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https://doi.org/10.5109/12965

出版情報：統計数理研究. 6 (1/2), pp.17-23, 1955-12. Research Association of Statistical Sciences

TABLES FOR TESTING RANDOMNESS BY MEANS OF LENGTHS OF RUNS

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As one of the non-parametric tests, there is a test of randomness by means of runs, which is an application of combinatorial probability theory.

Suppose that there are some elements of two kinds A, B. Let the numbers of A's and B's be m and n respectively. When these $(m+n)$ elements are arranged in the order, the total numbers of A- and B-runs are denoted by r and s, and the numbers of A-runs of length 1, 2, $\cdots$, m and of B-runs of length 1, 2, $\cdots$, n are denoted by $r_1, r_2, \cdots, r_m$ and $s_1, s_2, \cdots, s_n$, respectively. Of course, the following equalities must hold,

$$r=\sum_{i=1}^{m} i r_i, \qquad s=\sum_{j=1}^{n} j s_j.$$

The distribution of the total number of runs $n=r+s$ was computed in details by F. S. SWED and C. EISENHART [3]. Moreover the asymptotic distribution of u was investigated by A. WALD and J. WOLFOWITZ [4]. But, in the following, some tables for testing randomness by means of length of runs are computed.

Now, the probability $Q_1(t)$ that there appears at least one A-run of length t or longer is

$$Q_1(t)=\Pr.\{r_i\geqq 1;\ i\geqq t\} = 1-A_1\Big/\binom{m+n}{m},$$

where

$$A_1=\sum{}'\binom{n+1}{r}\sum_{i=1}^{i^*(r)}(-1)^i\binom{r}{i}\binom{m-i(t-1)-1}{r-1},$$

summation $\sum'$ being extended only over those values of r for which $\frac{m}{t-1}\leqq r\leqq m$ and $i^*(r)$ being the greatest integer i which satisfies the inequalities

$$0\leqq i\leqq r, \quad i\leqq\frac{m-r}{t-1}.$$

When $r>n+1$ in the above expression of A_1, we put $\binom{n+1}{r}=0$. Similarly, we can obtain the probability $Q_2(t)$ that there appears at least one B-run of length t or longer. $Q_2(t)$ has the form similar to $Q_1(t)$

Moreover, in the analogous way, the probability $Q(t)$ that there appears at least one A- or B-runs of length t or longer is

$$Q(t) = \Pr.\{r_i \geqq 1 \quad \text{or} \quad s_i \geqq 1 \quad \text{or both}; \quad i \geqq t\}$$
$$= 1 - A \Big/ \binom{m+n}{m},$$

where

$$A = \sum{}' \sum{}'' F(r, s)$$
$$\times \sum_{i=0}^{i^*(r)} (-1)^i \binom{r}{i} \binom{m - i(t-1) - 1}{r-1}$$
$$\times \sum_{j=0}^{j^*(s)} (-1)^j \binom{s}{j} \binom{n - j(t-1) - 1}{s-1},$$

summation $\sum'$; $\sum''$ being extended only over those values of r; s for which $\frac{m}{t-1} \leqq r \leqq m$; $\frac{n}{t-1} \leqq s \leqq n$, and $i^*(r)$; $j^*(s)$ being the greatest integer i; j which satisfies the inequalities

$$0 \leqq i \leqq r, \quad i \leqq \frac{m-r}{t-1}; \quad 0 \leqq j \leqq s, \quad j \leqq \frac{n-s}{t-1}$$

respectively. When $r > n+1$ (or $s > m+1$) in the above expression of A, we put $\binom{n+1}{r} = 0$ (or $\binom{m+1}{s} = 0$). And we define

$$F(r, s) = \begin{cases} 0 & \text{if} \quad |r - s| > 1, \\ 1 & \text{if} \quad |r - s| = 1, \\ 2 & \text{if} \quad |r - s| = 0. \end{cases}$$

These probabilities are investigated by A. M. MOOD [1] in the more general cases.

Now, we define t_5 and t_1 ($\tilde{t}_5$ and $\tilde{t}_1$) as the smallest integers t which make the $Q(t)$ ($Q_1(t)$) less than 0.05 and 0.01 respectively. F. MOSTELLER [2] computed these t-values in the cases of $m = n = 5,10,15,20$ and 25. But, as far as we have known, it seems that these t-values have never been computed in the cases of $m \neq n$.

