\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{(s+it)u}}{A(s+it)}dt - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{(s+it)u}}{A(s+it)}dt\right]\right\|_M$$

$$\leqq C_M^{\Lambda(n)} \left\| \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{(s+it)u}}{A(s+it)} dt - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(s+it)u}}{A(s+it)} dt \right\|_{\sigma^\Lambda M}$$

$$< C_M^{\Lambda(n)} \varepsilon$$

for $T \geqq T_0(\varepsilon, \sigma^\Lambda M)$, which will yield us the first part of the equality in (3.05), while the second part can be obtained by means of Lemma 3.1 and the assumption (3.04), hence our proof is completed.

Lemma 3.3. *In addition to the assumptions to Lemmas* 3.1 *and* 3.2, *let us assume that there is a function* $\theta(s+it)$ *defined over the strip* $S=[s+it;\ \alpha<s<\beta,\ -\infty<t<\infty]$ *such that, for every* s *in* $\alpha<s<\beta$,

$$(3.09) \qquad \int_{-\infty}^{\infty} \left| \frac{\theta_n(s+it)-\theta(s+it)}{A(s+it)} \right|^p dt < \infty .$$

Let us define an additive operation Λ *in such a way that we have*

$$(3.10) \qquad \Lambda_u[e^{\lambda u}] = \theta(\lambda)\, e^{\lambda u}$$

for every λ *belonging to the strip* S.

Then we have

$$(3.11) \qquad \lim_{n\to\infty} \Lambda_u^{(n)}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(s+it)u}}{A(s+it)} dt\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\theta(s+it)e^{(s+it)u}}{A(s+it)} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Lambda_u\left[\frac{e^{(s+it)u}}{A(s+it)}\right] dt .$$

Proof: This is immediate from Lemma 3.2, (3.09) and (3.10).

Lemma 3.4. *In addition to the assumptions to Lemma* 3.3, *let us assume that the function of* t

$$(3.12) \qquad \theta(s+it)[A(s+it)]^{-1}$$

tends to zero uniformly in s *belonging to the interval* $\alpha+\varepsilon \leqq s \leqq \beta-\varepsilon$, ε *being any assigned positive number, as* t *tends to* $+\infty$ *or to* $-\infty$.

Let $\theta(s+it)$ *be regular and analytic in every open strip included the strip* S.

Then the integral

$$(3.13) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\theta(s+it)}{A(s+it)} e^{(s+it)u} dt$$

is independent of s *such that* $\alpha<s<\beta$.

Proof: This is immediate from the Cauchy integral theorem applied to the rectangular domain R_T whose corner points are $\alpha+\varepsilon-iT$, $\alpha+\varepsilon+iT$, $\beta-\varepsilon-iT$ and $\beta-\varepsilon+iT$, and from the limiting process of making $|T|$ to infinity.

The combinations of these four Lemmas give us now

Theorem 3.1. *Let* $\{\Lambda_u^{(n)}, (n=1, 2, 3, \dots)\}$ *be a sequence of bounded linear translatable operations having their respective generating functions* $\{\theta_n(\lambda)\}$ *for which* (3.01) *hold true. Assume that*

(1°) $[A(s+it)]^{-1}$ *and* $\theta_n(s+it)[A(s+it)]^{-1}$ *belong to* $L^p(-\infty, \infty)$ *as a function of* t *for every* s *in* $\alpha<s<\beta$ *and for every positive integer* n.

(2°) *There is a function* $\theta(s+it)$ *which is regular and analytic in every closed strip included in* S *and for which* (3.09) *holds true for every* s *in* $\alpha\leqq s\leqq\beta$.

(3°) *The function* (3.12) *tends to zero uniformly in* s *belonging to every closed interval contained in the interval* (α, β).

Then we have the following assertions:

(*a*) *The function*

$$e^{v(u)}W(u)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\theta(s+it)}{A(s+it)}e^{(s+it)u}\,dt \tag{3.14}$$

exists and is independent of s *in* $\alpha<s<\beta$.