The purpose of this note is the computations of the following tables:

Table I. Table of t_5, in the cases of every pair (m, n), $1 \leqq m \leqq 25$, $1 \leqq n \leqq 25$.

Table II. Table of t_1, in the cases of every pair (m, n), $1 \leqq m \leqq 25$, $1 \leqq n \leqq 25$.

Table III. Table of $\tilde{t}_5$, in the cases of every pair (m, n), $1 \leqq m \leqq 25$, $1 \leqq n \leqq 25$.

Table IV. Table of $\tilde{t}_1$, in the cases of every pair (m, n), $1 \leqq m \leqq 25$, $1 \leqq n \leqq 25$.

Table I.

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1																									
2																									
3																									
4																									
5				5	5																				
6			6	6	6	6																			
7			7	7	6	6	6																		
8			8	8	7	7	7	7																	
9			9	8	8	7	7	7	7																
10		10	10	9	8	8	7	7	7	7															
11		11	10	10	9	8	8	8	7	7	7														
12		12	11	10	10	9	9	8	8	8	8	8													
13		13	12	11	10	10	9	9	8	8	8	8	8												
14		14	13	12	11	10	10	9	9	8	8	8	8	8											
15		15	13	12	11	11	10	10	9	9	9	8	8	8	8										
16		16	14	13	12	11	11	10	10	9	9	9	8	8	8	8									
17		17	15	14	13	12	11	11	10	10	9	9	9	9	8	8	8								
18		17	16	14	13	12	12	11	11	10	10	9	9	9	9	9	9	9							
19		18	17	15	14	13	12	12	11	11	10	10	9	9	9	9	9	9	9						
20		19	17	16	15	14	13	12	11	11	10	10	10	9	9	9	9	9	9	9					
21		20	18	16	15	14	13	13	12	11	11	10	10	10	10	9	9	9	9	9	9				
22		21	19	17	16	15	14	13	12	12	11	11	10	10	10	10	9	9	9	9	9	9			
23		22	20	18	16	15	14	13	13	12	12	11	11	10	10	10	10	9	9	9	9	9	9		
24		23	20	19	17	16	15	14	13	13	12	12	11	11	10	10	10	10	9	9	9	9	9	9	
25		24	21	19	18	17	15	14	14	13	12	12	12	11	11	10	10	10	10	9	9	9	9	9	9

Table II.

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1																									
2																									
3																									
4																									
5																									
6																									
7					7	7	7																		
8					8	8	8	8																	
9				9	9	8	8	8	8																
10				10	10	9	9	8	8	8															
11				11	10	10	9	9	9	9	9														
12			12	12	11	11	10	10	9	9	9	9													
13			13	13	12	11	11	10	10	9	9	9	9												
14			14	13	13	12	11	11	10	10	10	9	9	9											
15			15	14	13	13	12	11	11	11	10	10	10	10	10										
16			16	15	14	13	13	12	11	11	11	10	10	10	10	10									
17			17	16	15	14	13	13	12	12	11	11	10	10	10	10	10								
18			18	17	15	15	14	13	13	12	12	11	11	11	10	10	10	10							
19			19	17	16	15	14	14	13	13	12	12	11	11	11	10	10	10	10						
20			19	18	17	16	15	14	14	13	13	12	12	11	11	11	11	10	10	10					
21			20	19	18	17	16	15	14	14	13	13	12	12	11	11	11	11	11	10	10				
22			21	20	18	17	16	15	15	14	14	13	13	12	12	12	11	11	11	11	11	11			
23		23*	22	20	19	18	17	16	15	15	14	14	13	13	12	12	12	11	11	11	11	11	11		
24		24	23	21	20	19	18	17	16	15	15	14	13	13	13	12	12	12	11	11	11	11	11	11	
25		25	24	22	21	19	18	17	16	16	15	14	14	13	13	13	12	12	12	11	11	11	11	11	11

Table III.

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1																									
2																									
3									3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4					4	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3
5				5	5	5	5	5	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
6			6	6	6	6	5	5	5	5	5	5	5	4	4	4	4	4	4	4	4	4	4	4	4
7			7	7	6	6	6	6	5	5	5	5	5	5	5	5	5	5	4	4	4	4	4	4	4
8			8	7	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5	5	5	5	5	4	4
9			9	8	8	7	7	7	6	6	6	6	6	6	5	5	5	5	5	5	5	5	5	5	5
10		10	10	9	8	8	7	7	7	7	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5
11		11	10	10	9	8	8	8	7	7	7	7	6	6	6	6	6	6	6	6	6	5	5	5	5
12		12	11	10	10	9	9	8	8	8	7	7	7	7	6	6	6	6	6	6	6	6	6	6	5
13		13	12	11	10	10	9	8	8	8	8	7	7	7	7	7	7	6	6	6	6	6	6	6	6
14		14	13	12	11	10	10	9	9	8	8	8	8	7	7	7	7	7	7	6	6	6	6	6	6
15		15	13	12	11	11	10	10	9	9	8	8	8	8	8	7	7	7	7	7	7	6	6	6	6
16		16	14	13	12	11	11	10	10	9	9	9	8	8	8	8	7	7	7	7	7	7	7	7	6
17		17	15	14	13	12	11	11	10	10	9	9	9	8	8	8	8	8	7	7	7	7	7	7	7
18		17	16	14	13	12	12	11	11	10	10	9	9	9	9	8	8	8	8	8	7	7	7	7	7
19		18	17	15	14	13	12	12	11	11	10	10	9	9	9	9	8	8	8	8	8	8	7	7	7
20		19	18	16	15	14	13	12	11	11	10	10	10	9	9	9	9	8	8	8	8	8	8	8	7
21		20	18	16	15	14	13	13	12	11	11	10	10	10	9	9	9	9	9	8	8	8	8	8	8
22		21	19	17	16	15	14	13	12	12	11	11	10	10	10	10	9	9	9	9	8	8	8	8	8
23		22	20	18	16	15	14	13	13	12	12	11	11	10	10	10	10	9	9	9	9	9	8	8	8
24		23	20	19	17	16	15	14	13	13	12	12	11	11	10	10	10	10	9	9	9	9	9	8	8
25		24	21	19	18	16	15	14	14	13	12	12	12	11	11	10	10	10	10	9	9	9	9	9	9

Table IV.

m \ *n*	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1																									
2																									
3																						3*	3	3	3
4											4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5								5	5	5	5	5	5	5	5	5	5	5	4	4	4	4	4	4	4
6						6	6	6	6	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5
7					7	7	7	7	6	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5	5
8					8	8	8	7	7	7	7	6	6	6	6	6	6	6	6	6	6	5	5	5	5
9				9	9	8	8	8	8	7	7	7	7	7	6	6	6	6	6	6	6	6	6	6	6
10				10	10	9	9	8	8	8	8	7	7	7	7	7	7	7	6	6	6	6	6	6	6
11				11	10	10	9	9	9	8	8	8	8	8	7	7	7	7	7	7	7	6	6	6	6
12			12	12	11	11	10	10	9	9	9	8	8	8	8	8	7	7	7	7	7	7	7	7	7
13			13	13	12	11	11	10	10	9	9	9	9	8	8	8	8	8	8	8	7	7	7	7	7
14			14	13	13	12	11	11	10	10	10	9	9	9	8	8	8	8	8	8	8	7	7	7	7
15			15	14	13	13	12	11	11	11	10	10	10	9	9	9	9	8	8	8	8	8	8	8	7
16			16	15	14	13	13	12	11	11	11	10	10	10	9	9	9	9	9	8	8	8	8	8	8
17			17	16	15	14	13	13	12	12	11	11	10	10	10	10	9	9	9	9	9	8	8	8	8
18			18	17	15	15	14	13	13	12	12	11	11	11	10	10	10	9	9	9	9	9	9	8	8
19			19	17	16	15	14	14	13	13	12	12	11	11	11	10	10	10	10	9	9	9	9	9	9
20			19	18	17	16	15	14	14	13	13	12	12	11	11	11	10	10	10	10	10	9	9	9	9
21			20	19	18	17	16	15	14	14	13	13	12	12	11	11	11	11	10	10	10	10	9	9	9
22			21	20	18	17	16	15	15	14	14	13	13	12	12	11	11	11	11	10	10	10	10	10	9
23		23*	22	20	19	18	17	16	15	15	14	14	13	13	12	12	12	11	11	11	11	10	10	10	10
24		24	23	21	20	19	18	17	16	15	15	14	13	13	13	12	12	12	11	11	11	11	10	10	10
25		25	24	22	21	19	18	17	16	16	15	14	14	13	13	13	12	12	12	11	11	11	11	10	10

Remark. Table I and Table II are symmetrical about their main diagonal. And *-signs are denoted for the cases when the probabilities are exactly equal to 0.05 or 0.01.

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