(*b*) *We have*

$$e^{v(u)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{(s+it)u}}{A(s+it)}\,dt. \tag{3.15}$$

(*c*) *The limit*

$$\lim_{n\to\infty}\Lambda_u^{(n)}[e^{v(u)}] \tag{3.16}$$

exists for which (3.11) *holds true.*

(*d*) *If we define* $\Lambda_u[e^{v(u)}]$ *by*

$$\Lambda_u[e^{v(u)}]=\lim_{n\to\infty}\Lambda_u^{(n)}[e^{v(u)}], \tag{3.17}$$

then we have the consequences: (i) *The definition* (3.17) *is independent of a particular choice of a sequence of bounded linear translatable operations* $\{\Lambda_u^{(n)}\}$; (ii) *The relation holds true that*

$$\Lambda_u[e^{v(u)}]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\theta(s+it)e^{(s+it)u}}{A(s+it)}\,dt. \tag{3.18}$$

(*e*) *The function*

$$W(u)\equiv e^{-v(u)}\Lambda_u[e^{v(u)}] \tag{3.19}$$

is the unique unbiased estimate for $\theta(\tau)$ *based upon the sufficient statistic* u *for* τ. *Furthermore it is with the minimum variance provided that the variance of* $W(u)$ *exists.*

Proof: The assertion (e) is the only one to be proved now.
What we have to establish is the relation

$$(3.20) \qquad A(\tau)\int_{-\infty}^{\infty} W(u)e^{-\tau u+v(u)}\,du=\theta(\tau),$$

which can be obtained if we have

$$(3.21) \qquad \int_{-\infty}^{\infty} A_u[e^{v(u)}]\,e^{-\tau u}\,du=\frac{\theta(\tau)}{A(\tau)},$$

which is however immediate from (3.19) in virtue of the Laplace transform, completing our proof.

Example 3.1. The differentiation D is an additive translatable operation defined over the subset of $\mathfrak{L}^2(E)$ consisting of differentiable functions, but it is not bounded in our sense. On the other hand

$$(3.32) \qquad \Delta_n^{h_n}[f(u)]=h_n^{-1}\{f(u+h_n)-f(u)\}$$

is a bounded linear translatable operation whose generating function is $(e^{h_n\lambda}-1)/h_n$.

For a sequence of non-negative real numbers $\{h_n\}$ such that $\lim_{n\to\infty} h_n=0$, the conditions in our Theorem 3.1 are satisfied in some class of functions $[A(s+it)]^{-1}$, that is equivalent to say, $e^{v(u)}$. For instance for the case (4.07).

§ 4. Loss function approach in estimating value of a known function of an unknwon parameter.

Let $\theta(\lambda)$ be a known function defined over a finite interval, which we may and we shall assume without any loss of generality to be the interval $0\leqq\lambda\leqq 2\pi$. Let $\theta(\lambda)$ be a function belonging to $L^2(0, 2\pi)$ with the expansion

$$(4.01) \qquad \theta(\lambda)\sim\sum_{k=0}^{\infty} a_k e^{i\lambda k}\ .$$

Since this function θ is known to us, we may and we shall assume the sequence $\{a_k\}$ is known to us. On the other hand the value of λ itself is unknown to us, and we are now concerned with the case when λ can be estimated by means of a certain sufficient unbiased statistic u whose probability density function is given by (1.01). Now the results obtained by WASHIO, MORIMOTO and IKEDA [1] § 5 lead us in particular that the following assertions hold under certain restrictions given in their paper:

(1°) The unbiased sufficient statistic for estimating $e^{i\lambda k}$ for each k is given by

$$(4.02) \qquad \begin{aligned} W_k(u) &\equiv e^{-v(u)}T_u^{ik}[e^{v(u)}] \\ &= e^{-v(u)}e^{v(u+ik)}\,. \end{aligned}$$

(2°) The unbiased sufficient statistic for $\theta(\lambda)$ is given by

$$\sum_{k=0}^{\infty} a_k W_k(u) \tag{4.03}$$

provided that the convergence of the series (4.03) secures us the change of the order of the summation of this series with the integration of $e^{-\tau u+v(u)}$ with respect to μ measure, that is

$$\int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} a_k W_k(u)\right) e^{-\tau u+v(u)} d\mu(u) \tag{4.04}$$
$$= \sum_{k=0}^{\infty} a_k \int_{-\infty}^{\infty} W_k(u) e^{-\tau u+v(u+ik)} d\mu(u).$$

However we have a hesitation for appealing to the statistic (4.03) in estimating the value $\theta(\lambda)$, because it is not practical to adopt an infinite series. Indeed there may be a possibility of appealing to biased estimates of the form of a finite sum

$$g_N(u) = \sum_{k=0}^{N} a_k W_k(u), \tag{4.05}$$

Our criterion in choosing a suitable estimate among biased or unbiased estimates for $\theta(\lambda)$ is to make use of the average loss function defined by

$$W(\theta, g_N) \tag{4.06}$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \underset{(\lambda)}{E_u} \left\{ \left| \theta(\lambda) - \sum_{k=0}^{N} a_k w_k(u) \right|^2 \right\} d\lambda .$$

In what follows we are concerned with a special case when we have

$$A(\lambda) e^{-\lambda u+v(u)} d\mu(u) \tag{4.07}$$
$$= \frac{1}{\sqrt{2\pi n}} e^{-\frac{n}{2}\lambda^2} e^{-\lambda u-\frac{u^2}{2n}} du ,$$

that is, u is the unbiased sufficient estimater with minimum variance for the population mean λ of the normal distribution $N(\lambda, 1)$, which is derived from a sample of size n.

First we can observe

Lemma 4.1. *Let us put*

$$\theta_N(\lambda) = \sum_{k=0}^{N} a_k e^{i\lambda k} . \tag{4.08}$$

Then $g_N(u)$ is the unbiased sufficient statistic with the minimum variance for estimating $\theta_N(\lambda)$ and we have

$$W(\theta_N, g_N) = \sum_{k=1}^{N} |a_k|^2 (e^{\frac{k^2}{n}} - 1) . \tag{4.09}$$

Proof: What we have to show here is to prove (4.09). Now in virtue of the definition

$$(4.10)\qquad W(\theta_N, g_N) = \frac{1}{2\pi}\int_0^{2\pi} \underset{(\lambda)}{E_u}\Big[|\theta_N(\lambda) - \sum_{k=0}^{N} a_k W_k(u)|^2\Big]\, d\lambda = \sum_{k=0}^{N}\sum_{l=0}^{N} a_k \bar{a}_l \frac{1}{2\pi}\int_0^{2\pi} \underset{(\lambda)}{E_u}\{h_{kl}(u)\}\, d\lambda\,,$$

where we have put for a moment

$$(4.11)\qquad \underset{(\lambda)}{E_u}\{h_{kl}(u)\} \equiv \underset{(\lambda)}{E_u}\{(W_k(u) - e^{ik\lambda})\overline{(W_l(u) - e^{il\lambda})}\}\,.$$

But the results obtained by WASHIO, MORIMOTO and IKEDA [1] give us in particular

$$(4.12)\qquad \underset{(\lambda)}{E_u}\{h_{kl}(u)\} = e^{i\lambda(k-l)}(e^{\frac{kl}{n}} - 1)\,.$$

Consequently the combination of (4.12) with (4.10) leads us to

$$(4.13)\qquad W(\theta_N, g_N) = \sum_{k=0}^{N} |a_k|^2 \frac{1}{2\pi}\int_0^{2\pi} (e^{\frac{k^2}{n}} - 1)\, d\lambda = \sum_{k=0}^{N} |a_k|^2 (e^{\frac{k^2}{n}} - 1)\,,$$

as was to be proved.

Lemma 4.2. *We have*

$$(4.14)\qquad W(\theta, g_N) = W(\theta_N, g_N) + W(\theta, \theta_N) = \sum_{k=1}^{N} |a_k|^2 (e^{\frac{k^2}{n}} - 1) + \sum_{j=N+1}^{\infty} |a_j|^2\,.$$

Proof: We have

$$(4.15)\qquad W(\theta, g_N) = \frac{1}{2\pi}\int_0^{2\pi} \underset{(\lambda)}{E_u}[|(\theta(\lambda) - \theta_N(\lambda) + \theta_N(\lambda) - g_N(u))|^2]\, d\lambda$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underset{(\lambda)}{E_u}[|\theta_N(\lambda) - g_N(\lambda)|^2]\, d\lambda + \frac{1}{2\pi}\int_0^{2\pi} \underset{(\lambda)}{E_u}[|\theta(\lambda) - \theta_N(u)|^2]\, d\lambda$$

$$+ 2\frac{1}{2\pi}\int_0^{2\pi} \underset{(\lambda)}{E_u}\{R(\theta(\lambda)-\theta_N(\lambda))(\theta_N(\lambda)-g_N(u))\}\,d\lambda$$

$$= W_1 + W_2 + W_3, \quad \text{say.}$$

But it can be readily seen that

$$W_1 = W(\theta_N,\ g_N) = \sum_{k=1}^{N} |a_k|^2 (e^{\frac{k^2}{n}} - 1) \tag{4.16}$$

$$W_2 = \sum_{j=N+1}^{\infty} |a_j|^2 \tag{4.17}$$

and

$$W_3 = \frac{1}{\pi}\int_0^{2\pi} R\{(\theta(\lambda)-\theta_N(\lambda))\underset{(\lambda)}{E_u}(\theta_N(\lambda)-g_N(u))\}\,d\lambda \tag{4.18}$$

$$= 0,$$

which complete our proof.

The direct combination of these two Lemmas obtained just now yields us

Theorem 4.1. *The value of the non-negative integer N which minimizes $W(\theta, g_N)$ is given by the integer N_0 which is the largest non-negative integer N_0 among N such that*

$$N < (n \log 2)^{\frac{1}{2}}. \tag{4.19}$$

KYUSHU UNIVERSITY.

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